ISRAEL MATHEMATICAL CONFERENCE PROCEEDINGS, VOL. 7, 1993

QUASI-LIE BIALGEBRA STRUCTURES OF sl_2, WITT AND VIRASORO ALGEBRAS

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Abstract. The cohomology group $H^1(L, L \wedge L)$ is calculated for the cases when L is an $sl(2)$, Witt or Virasoro Lie algebra (both modular and non-modular). This allows to classify quasi-Lie bialgebra structures on these algebras.

1. Introduction

The question of calculating quasi-Lie bialgebra structures on a given Lie algebra L can be divided into two parts:

(i) first calculate the cohomology group $H^1(L, L \wedge L)$; then

(ii) for a given cocycle $\psi \in Z^1(L, L \wedge L)$, check whether the cohomology class of the cocycle

$$\mathrm{Alt}(1 \otimes \psi) \otimes \psi \in Z^1(L, L \wedge L \wedge L)$$

is trivial.

If the answer to the last question is "yes", i.e.

$$\mathrm{Alt}(1 \otimes \psi) \otimes \psi = d\omega,$$

for some $\omega \in C^0(L, L \wedge L \wedge L) = L \wedge L \wedge L$, then the triple (L, ψ, ω) is called a quasi-Lie bialgebra of L [**1**].

If $\omega = 0$, then the quasi-Lie bialgebra is called a Lie bialgebra on L. A quasi-Lie bialgebra (L, ψ, ω) has coboundary type if $\psi = dr$ is a coboundary for some $r \in L \wedge L$.

According to quantum deformation ideology [**2**], the question of deformation of a Lie bialgebra is more correct than the question of deformaton of a Lie algebra. In the first approach, a quantum deformation of L is just the same as a Lie bialgebra on L.

1991 *Mathematics Subject Classification.* 17B37, 17B50, 17B56, 17B68.
Key words and phrases. Cohomology of Lie algebra, quasi-Lie bialgebra.

To a quasi-Lie bialgebra (L,ψ,ω), one can a associate a Lie algebra $D(L) = L+L'$ (the double of L), where L' is the coadjoint L-module endowed by multiplication induced by ψ and the multiplication between L and L' is defined by ψ and ω.

In our paper we study nontrivial quasi-Lie bialgebra structures on the simple three-dimensional Lie algebra, the Witt algebra and its central extensions. The characteristic p of the ground field P may be zero or positive. In [3], [4] coboundary type bialgebra structures on the Virasoro algebra are studied. We supplement this result proving that any quasi-Lie bialgebra structure on such an algebra will have coboundary type, except the following three cases:

$$p = 2, \quad L = sl_2, \qquad\qquad p = 5, 7, \quad L = W_1.$$

In the latter cases, the doubles of Witt algebras give us examples of simple Lie algebras having extremely short filtration for $p = 5, 7$. Recall that according to A.I. Kostrikin's and A.A. Premet's results any simple Lie algebra of characteristic $p > 7$ has long or short filtration. Examples of simple Lie algebras with extremely short filtrations in characteristic 2 or 3 were known earlier.

2. The main result

Let P be the ground field and p, the characteristic of P. For the set $\mathcal{X}$, by $\langle\mathcal{X}\rangle$ we will denote its linear span over P. Let L be one of the following Lie algebras

$$sl_2 = \langle e_-, e_0, e_+ | [e_-, e_+] = e_0,\ [e_0, e_\pm] = \pm e_\pm\rangle, \quad p \geq 0$$

(this algebra is simple for any p; if $p \neq 2$ this is really the traceless 2×2 matrix algebra);

$$W_1^+ = \langle e_i | [e_i, e_j] = (j-i)e_{i+j},\ -1 \leq i, j,\ i, j \in \mathbb{Z}\rangle,\ p = 0$$

(one-sided Witt algebra isomorphic to a Lie algebra of formal vector fields on the line);

$$W_1 = \langle e_i | [e_i, e_j] = (j-i)e_{i+j},\ i, j \in \mathbb{Z}\rangle,\ p = 0$$

(two-sided Witt algebra isomorphic to a Lie algebra of vector fields on the circle);

$$W_1 = \langle e_\alpha | [e_\alpha, e_\beta] = (\beta-\alpha)e_{\alpha+\beta},\ \alpha, \beta \in \mathbb{Z}/p\mathbb{Z}\rangle,\ p > 2$$

