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Remarks on Multiplicative Lattices

I. Kaplansky

1. Krull's theorem. I take it in the form: if a commutative ring has no nilpotent elements then the intersection of the prime ideals is 0.

There are two noncommutative generalizations.

A. If there are no nilpotent elements then the intersection of the completely prime ideals is 0 (I is completely prime if $xy \in I \Rightarrow x$ or y in I; in other words R/I has no zero-divisors).

B. If there are no nilpotent ideals then the intersection of the prime ideals is 0. ("Ideal" shall always mean "two-sided ideal". I is prime if $JK \subset I \Rightarrow J$ or $K \subset I$, J and K ideals; in other words R/I has no ideal zero-divisors.)

The proofs are quite different. I haven't thought about moving A to multiplicative lattices; presumably it would require appropriate assumptions about principal ideals, if it could be done at all. B moves to lattices nicely. But before doing this I ~~now~~ allow nonassociativity.

~~Now A fails. For a miniature example take the 2-dimensional algebra over GF(2) with basis~~

~~u~~ To avoid a boring fuss here over the meaning of nilpotence, I modify the hypothesis to: no elements (resp. ~~of~~ ideals) with square 0. (This formally strengthens the affirmative theorems though maybe it shortchanges the counterexamples.)

A fails. Take the 2-dimensional algebra R over the complex numbers with basis u, v and $u^2 = u$, $v^2 = v$, $uv = vu = u + v$. Then R is simple, there are zero-divisors but no elements with square 0.

B works! When I noticed this recently I believe it forced me to give a proof simpler than the usual associative (noncommutative) one. The next thought: since this is in particular a theorem on Lie rings, there ought to be a corresponding one for groups. There is, and it's in the literature (Schenkman, Proc. Amer. Math. Soc. 9 (1958), 375–384.) (My thanks to Dan Anderson for this reference.)

At this point a unification via multiplicative lattices ~~seemed~~ seemed like a good idea, and it works fine.

The setup is a complete ~~multiplicative~~ lattice L with a multiplication (no commutativity, associativity or any substitute). The proposed hypotheses that $x^2 = 0$ implies $x = 0$, and the proposed conclusion that the intersection of the primes is 0 forthwith make sense.

$xy \leq x \cap y$ and also

I assume distributivity of multiplication into union: $(x \cup y)z = xz \cup yz$ and $z(x \cup y) = zx \cup zy$. And, in order to allow Zornification, ~~we~~ assume that every element is a union of compact ones.

Example 1. In any ring take the ideals, with product = the ideal generated by the ordinary product.

Example 2. In any group take the normal subgroups, with product = the commutator.

At last I give the proof. Assume on the contrary that the intersection x of the primes is nonzero. I construct a descending sequence $\{x_i\}$ of nonzero compact elements. Take x_1 compact $\leq x$. Having constructed x_n, take x_{n+1} compact with $x_{n+1} \leq x_n^2$. Now let p be maximal with respect to not containing any x_i. If p is not prime take

$yz \leq p$ with neither y nor $z \leq p$. By replacing y, z by $y \cup p$, $z \cup p$ can assume y, z properly contain p. Then $y \supset$ ~~some~~ some x_i, $z \supset x_j$. Say $j \geq i$. Then y and z both contain x_j. Hence $p \geq yz \geq x_j^2 \geq x_{j+1}$, contradiction

2. Lie lattices

For a more serious ~~[illegible]~~ unification of suitable theorems on groups and Lie rings, the appropriate axiom is a "Jacobi inequality":

$$xy.z \leq yz.x \cup zx.y.$$

Assume commutativity as well. The ideals in a Lie ring and the normal subgroups of a group fulfil this axiom. I find this to work better than the unification proposed by Higgins (Proc. Lon. Math. Soc. vol 6 (1956).)

The target theorem is that the union of two nilpotent elements is nilpotent.

Remark. Solvability has an obvious meaning and -- without any Jacobi axiom -- it is very easy that the union of two solvable elements is solvable.

I recapitulate that a ~~[illegible]~~ lattice is assumed, also $xy \leq x \cap y$, and distributivity of multiplication into union.

The meaning of nilpotency first cries for attention. Define x^n inductively by $x^n = x.x^{n-1}$. Say x is nilpotent if some $x^n = 0$. It then turns out that any product of r x's, $r \geq n$, is 0. And more.

Prop. Let y be a product in any association of m elements, r of which are x's. Then $y \leq x^r$.

2. Unions of nilpotent ideals. Looking for more material to unify I gazed at the propositions: (a) in a Lie ring the union of two nilpotent ideals is nilpotent, (b) in a group the union of two nilpotent normal subgroups is nilpotent. ~~Initially a~~ The ~~thought a Jacobi identity~~ unification proposed by Higgins (Proc. Lon. Math. Soc. vol. 6 (1956)) seems to me awkward. Multiplicative lattices look more promising.

