Lie and Jordan Superalgebras

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1. Introduction. Lie superalgebras surfaced in mathematics some twenty years ago and made a number of further appearances after that. But the problem of their structure was ignored until physicists, using the term "supersymmetry", declared the problem to be interesting. I believe the first reference made by a physicist was in [8]. For the state of the art as of about early 1975 see [1].

Around 1974-5 a number of mathematicians vigorously attacked the structure problem. Victor Kac won the palm when he announced a complete solution in [4]. Details are available in ~~the preprint~~ [6].

The analogous problem for Jordan superalgebras was formulated and solved within a year. Kac and I independently completed the work (by different methods) in June 1976; a slight discrepancy between the results was promptly resolved.

I now present a summary of these two classification theorems.

Remark. I have joined a growing trend by switching from "graded" to "super". An overriding consideration is the matter of ambiguity. For instance, ordinary Lie algebras which happen to carry a grading also occur and are also of interest.

2. Superalgebras. Grading by the integers will have to receive appropriate attention in due course, but here

the grading will be mod 2. So a super-algebra A is an algebra which is a vector space direct sum $A_0 + A_1$ in such a way that the product satisfies $A_i A_j \subset A_{i+j}$ (subscripts read mod 2). Mathematicians call the two parts "even" and "odd"; physicists refer to them as "Bose" and "Fermi".

Except for a few remarks in connection with Jordan superalgebras, all algebras in this note are finite-dimensional.

Simple associative superalgebras were classified some time ago by Wall [9]. Let us stick to the case of an algebraically closed field. Then there are just two cases. In the first, the algebra is all n by n matrices with the grading as indicated:

(1)
$$\left(\begin{array}{c|c} \text{even} & \text{odd} \\ \hline \text{odd} & \text{even} \end{array}\right)$$

A basis-free description is: all linear transformations on a super vector space (there is a natural induced grading on the linear transformations). The second kind, in matrix form with square blocks of the same size, consists of all matrices of the form

(2)
$$\begin{pmatrix} A & B \\ B & A \end{pmatrix},$$

with parity as in (1).

Remark. There is an additional point about the last mentioned algebra: as an ordinary ungraded algebra is is not simple; for instance in characteristic $\neq 2$ the element

$$(3) \qquad \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

is a nontrivial central idempotent. But it is simple (should one say "supersimple"?) in the sense, appropriate in this context, that it has no nontrivial **homogeneous** ideals. As a matter of fact, this is the only place in the present note where this subtlety arises; all other simple algebras are really and truly simple.

3. Lie superalgebras. In an algebra the bracket $[xy]$ is defined as xy - yx. Now take a superalgebra A, and assume that x and y are homogeneous. It is an old story that interchanging two odd elements calls for a minus sign. So: the super definition of $[xy]$ is xy - yx except when x and y are both odd, in which case xy + yx is substituted.

Take A to be associative. A subspace of A closed under [ ] is then a "concrete" Lie superalgebra.

In the ordinary case the key identity is named for Jacobi:

$$[[xy]z] + [[yz]x] + [[zx]y] = 0.$$

How should this be changed in the super context? There is a rule that can be given once for all. It is to be understood that the identity under scrutiny is multilinear; it can involve multiplication, the bracket [ ], and the Jordan brace { } introduced below.

Principle for the passage from an ordinary identity to a super identity: change the sign of each term in which the odd vaiables have undergone an odd permutation.

The proof of this principle is easy but I shall not present it here. Applied to the Jacobi identity, it shows that there is activity only when exactly two of x, y, z are odd; then one term in the Jacobi identity acquires a minus sign. It's a tiny change but enough to make Lie superalgebras a brand new subject!

It is a matter of indifference whether one starts the theory with concrete Lie superalgebras or with (graded) anticommutativity and the (twisted) Jacobi identity as axioms. In other words, the Birkhoff-Witt theorem holds. For that matter, the Ado theorem is also valid: any finite-dimensional Lie superalgebra can be faithfully represented within a finite-dimensional associative superalgebra.

5. Lie superalgebras derived from associative ones. Let A be a simple associative superalgebra. Then, under [ ], A is a Lie superalgebra. It is almost simple -- at worst it may have to be trimmed by one dimension at top and/or bottom.

The details work out as follows. In the first type of algebra, where A consists of all n by n matrices with a grading given by (1), one drops to the matrices of trace 0. An instant word of warning is needed: the correct trace is of course a "supertrace", equal to Tr(A) - Tr(D) for

$$\begin{pmatrix} \underset{\text{even}}{A} & \underset{\text{odd}}{B} \\ \underset{\text{odd}}{C} & \underset{\text{even}}{D} \end{pmatrix}$$

There is still trimming needed at the bottom in the case where all the blocks have the same size, for then the scalar matrices are still present and constitute a one-dimensional center that needs to be divided out.