(modular Witt algebra of dimension p);

$$V_1 = \langle e_i, z \mid [e_i, e_j] = (j-i)e_{i+j} + \delta_{i+j,o}(i^3 - i)z, [e_i, z] = 0,\ i, j \in \mathbb{Z}\rangle,\ p = 0$$

(Virasoro algebra isomorphic to a nontrivial central extension of the two-sided Witt algebra);

$$V_1 = \langle e_\alpha, z \mid [e_\alpha, e_\beta] = (\beta-\alpha)e_{\alpha+\beta} + \delta_{\alpha+\beta,o}(\alpha^3-\alpha)z,\ [e_\alpha, z] = 0,\ \alpha, \beta \in \mathbb{Z}/p\mathbb{Z}\rangle,\ p = 2$$

(modular Virasoro algebra isomorphic to a nontrivial central extension of the modular Witt algebra);

$$W_1^+ \oplus Z,\ W_1 \oplus Z,\ p \geq 0$$

(trivial central extensions of Witt algebras).

Let

$$f_i = e_i/(i+1)!, \ -1 \leq i,$$

then

$$[f_i, f_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) f_{i+j}, \ -1 \leq i, j$$

Since the structure constants are integers, we can consider their reductions modulo p, getting in this way the modular infinite-dimensional Lie algebra W_1^+ (mod p). It is easy to see that setting

$$f_i = \sum_{\alpha \in Z/pZ} \alpha^{-1-i} e_\alpha \leq i \leq p-2,$$

we can get an imbedding

$$W_1 = \langle f_i | -1 \leq i \leq p-2 \rangle \subset W_1^+ \quad (\text{mod } p).$$

In this basis the multiplication in the modular Virasoro algebra is given as

$$[f_i, f_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i}\right) f_{i+j} + \delta_{i+j,p}(-1)^i z.$$

We endow the infinite-dimensional Lie algebras

$$W_1, W_1^+, V_1$$

with the filtration topology taking as a base of the neighborhoods of the zero subspaces

$$\mathcal{L}_1 = \langle e_j | j \geq i \rangle, \ i \in \mathbb{Z}.$$

For

$$(L \wedge L)_a = \langle e_j \wedge e_{a-j} | j \in \mathbb{Z}, j < \frac{a}{2}, -1 \leq j \ (\text{if } L = W_1^+) \rangle, \ a \in \mathbb{Z}$$

in the case $L = W_1$, we allow infinite sums of type $\sum_{i<\frac{a}{2}} \lambda_i e_i \wedge e_{a-i}$.

THEOREM. *Let L be one of the following Lie algebras: sl_2 ($p \geq 0$), W_1^+ ($p = 0$), W_1 ($p \neq 2$), V_1 ($p \neq 2$), $W_1^+ \oplus Z$, $W_1 \oplus Z$ ($p \geq 0$).*

(i) *Then $H^1(L, L \wedge L) = 0$, except for the following cases:*

$$p = 2, \ L = sl_2,$$

$H^1(L, L \wedge L)$ is two-dimensional and the basic cocycles ψ_-, ψ_+ can be given by the formulas

$$\psi_-(e_-) = e_0 \wedge e_+, \ \psi_-(e_0) = 0, \ \psi_-(e_+) = 0,$$
$$\psi_+(e_-) = 0, \ \psi_+(e_0) = 0, \ \psi_+(e_+) = e_- \wedge e_0;$$
$$\textit{for} \quad p = 5, 7, \quad L = W_1.$$

$H^1(L, L\wedge L)$ is one-dimensional and nonzero values of the basic cocycle ψ_p on f_i, $-1 \le i \le p-2$, can be obtained from the following cochain $\psi \in C^1(W_1^+, W_1^+ \wedge W_1^+)$ reducing modulo p (in the case $p = 5$ the last two lines should be omitted):

$$\begin{aligned}
\psi(f_{-1}) &= 0,\\
\psi(f_0) &= 0,\\
\psi(f_1) &= f_{-1}\wedge f_2 - f_0\wedge f_1,\\
\psi(f_2) &= 2f_{-1}\wedge f_3 - f_0\wedge f_2,\\
\psi(f_3) &= 5f_{-1}\wedge f_4 - 3f_0\wedge f_3 + 2f_1\wedge f_2,\\
\psi(f_4) &= 10f_{-1}\wedge f_5 - 5f_0\wedge f_4 + 2f_1\wedge f_3,\\
\psi(f_5) &= -4f_0\wedge f_5 - f_1\wedge f_4 + 3f_2\wedge f_3,
\end{aligned}$$

