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# A More Extended Sketch of GJA's

1. The "IG". Suppose we have an associative identity, ~~built out of as~~ a linear combination of terms, each built out of multiplication, [ ] and { }. Declare some of the variables odd. Reinterpret [ ] and { } graded style, i.e. ~~change~~ "interchange" them if both their variables are odd. (Assume each term "amounts" to the same $x_1 \ldots x_n$.) Taking $x_1, \ldots, x_n$ as the standard order, stick a minus sign

on a term if in it the odd elements have undergone an odd permutation. The claim is that we have a valid graded identity.

*Stab at proof.* Expand ~~the~~ out the ungraded identity. We get terms which are permutations of $x_1 \dots x_n$. For each permutation the sum of the coefficients is 0. Now grade it. Look, for instance at what happens to $\{\alpha \beta\}$ when $\alpha$ and $\beta$ are both odd. Ungraded, it is $\alpha\beta + \beta\alpha$, Graded: $\alpha\beta - \beta\alpha$. The switch from

$\alpha\beta$ to $\beta\alpha$ involves $rs$ transpositions of odd variables, where $\alpha$ has $r$ odd variables and $\beta$ $s$. Both $r$ and $s$ are odd. The ~~two~~ minus signs compensate!

2. Jordan axioms, etc.

The ungraded Jordan axiom, in linearized form, is

$$ab.cd + ac.db + ad.bc =$$

$$(bc.a)d + (cd.a)b + (db.a)c$$

The left side is pretty symmetric

~~Working off~~

Particularly important is the

proposition (graded) that $a, b$ odd $\Rightarrow$
$L_a L_b + L_b L_a$ is a derivation, or its collapsed version that $a$ odd $\Rightarrow L_a^2$ is a derivation.

Note. On an even element $L_a = R_a$. On an odd element $L_a = -R_a$. In any event $L_a^2 = R_a^2$.

Write $D = L_a^2$. We need

$$D(xy) = Dx.y + x.Dy.$$

~~except~~
there is no "except" since $D$ is even.

$$\cancel{a.(a.xy) \overset{?}{=} a^2x\, y + x.a^2y}$$

$$a(a.xy) \overset{?}{=} (a.ax)y + x(a.ay)$$

This is a consequence of I G.

3. <u>A reference</u>. R. D. Schafer showed me that the theorem about the stability of the radical of a Jordan algebra of char. 0 is in a Jac. paper

4. <u>The first theorem</u>. $J = K + L$ graded Jordan algebra of char 0. Assume $J$ has a unit element $1$ and $K = 1 +$ radical $N$. Then $N + NL$ is an ideal.

Proof. Stability under $K$ is easy. $KN \subset N$ anyhow. For $K . NL$ we can drop to $N . NL$ to see $K . NL \subset NL$ (note $NL \subset L$) since $1$ is harmless.

For stability under $\cancel{K}$ $L$, all we need is $\cancel{N\,NL}$ $NL.L \subset N$.

This we do as follows. If $NL = L$, a kind of Nakayama's lemma shows that $L = 0$. So $NL \neq L$. Take $a \in L$, $a \notin NL$. Take $z \in L$. We plan to show that $NL.z \subset N$. It is enough to prove that $(NL.z)a \in NL$, for if $NL.z$ contains $1+n$, then we would have $(1+n)a \in NL$, $a \in NL$. Now

$$(NL.z)a + (NL.a)z \subset NL.$$

For $R_z R_a + R_a R_z$ is a derivation (even). This derivation sends $L$ into $L$, $N$ into $N$ (for the latter see above). So it is enough to prove $(NL.a)z \subset NL$. Of course for this $NL.a \subset N$ will do. We do have $(NL.a)a \subset NL$ (derivation again). Therefore $NL.a \subset N$. Done.

5. <u>Semisimplicity of K</u>. Now we launch the long proof that $J$ semisimple $\Rightarrow$ $K$ semisimple (field alg. closed and of characteristic 0). The method uses a Peirce decomposition heavily (as well as the result in §4). We take a maximal set $\{e_i\}$ of orthogonal primitive idempotents in $K$. In the event that $J$