The algebra of matrices of the form (2) always needs trimming both above and below. One then reaches a Lie superalgebra bearing the names of Gell-Mann, Michel, and Radicati. The reader might find it interesting to compare the description I have just given with the original one which went as follows: the even and odd parts both are n by n matrices of trace 0 and the operation is commutation, except between two odd elements, in which case it is the Jordan brace, adjusted by a scalar so as to maintain trace 0. The description given here has two advantages: the Jacobi identity is built in, and the one-dimensional extensions at top and bottom give useful additional information.

trimming below means dividing out by center (so A can be taken tr 0) trimming above means dividing out by the idempotent (3) (so B can be taken trace 0)

The discussion of Lie superalgebras derived from associative ones continues in the context of algebras with an involution *. Once again a "super" change is needed. In addition to requiring that * is linear and that $x^{**} = x$ for all x, we need to modify $(xy)^* = y^*x^*$; if x and y are both odd this changes to $(xy)^* = -y^*x^*$.

It turns out that the algebra of matrices of the form (2) does not admit an involution, so we put it aside. We turn to matrices of the form (1). Now an involution induces an involution on the even subalgebra. There are now two possibilities.

(a) The involution may send each block into itself (this has to be the case if the two blocks have different sizes).

(b) The involution interchanges the two blocks (necessarily of the same size).

The passage to a Lie superalgebra takes place by dropping to the set of skew elements, an element x being called <u>skew</u> if $x^* = -x$. In case (a) this leads to a Lie superalgebra called <u>orthosymplectic</u>. In matrix style, one has matrices of the form

$$\begin{pmatrix} \text{skew-symmetric} & \text{———} \\ \text{———} & \text{skew-symplectic} \end{pmatrix}$$

the off-diagonal (in general rectangular) blocks being suitable "intertwining" matrices.

A basis-free description of the orthosymplectic algebra is also desirable, and in doing this one introduces another concept of interest -- the super version of inner product spaces. Consider the ordinary notion of a symmetric inner product (,). The requirement of symmetry says that $(x, y) = (y, x)$. Now start with a graded vector space $V = V_0 + V_1$. Naturally we take V_0 and V_1 orthogonal. We modify the symmetry by requiring $(x, y) = -(y, x)$ when x and y are odd. In other words, V_0 is to carry a symmetric inner product, V_1 a skew-symmetric one, the two structures being thus unified. Now consider what it should mean for a linear transformation T on V to be skew relative to the inner product. The ordinary requirement is

$$(Tx, y) = -(x, Ty).$$

The general principle applies, asserting that this is to be modified to

$$(Tx, y) = (x, Ty)$$

in case x and T are both odd.

There remains one more algebra to be collected, corresponding to case (b) above, with an involution. Kac's name for the resulting algebra is P(m). In matrix style it consists of all 2m by 2m matrices of the form

$$\begin{pmatrix} A & B \\ C & -A' \end{pmatrix},$$

where A has trace 0, B is symmetric, and C is skew.

5. Lie superalgebras of the Cartan pseudogroup type. For readers who are not familiar with Cartan's infinite pseudogroups I offer a little background. Let k be a field of characteristic 0. Let L be the Lie algebra of derivations of k[x], x being an indeterminate. Such a

derivation is determined by the image of x, and that image can be anything. Write D_i for the derivation sending x into x^{i+1}. Then the D_i's ($i = -1, 0, 1, 2, \ldots$) form a basis of L, with multiplication table

$$[D_i D_j] = (i - j)D_{i+j}.$$

(The similarilty to the Witt algebra in characteristic p will no doubt be observed by many readers.) It is a fact that L is simple.

Now generalize by replacing $k[x]$ by $k[x_1, \ldots, x_n]$. The result is an infinite family of infinite-dimensional simple Lie algebras. This is essentially the first of four families discovered by E. Cartan in connection with a problem in differential geometry. The remaining three families consist of subalgebras determined in appropriate ways by differential forms.

Let us switch to the super setup. One should ask oneself: what is the appropriately changed version of $k[x_1, \ldots, x_n]$? ~~Now a polynomial ring~~ can be viewed as being as free as possible among commutative rings over the field k. What we must do is give "commutative" its graded meaning. If we wish to reach a finite-dimensional algebra we have to take all the generators to be odd. What results is simply a Grassman algebra. So a candidate for a simple Lie superalgebra is the algebra of derivations of a Grassman algebra. This is correct, but one more super warning is in order: the rule

$$D(xy) = Dx.y + x.Dy$$

for a derivation needs to have the sign of x.Dy changed in the event that D and x are both odd.

With three additional families suitably determined by super differential forms, the analogues of the pseudogroups are on hand.

6. <u>The exceptions</u>. The list of simple Lie superalgebras is completed by a 40-dimensional one, a 31-dimensional one, and an infinite family of 17-dimensional ones. Here is a table.