(ii) *$(sl_2, \psi_-, 0)$ and $(sl_2, \psi_+, 0)$ are Lie bialgebras.*

For $\omega = 2f_{-1}\wedge f_0\wedge f_1$, $p = 5, 7$, the triple (W_1, ψ_p, ω) forms a quasi-Lie bialgebra. In the case $p = 5$ the double $D(W_1)$ is isomorphic to the 10-dimensional classical simple Lie algebra B_2, and in the case $p = 7$, the double $D(W_1)$ is isomorphic to 14-dimensional exceptional simple Lie algebra of type G_2.

REMARK 1. Note that the 7-dimensional Witt algebra

$$W_1 = \langle f_i | -1 \le i \le 5\rangle$$

with nonzero terms in the multiplication table given as follows

	f_0	f_1	f_2	f_3	f_4	f_5
f_{-1}	f_{-1}	f_0	f_1	f_2	f_3	f_4
f_0		f_1	$2f_2$	$3f_3$	$4f_4$	$5f_5$
f_1			$2f_3$	$5f_4$	$-5f_5$	
f_2				$5f_5$		

is a Lie algebra not only over the field of characteristic 7, but also over the ring $\mathbb{Z}/14\mathbb{Z}$. The cochain ψ_{14} obtained from ψ modulo 14 will be also a cocycle over the ring $\mathbb{Z}/14\mathbb{Z}$. So, we obtain G_2 as a double of 7-dimensional Witt algebra over $\mathbb{Z}/14\mathbb{Z}$.

REMARK 2. Let Q be a Lie algebra and Q_0, a subalgebra of Q. Construct a Weisfeiler filtration

$$Q = Q_{-q} \supset \cdots \supset Q_{-1} \supset Q_0 \supset Q_1 \supset \cdots \supset Q_r \supset 0$$

where

$$Q_{i+1} = \langle x \in Q_i | [Q, x] \subseteq Q_i\rangle,\ i \ge 0$$
$$Q_{-1}/Q_0 \text{ is an irreducible } Q_0\text{-module},$$
$$Q_{-i-1} = [Q_{-1}, Q_{-i}],\ i \ge 1.$$

The filtration is long if $r \geq 2$, short if $r = 1$, and extremely short if $r = 0$. For classical simple Lie algebras ($p = 0$ or $p > 7$), any filtration will be short. For simple Lie algebras of Cartan type ($p > 7$) the filtration will be long.

Let us prove that imbeddings

$$W_1 \subset D(W_1),\ p = 5 \text{ or } 7$$

give us an example of extremely short filtrations. Quotient-modules $D(W_1)/W_1$ as W_1-modules are coadjoint modules; in particular, they are irreducible. Since W_1 is a simple Lie algebra and a first prolongation $(D(W_1))_1$ will be an ideal in the zero component $(D(W_1))_0$, we get a two-term filtration

$$D(W_1) = (D(W_1))_{-1} \supset (D(W_1))_0 = W_1 \supset 0.$$

Since, according to the theorem, $D(W_1)$ will be a simple Lie algebra, we obtain an example of simple Lie algebras in characteristic $p = 5, 7$ with extremely short filtrations.

3. Preliminary facts

Let Q be a Lie algebra and M is a Q-module. Suppose that H is a Cartan subalgebra and Q and M are semisimple H-modules. Let $C^*(Q, M) = \bigoplus_k C^k(Q, M)$ be the standard cochain complex. Recall that

$$C^0(Q, M) = M,$$
$$C^1(Q, M) = \langle \text{linear maps } \alpha : Q \to M \rangle$$
$$C^2(Q, M) = \langle \text{skew-symmetric bilinear maps } \beta : Q \times Q \to M \rangle$$

The coboundary operator

$$d : C^k(Q, M) \to C^{k+1}(Q, M)$$

for small k is defined by the formulas

$$dm(x) = x(m), \qquad k = 0,$$
$$d\alpha = -\alpha[x, y] + x\alpha(y) - y\alpha(x), \quad k = 1.$$

Let

$$Z^k(Q, M) = \langle \varphi \in C^k(Q, M) | d\varphi = 0 \rangle \text{ (subspace of cocycles)},$$
$$B^k(Q, M) = \langle d\varphi | \varphi \in C^{k-1}(Q, M) \rangle \text{ (subspace of coboundaries)},$$

and

$$H^k(Q, M) = Z^k(Q, M)/B^k(Q, M)$$

be the k-cohomology of a Lie algebra Q with coefficients M.