Initially I thought a Jacobi identity axiom was crucial. On closer inspection the key is seen to be simply that the product of two ideals is an ideal.

Proposed setup: a lattice with 0, a commutative multiplication, and the axioms $xy \leqq x$, $(x \cup y)z = xz \cup yz$. Now one must agree on a definition of "nilpotent". I say x is nilpotent if for some n the product of n x's in any association is 0. It then follows (exercise) that any product containing n x's vanishes

Remark. Maybe you'd prefer to define x^n inductively by $x^n = x x^{n-1}$ and say that some $x^n = 0$. But then any product of ~~...~~ 2^n x's is 0 (exercise).

Th. x, y nilp $\Rightarrow$ $x \cup y$ nilp.

Pf. Indeed if m, n work for x, y then $m+n-1$ works for $x \cup y$.

The obvious application is to rings in which I, J ideals $\Rightarrow IJ+JI$ is an ideal; take the product to be $IJ+JI$. This covers Lie rings and alternative rings or more generally the "shrinkable" rings of Albert (Portugaliae 8, 1949)

Some more mileage can be squeezed out. In any nonassociative ring define $I \circ J$ to be the ideal generated by $IJ + JI$. Use this as a multiplication, as above. This gives rise to a notion I call <u>ultranilpotent</u>. Always: the sum of two ultranilpotent ideals is ultranilpotent. ~~((Still more generally, put $x \circ y = xy \cup yx$ in a general multiplicative lattice))~~

Query: in a Jordan algebra does nilpotence imply ultranilpotence? Failing this, is the sum of two nilpotent ideals nilpotent?

A little example is needed at this point. Take the 4-dim comm. alg A with basis u_1, u_2, v, w and $vu_1 = u_2$, $wu_2 = u_1$, all other products 0. Let I be spanned by ~~v, u_1, u_1~~ u_1, u_2, v, J by ~~v, w, u_2~~ u_1, u_2, w. Then I and J are ideals. I^2 is spanned by u_2, $I^3 = 0$. $I + J = A$, A^2 is spanned by u_1, u_2, $A^2 = A^3$, A is not nilpotent. I is not ultranilpotent.

3. <u>Lie multiplicative lattices</u>. Here is a theorem suitable for ~~latt~~ multiplicative lattice treatment. It was proved by Kac for Lie algebras (Izv. 32 (1968)) and then for Lie superalgebras (Adv. in Math. 26 (1977)). It was in trying to unify these that I moved the setting to Lie multiplicative lattices.

Picture to have in mind: the additive subgroups of a Lie ring. Then a graded Lie ring; take homogeneous additive subgroups. In a Lie superalgebra (graded mod 2) homogeneous spaces. There can be another grading on top of the "super" grading.

Axioms: a commutative multiplicative lattice. $(x \cup y)z = xz \cup yz$. And

$$xy.z \leq yz.x \cup zx.y.$$

Note that $xy \leq x$ is not assumed. Contrariwise, call x an ideal if $xy \leq x$ for all y. Note that the product of two ideals is an ideal. More generally, call y x-invariant if $xy \leq y$ and note:

Lemma y, z x-inv. $\Rightarrow yz$ x-inv.

Here is a little ~~[illegible]~~ prelude. Call a product simple if it is obtained by successive multiplications by a single element. E.g. $(ab.c)d$ is simple and $ab.cd$ is not.

Prop. Let y be a product of $x_1, \ldots, x_n$ in some association. Then $y \leq$ the ~~sum~~ union of all simple products of the x_i's.

Pf. Induction on n. Let $y = uv$ be the last multiplication forming y. Let length $u \leq$ length v and make a second induction on length u, say k. Since $u \leq$ sum of simple products of length k we can assume $u = x_i u'$. Then

$$y \leq x_i.u'v \cup u'.x_i v. \quad \text{Done.}$$

Define x^n inductively by $x^n = xx^{n-1}$. Cor.: any product of n x's is $\leq x^n$. In particular (assuming there is a 0 in the lattice) if $x^n = 0$ then any product of n x's is 0.

Now for Kac's theorem. Assume given x, y, z satisfying $x^2 \leq x$, $xy \leq y$, $xz \leq z$, $yz \leq x$. ~~The~~ One is imitating what happens if there is a Z-grading and the degrees of x, y, z are $0, 1, -1$.

Conclusion: any product of x's, y's and z's is $\leqq$ one of x, a power of y, a power of z.

Pf. In view of the lemma and the proposition, as well as symmetry, it will suffice to ~~examine~~ prove $y^n z \leq y^{n-1}$.

$$y^n z = y\, y^{n-1} \cdot z \leqq yz \cdot y^{n-1} \cup y^{n-1} z \cdot y$$

$$\leqq x y^{n-1} \cup y^{n-2} y \quad \text{by induction on } n$$

$$\leqq y^{n-1}.$$