Dimension	Even part	Odd part as a representation space for the even part
17 = 9 + 8	$A_1 \oplus A_1 \oplus A_1$	Tensor product of 3 copies of the 2-dimensional representation of A_1
31 = 17 + 14	$G_2 \oplus A_1$	Tensor product of the 7-dimensional representation of G_2 and the 2-dimensional representation of A_1.
40 = 24 + 16	$B_3 \oplus A_1$	Tensor product of the (8-dimensional) spin representation of B_3 with the 2-dimensional representation of A_1

Here A_1 is the 3-dimensional simple Lie algebra, G_2 the exceptional 14-dimensional one, and B_3 the algebra of all 7 by 7 skew-symmetric matrices. The information in the table determines the Lie superalgebra in question except for a description of how the odd part multiplies into the even part. This element of structure turns out to be uniquely determined in the case of the 31-dimensional and 40-dimensional algebras, but there is a free parameter in the 17-dimensional case.

7. Historic note. Shortly after I began work on Lie superalgebras in the Spring of 1975 I noted the resemblance between the super case and characteristic p (the possible vanishing of the Killing form, the need to trim certain algebras at the bottom as well as the top, the appearance of analogues of Cartan's pseudogroups, etc.). It therefore seemed reasonable to me to launch, as the first project, the study of simple Lie superalgebras having a nondegenerate form arising from a representation, as Seligman [7] had done in his pioneering work on characteristic p. This study was essentially complete by the end of the summer, 1975. It turned out finally that this modest project happened to catch all the exceptional algebras.

In his announcement [4] Kac omitted the 17-dimensional and 31-dimensional algebras. This was remedied in the English translation and in a correction [5].

8. Jordan superalgebras. The theory of ordinary Jordan algebras has acquired a good deal of importance in its own right, and has shed additional light on Lie algebras. It therefore seems appropriate to study Jordan superalgebras.

The Jordan brace $\{xy\} = xy + yx$ is of course reinterpreted as xy - yx when x and y are both odd. We forthwith have the concept of a concrete or "special" Jordan superalgebra. Here it does make for a genuine broadening to start again with axioms. The first axiom is commutativity, graded style. There is no identity of degree 3. The linearized version of the ordinary identity of degree 4 is

$$ab.cd + ac.db + ad.bc = (bc.a)d + (cd.a)b + (db.a)c.$$

The principle for transforming ordinary identities into superidentities applies (and there are several cases to be distinguished).

Here, in sketchy form, is the list of simple Jordan superalgebras (over an algebraically closed field of characteristic 0).

I. Those derived from associative simple superalgebras, in a fashion quite analogous to the way Lie superalgebras were derived in §4.

II. Quadratic algebras: a unit element adjoined to a vector space carrying a nondegenerate symmetric form, where "symmetric form" has the super meaning discussed in §4. The simplest example is 3-dimensional, with basis 1, a, b (a and b odd) satisfying $a^2 = 0 = b^2$, $ab = 1 = -ba$. This 3-dimensional algebra (related to the Heisenberg algebra) is an example where the following holds: it is not representable within a finite-dimensional associative superalgebra, but it is representable in an infinite-dimensional one.

III. An infinite family of 4-dimensional algebras.

IV. A 3-dimensional algebra with no unit element. A basis for it consists of e (even), a, b (odd) with $ea = ae = a/2$, $eb = be = b/2$, $e^2 = e$, $a^2 = 0 = b^2$, $ab = e = -ba$. Call this algebra T.

V. A certain 10-dimensional algebra. This algebra shows signs of playing the role occupied by the 27-dimensional excetional algebra in ordinary Jordan theory. It admits no associative representation -- not even an infinite-dimensional one. But the question as to whether it is a homomorphic image of a special Jordan superalgebra remains unresolved. An attack on this problem via the super version of Glennie's identity [3, pages 49-51] has thus far been inconclusive.

Nil Jordan superalgebras are solvable but not necessarily nilpotent. So the logical candidate for a radical is the maximal solvable ideal. This carries with it a definition of semisimplicity as the absence of solvable ideals. But there is a snag: semisimple Jordan superalgebras need not be direct sums of simple ones. To see this, just adjoin a unit element to T. However, hopes for an effective structure theory need not be abandoned. Every semisimple algebra is uniquely a direct sum of indecomposable semisimple ones, and it turns out that an indecomposable semisimple Jordan superalgebra is either simple or the result of adjoining a unit element to a direct sum of copies of T.

All the above referred only to characteristic 0. For characteristic p the theory of Jordan superalgebras is developing as this is being written. McCrimmon discovered some additional simple Jordan superalgebras in early 1977. In a letter dated April 21, 1977 Kac made some interesting conjectures which it would perhaps be premature to reproduce.

9. <u>Alternative superalgebras</u>. Every theory of a class of algebras defined by identities has a super variant. For ordinary algebras there is -- in addition to Lie algebras and Jordan algebras -- one more important class: <u>alternative</u> algebras. The defining identity says that the associator

$$xy.z - x.yz$$

is a skew-symmetric function of x, y, z. Associativity is a special case. In the ordinary theory there is exactly one more simple algebra: the 8-dimensional Cayley algebra. I made a hasty investigation of simple alternative superalgebras and tentatively report that they are associative.

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