The first cohomology space $H^1(Q, M)$ has many interpretations. For example, $H^1(Q, Q)$ is isomorphic to the space of outer derivations of Q and $H^1(Q, Q \wedge Q)$ is responsible for the quasi-Lie bialgebra structures on Q. Recall that

$$H_1(Q, P) \simeq Q/[Q, Q].$$

LEMMA 1. *Let*

$$C_0^*(Q,M) = \langle \varphi \in C^*(Q,M) | h\varphi = 0, \ \forall h \in H \rangle$$

be the subcomplex of invariants under the action of H. *Then the cohomology of* $C_0^*(Q,M)$, *denoted by* $H_0^*(Q,M)$, *is isomorphic to* $H^*(Q,M)$.

In the sequel, L denotes one of the Lie algebras described in Section 2.

In our cases H is equal either to $\langle e_0\rangle$ or $\langle f_0\rangle$ and thus one-dimensional or equal to $\langle e_0, z\rangle$ ($\langle f_0, z\rangle$) and thus two-dimensional (in the central extension case). The Cartan decompositions are

$$L = \bigoplus_i L_i, \qquad L_i = \langle e_i\rangle \text{ or } \langle f_i\rangle, \ (i \geq -1)$$
$$L \wedge L = \bigoplus_a (L\wedge L)_a, \qquad (L\wedge L)_a = \langle e_i \wedge e_{a-i}\rangle, \ (i < \frac{a}{2})$$

All of our cochains will satisfy conditions

$$\alpha(e_a) \in (L\wedge L)_a, \quad a \in C^1(L, L\wedge L) \qquad (1)$$
$$r \in (L\wedge L)_0.$$

Let

$$\mathcal{L}_1^+ = \langle e_i \mid i \geq 1\rangle = \langle f_i \mid i \geq 1\rangle \quad (i \leq p-2, \text{ if } p \neq 0)$$
$$\mathcal{L}_1^- = \langle e_i \mid i \leq -1\rangle \quad (i \leq p-2, \quad \text{if } p \neq 0).$$

LEMMA 2. $H_1(\mathcal{L}_1^+, P)$ *is 2-dimensional and the classes of the elements* f_1, f_2 *form a basis.* $H_1(\mathcal{L}_1^-, P)$ *is 2-dimensional and a basis of cohomology classes is provided by the elements* e_{-1}, e_{-2}.

PROOF. If $i > 2$, then

$$e_i = \frac{1}{(i-2)}[e_1, e_{i-1}] \in [\mathcal{L}_1^+, \mathcal{L}_1^+],$$
$$e_{-i} = \frac{1}{(2-i)}[e_{-1}, e_{-i+1}] \in [\mathcal{L}_1^-, \mathcal{L}_1^-].$$

It is obvious that

$$e_1, e_2 \notin \langle [e_i, e_j] \mid i+j = 1,2 \quad 0 < i,j\rangle,$$
$$e_{-1}, e_{-2} \notin \langle [e_{-i}, e_{-j}] \mid i+j = 1,2, \quad 0 < i,j\rangle. \quad \square$$

Let

$$C(L) = \langle x \in L\wedge L \mid [f_{-1}, X] = 0\rangle.$$

LEMMA 3. *For $L = W_1^+$ $(p = 0)$, W_1, or V_1 $(p \neq 0)$, the following is true:*

$$C(L) \subseteq \bigoplus_{k \geq 0} (L \wedge L)_{2k-1},$$

namely, $C(L)$ is generated by elements of type

$$\sum_{i=-1}^{k-1} (-1)^i f_i \wedge f_{2k-1-i}.$$

For $L = W_1$ or V_1 $(p = 0)$, the following is true:

$$C(L) \subseteq \bigoplus_{a \leq -3} (L \wedge L)_a \oplus \bigoplus_{k \geq 0} (L \wedge L)_{2k-1}.$$

PROOF. Let

$$X = \sum_{i < \frac{a}{2}} \lambda_i e_i \wedge e_{a-i} \in C(L) \cap (L \wedge L)_a.$$

Represent X as a sum of X' and X'', where

$$X' = \sum_{-1 \leq i < \frac{a}{2}} \mu_i f_i \wedge f_{a-i}, \quad \mu_i = \lambda_i (i+1)!(a-i+1)!,$$

$$X'' = \sum_{i < -1, i < \frac{a}{2}} \lambda_i e_i \wedge e_{a-i} \quad (if \quad L = W_1 \text{ or } V_1 \ (p = 0)).$$

Then

$$X, X'' \in C(L).$$

If $a = 2k$, $k \geq 0$, then

$$[f_{-1}, X'] = (\mu_{-1} + \mu_0) f_{-1} \wedge f_{2k} + \cdots + (\mu_{k-2} + \mu_{k-1}) f_{k-2} \wedge f_{k+1} + \mu_{k-1} f_{k-1} \wedge f_k,$$

therefore,

$$X' \in C(L) \Rightarrow \mu_{k-1} = 0, \quad \mu_{k-2} = 0, \ldots, \mu_{-1} = 0 \Rightarrow X' = 0.$$

If $a = 2k - 1$, $k \geq 0$, then

$$[f_{-1}, X'] = \sum_{i=-1}^{k-2} (\mu_i + \mu_{i+1}) f_i \wedge f_{2k-2-i},$$

that is why,

$$X' \in C(L) \Rightarrow X' = \sum_{i=-1}^{k-1} (-1)^i \mu_0 f_i \wedge f_{2k-1-i}.$$

So, for $L = W_1^+$ $(p = 0)$, W_1 or V_1 $(p > 0)$ the lemma is proved.

Let us consider the condition

$$[f_{-1}, X''] = 0.$$

Take the maximal $j \leq -2$ such that $\lambda_j \neq 0$. Since

$$[f_{-1}, X''] - \lambda_j e_j \wedge [f_{-1}, e_{a-j}] \in \langle e_s \wedge e_{a-s} \mid s < \frac{a}{2}, \ s < j \rangle,$$

we have

$$[f_{-1}, e_{a-j}] = 0,$$

from what follows, $a = j - 1 \leq -3$. □

For the Lie algebra Q, the subalgebra N and the Q-module M, let

$$C^1(Q, N, M) = \langle \alpha : Q \to M \mid \alpha(x) = 0,\ \forall x \in N \rangle,$$
$$Z^1(Q, N, M) = C^1(Q, N, M) \cap Z^1(Q, M).$$

LEMMA 4. *For $L \supsetneqq W_1^+$, we have $Z^1(L, W_1^+, L \wedge L) = 0$.*

PROOF. Let $\alpha \in Z^1(L, W_1^+, L \wedge L)$ and i be the maximal integer such that $\alpha(e_i) \neq 0$. Then $i \geq -2$. Moreover,

$$e_i \notin \langle [e_j, e_s] \mid j + s = i, j > i, \quad s > i \rangle,$$

because

$$d\alpha(e_j, e_s) = 0, \quad j + s = i, \quad j > i, \quad s > i \Rightarrow \alpha([e_j, e_s]) = 0.$$

So, by lemma 2, $i = -2$. On the other hand,

$$d\alpha(e_{-1}, e_i) = 0 \Rightarrow [e_{-1}, \alpha(e_i)] = 0,$$

and by Lemma 3 and according to (1), $i \leq -3$, contradiction. □

LEMMA 5. ($L \neq sl_2$). *Any cocycle $\alpha \in Z^1(L, L \wedge L)$ is cohomologous to some cocycle ψ such that $\psi(e_{-1}) = 0$.*

PROOF. According to (1), for some $\lambda_i \in P$, $i \geq 1$,

$$\alpha(e_{-1}) = \sum_{i \geq 1} e_{-1} \wedge e_{i-1}.$$

Let

$$r = \sum_{i \geq 1} \mu_i e_{-i} \wedge e_i \in (L \wedge L)_0,$$

where

$$\mu_1 = \frac{\lambda_1}{2}, \quad \mu_i = \sum_{j=2}^{i} \frac{\lambda_j}{(i+1)i \ldots (j+1)}, \quad i \geq 2.$$

Then

$$[e_{-1}, r] = \alpha(e_{-1}).$$

So, the cocycle $\psi = \alpha - dr$ satisfies the condition $\psi(e_{-1}) = 0$. □

4. Proof of the theorem in case $L = sl_2$.

The exterior power $L \wedge L$ as an L-module is isomorphic to the adjoint L-module:

$$e_- \mapsto e_- \wedge e_0, \quad e_0 \mapsto e_- \wedge e_+, \quad e_+ \mapsto e_0 \wedge e_+.$$

So, $H^1(L, L \wedge L)$ is isomorphic to the space of outer derivations Out $L = H^1(L, L)$.

If $p \neq 2$, the Lie algebra sl_2 has a nondegenerate Killing form, and standard reasonings using the Casimir element show that

$$H^1(L, L) = 0, \quad p \neq 2.$$

Let $p = 2$. For any prime p and $x \in L$, the endomorphism $(\mathrm{ad}\ x)^p$ is a derivation. If for some $\lambda_-, \lambda_+ \in P$, the derivation $D = \lambda_-(\mathrm{ad}\ e_-)^2 + \lambda_+(\mathrm{ad}\ e_+)^2$ is interior, i.e. $D = \mathrm{ad}\ X$, $X \in L$, then

$$D(e_-) = \lambda_+ e_+ = [X, e_-] \Rightarrow \lambda_+ = 0,$$
$$D(e_+) = \lambda_- e_- = [X, e_+] \Rightarrow \lambda_- = 0.$$

This means that $(\mathrm{ad}\ e_-)^2$ and $(\mathrm{ad}\ e_+)^2$ are linearly independent outer derivations.

For any $F \in Z_0^1(L, L)$, we have

$$dF(e_0, e_\pm) = 0 \Rightarrow [e_\pm, F(e_0)] = 0 \Rightarrow F(e_0) = 0.$$
$$dF(e_-, e_+) = 0 \Rightarrow [e_-, F(e_+)] = [e_+, F(e_-)].$$

Therefore, for some $a, b, c \in P$,

$$F(e_+) = ae_- + be_+, \quad F(e_-) = -be_- + ce_+.$$

In other words,

$$F = a(\mathrm{ad}\ e_-)^2 + c(\mathrm{ad}\ e_+)^2 + b\ \mathrm{ad}\ e_0.$$

So, Out L is two-dimensional and the classes of the derivations $(\mathrm{ad}\ e_-)^2$, $(\mathrm{ad}\ e_+)^2$ form a basis.

To conclude the proof, it is enough to notice that these derivations correspond to cocycles ψ_-, ψ_+ for $H^1(L, L \wedge L)$.

5. Proof of the theorem in the case $L \neq sl_2$

Recall that for $-1 \leq i$ (in the $p > 2$ case, $i \leq p - 2$) instead of e_i we take f_i and the multiplication between f_i, f_j is defined by binomial coefficients.

Let $0 \neq \alpha \in Z_0^1(L, L \wedge L)$. By Lemma 5 there exists a cocycle β cohomologous to α such that

$$\beta(f_{-1}) = 0.$$

If $\beta(f_i) = 0$, for all $i \geq -1$, then by Lemma 4, $\beta = 0$.

If L has a central element z, then

$$d\beta(z, X) = 0,\ \forall X \in L \Rightarrow [X, \beta(z)] = 0 \Rightarrow \beta(z) = 0.$$

So, one can find a minimal i such that $\beta(f_i) \neq 0$. By Lemma 3, i is an odd number. By Lemma 2, this is possible only if $i = 1$. Furthermore,

$$d\beta(f_{-1}, f_j) = 0 \Rightarrow [f_{-1}, \beta(f_j)] = \beta(f_{j-1}).$$

The last conditions for $j = 1, 2, 3, 4, 5$ give us

$$\beta(f_{-1}) = 0$$
$$\beta(f_0) = 0$$
$$\beta(f_1) = tf_{-1} \wedge f_2 - tf_0 \wedge f_1$$
$$\beta(f_2) = 2tf_{-1} \wedge f_3 - tf_0 \wedge f_2$$
$$\beta(f_3) = af_{-1} \wedge f_4 + (2t - a)f_0 \wedge f_3 + (-3t + a)f_1 \wedge f_2$$
$$\beta(f_4) = (-5t + 3a)f_{-1} \wedge f_5 + (5t - 2a)f_0 \wedge f_4 + (-3t + a)f_1 \wedge f_3$$
$$\beta(f_5) = bf_{-1} \wedge f_6 + (-5t + 3a - b)f_0 \wedge f_5 + (10t - 5a + b)f_1 \wedge f_4 + (-13t + 6a - b)f_2$$

for certain $t, a, b \in P$.

Now the conditions $d\beta(f_1, f_2) = 0$, $d\beta(f_1, f_4) = 0$ and $d\beta(f_2, f_3) = 0$ give us respectively:

$$a = 5t, \quad b = 154t, \quad 5b = 70t.$$

If $p \neq 3, 5, 7$, these equalities imply that $\beta(f_j)$ vanishes for the small values of j and, hence, for all j.
In the case $p = 3$, $W_1 \simeq sl(2)$, and this case was already considered.
In the cases $p = 5, 7$, we get $b = 0$ and the cocycle values as indicated in the statement of the theorem (the fact that this is really a cocycle is verified by a direct check).

For $r \in (L \wedge L)_0 = \langle f_{-1} \wedge f_1 \rangle$, $L = W_1$, $p = 5, 7$, we have

$$dr(f_1) \in \langle [f_1, f_{-1} \wedge f_1] \rangle = \langle f_0 \wedge f_1 \rangle.$$

This means that $\psi(f_1) \neq dr(f_1)$. In other words, ψ is not a coboundary.

So,

$$\dim H^1(L, L \wedge L) = \begin{cases} 1, & \text{if } p = 5, 7, \ L = W_1, \\ 0, & \text{otherwise} \end{cases}$$

Now we would like to prove the isomorphism

$$D(W_1) \cong B_2, \quad p = 5.$$

Let $\{f_i' \mid (f_i', f_j) = \delta_{i,j}, -1 \leq i, j \leq p - 2\}$ is the dual basis in the coadjoint W_1-module W_1'. The multiplication in $W_1 + W_1'$, corresponding to $\varepsilon\psi$, $2\varepsilon^2 f_{-1} \wedge f_0 \wedge f_1$, where ε is infinitely small, is defined by the tables

	f_{-1}	f_0	f_1	f_2	f_3
f_{-1}	0	f_{-1}	f_0	f_1	f_2
f_0		0	f_1	$2f_2$	$3f_3$
f_1			0	$2f_3$	0
f_2				0	0

	f'_{-1}	f'_0	f'_1	f'_2	f'_3
f_{-1}	$-f'_0$	$-f'_1$	$-f'_2$	$-f'_3$	0
f_0	f'_{-1}	0	$-f'_1$	$-2f'_2$	$2f'_3$
f_1	εf_2	$-\varepsilon f_1 + f'_{-1}$	$\varepsilon f_0 + f'_0$	$-\varepsilon f_{-1}$	$-2f'_2$
f_2	$2\varepsilon f_3$	$-\varepsilon f_2$	f'_{-1}	$\varepsilon f_0 + 2f'_0$	$-2\varepsilon f_{-1} + 2f'_1$
f_3	0	$2\varepsilon f_3$	$2\varepsilon f_2$	$-2\varepsilon f_1 + f'_{-1}$	$-2\varepsilon f_0 - 2f'_0$
f'_{-1}	0	$2\varepsilon^2 f_1$	$-2\varepsilon^2 f_0$	$\varepsilon f'_1$	$2\varepsilon f'_2$
f'_0	$-2\varepsilon^2 f_1$	0	$2\varepsilon^2 f_{-1} - \varepsilon f'_1$	$-\varepsilon f'_2$	$2\varepsilon f'_3$
f'_1	$2\varepsilon^2 f_0$	$-2\varepsilon^2 f_{-1} + \varepsilon f'_1$	0	$2\varepsilon f'_3$	0
f'_2	$-\varepsilon f'_1$	$\varepsilon f'_2$	$-2\varepsilon f'_3$	0	0
f'_3	$-2\varepsilon f'_2$	$-2\varepsilon f'_3$	0	0	0

The required isomorphism can be given in the following way:

$$e_{-2\alpha-\beta} = -f_3, \quad e_{-\alpha-\beta} = \varepsilon^{-1} f'_{-1} - 2f_1, \quad e_{-\beta} = \varepsilon^{-1} f'_1 + 2f_{-1},$$
$$e_{-\alpha} = f_2, \quad h_\alpha = 2\varepsilon^{-1} f'_0 + f_0, \quad h_\beta = -3\varepsilon^{-1} f'_0 + 3f_0, \quad e_\alpha = \varepsilon^{-1} f'_2,$$
$$e_\beta = \varepsilon^{-1} f'_{-1} + f_1, \quad e_{\alpha+\beta} = \varepsilon^{-1} f'_1 - f_{-1}, \quad e_{2\alpha+\beta} = 2\varepsilon^{-1} f'_3.$$

The case

$$D(W_1) \cong G_2, \quad p = 7$$

is considered analogously.

For multiplication in W_1 see remark 1. The multiplication between W'_1 and $W_1 + W'_1$ is given by the table

	f_{-1}	f_0	f'_1	f'_2	f'_3	f'_4	f'_5
f_{-1}	$-f'_0$	$-f'_1$	$-f'_2$	$-f'_3$	$-f'_4$	$-f'_5$	0
f_0	f'_{-1}	0	$-f'_1$	$-2f'_2$	$-3f'_3$	$-4f'_4$	$-5f'_5$
f_1	εf_2	$-\varepsilon f_1 + f'_{-1}$	$\varepsilon f_0 + f'_0$	$-\varepsilon f_{-1}$	$-2f'_2$	$-5f'_3$	$+5f'_4$
f_2	$2\varepsilon f_3$	$-\varepsilon f_2$	f'_{-1}	$\varepsilon f_0 + 2f'_0$	$-2\varepsilon f_{-1} + 2f'_1$	0	$-5f'_3$
f_3	$5\varepsilon f_4$	$-3\varepsilon f_3$	$2\varepsilon f_2$	$-2\varepsilon f_1 + f'_{-1}$	$3\varepsilon f_0 + 3f'_0$	$-5\varepsilon f_{-1} + 5f'_1$	$5f'_2$
f_4	$-4\varepsilon f_5$	$-5\varepsilon f_4$	$2\varepsilon f_3$	0	$-2\varepsilon f_1 + f'_{-1}$	$5\varepsilon f_0 + 4f'_0$	$4\varepsilon f_{-1} - 5f'_1$
f_5	0	$-4\varepsilon f_5$	$-\varepsilon f_4$	$3\varepsilon f_3$	$-3\varepsilon f_2$	$+\varepsilon f_1 + f'_{-1}$	$4\varepsilon f_0 + 5f'_0$
f'_{-1}	0	$2\varepsilon^2 f_1$	$-2\varepsilon^2 f_0$	$\varepsilon f'_1$	$2\varepsilon f'_2$	$5\varepsilon f'_3$	$-4\varepsilon f'_4$
f'_0		0	$2\varepsilon^2 f_{-1} - \varepsilon f'_1$	$-\varepsilon f'_2$	$-3\varepsilon f'_3$	$-5\varepsilon f'_4$	$-4\varepsilon f'_5$
f'_1			0	$2\varepsilon f'_3$	$2\varepsilon f'_4$	$-\varepsilon f'_5$	0
f'_2				0	$3\varepsilon f'_5$	0	0

For $[f'_i, f'_j]$ we should use the skew-symmetry condition if $i > j$ and set it equal to zero, if $3 \leq i, j$. The required isomorphism can be given by the rules

$$e_{3\alpha-2\beta} = 5f_5, \quad e_{-3\alpha-\beta} = -5(\varepsilon^{-1}f'_{-1} + f_1), \quad e_{-2\alpha-\beta} = 5f_2,$$
$$e_{-\alpha-\beta} = 3f_3, \quad e_{-\alpha} = -\varepsilon^{-1}f'_1 + f_{-1}, \quad e_{-\beta} = f_4,$$
$$h_\alpha = \varepsilon^{-1}f'_0 + 2f_0, \quad h_\beta = 4\varepsilon^{-1}f'_0 + 5f'_0, \quad e_\beta = \varepsilon^{-1}f'_4,$$
$$e_\alpha = 3\varepsilon^{-1}f'_{-1} + f_1, \quad e_{\alpha+\beta} = -\varepsilon^{-1}f'_3, \quad e_{2\alpha+\beta} = -\varepsilon^{-1}f'_1,$$
$$e_{3\alpha+\beta} = -3\varepsilon^{-1}f'_1 + f_{-1}, \quad e_{3\alpha+2\beta} = \varepsilon^{-1}f'_5$$

Acknowledgements

I would like to thank W. Michaelis for bringing to my attention the question of the calculation of $H^1(L, L \wedge L)$ for Witt algebras and its central extensions. I am deeply grateful to S. Shnider for his attention to my work and invitation to the conference.

REMARK. Since submitting this paper, the author has calculated the first cohomology group $H^1(L, L \wedge L)$ for Witt and Hamiltonian algebras of many variables. This group is not trivial in the Hamiltonian case.

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