THE IRREDUCIBLES OF THE EXTERIOR POWER OF THE SPACE OF MATRICES

ARKADY ONISHCHIK

Abstract. The exterior power of the space of $r \times s$-matrices is decomposed into irreducibles. The decomposition is known, but its simple deduction here is new. Besides, the highest weight vectors of the irreducibles are explicitly given.

Let ρ_1 and ρ_2 be the tautological linear representations of the Lie groups $\mathrm{GL}(r)$ and $\mathrm{GL}(s)$ in V and W, respectively. The decomposition of $\Lambda(\rho_1 \otimes \rho_2^*)$ into irreducible $\mathrm{GL}(r) \times \mathrm{GL}(s)$-modules is known, see [H]. I offer a simple deduction of this decomposition, and explicitly describe the highest weight vectors of the irreducibles.

We can assume that $\rho := \rho_1 \otimes \rho_2^*$ acts on the space $\mathrm{Mat}(r, s)$ of $r \times s$-matrices with r rows and s columns:

$$\rho_{A,B}(X) := AXB^{-1} \text{for any } A \in \mathrm{GL}(r),\ B \in \mathrm{GL}(s) \text{ and } X \in \mathrm{Mat}(r, s).$$

Let e_{ij}, E_{ij}, F_{ij} denote the elements of the standard bases (matrix units) in the spaces $\mathrm{Mat}(r, s)$, $\mathfrak{gl}(r)$, and $\mathfrak{gl}(s)$, respectively. In $\mathfrak{gl}(r)$ and $\mathfrak{gl}(s)$, take maximal tori consisting of diagonal matrices. Let λ_i for $1 \leq i \leq r$ and μ_k for $1 \leq k \leq s$ be the weights of ρ_1 and ρ_2, respectively. The roots $\lambda_i - \lambda_j$ for $i < j$ and $\mu_k - \mu_l$ for $k < l$ will be considered *positive*.

Obviously, any monomial

$$(1) \qquad v = ce_{i_1 j_1} \wedge \cdots \wedge e_{i_d j_d} \in \Lambda^d(\mathrm{Mat}(r, s)), \text{ where } c \in \mathbb{C}^\times,$$

is a weight vector of $\Lambda^d \rho$ of weight

$$\Lambda = \lambda_{i_1} + \cdots + \lambda_{i_d} - \mu_{j_1} - \cdots - \mu_{j_d}.$$

The monomial v will be called *normal* if it is of the form

$$ce_{1,s-p_1+1} \wedge \cdots \wedge e_{1,s} \wedge e_{2,s-p_2+1} \wedge \cdots \wedge e_{2,s} \wedge \cdots \wedge e_{r,s-p_r+1} \wedge \cdots \wedge e_{r,s},$$

2010 *Mathematics Subject Classification.* Primary 22E46, 22E70; Secondary 81S99, 51P05.

Key words and phrases. Decomposition into irreducibles, exterior power of the matrix representation.

where $s \geq p_1 \geq \cdots \geq p_r \geq 0$. Then, $d = \sum_{1\leq i\leq r} p_i$.

It is convenient to encode the monomial (1) by a diagram of the Young tableau type, the (i_k, j_k)th cells of the $r \times s$-table corresponding to e_{i_k,j_k} for $k = 1, \ldots, d$. Any normal monomial corresponds to the diagram whose ith row contains p_i cells (several last cells can be empty), and the last cells of all rows are in the sth column. Let q_j be the tally of cells in the jth column of the diagram; then, $0 \leq q_1 \leq \cdots \leq s \leq r$. Obviously, the weight of a given normal monomial is of the form

$$\Lambda = p_1\lambda_1 + \cdots + p_r\lambda_r - q_1\mu_1 - \cdots - q_s\mu_s, \tag{2}$$

and its degree is equal to $d = \sum_{1\leq i\leq r} p_i = \sum_{1\leq j\leq s} q_j$. Clearly, the normal monomial is uniquely (up to a proportionality) determined by its weight.

Example. The diagram

x	x	x
	x	x
	x	x

represents the normal monomial

$$v = e_{1,2} \wedge e_{1,3} \wedge e_{1,4} \wedge e_{2,3} \wedge e_{2,4} \wedge e_{3,3} \wedge e_{3,4} \in \Lambda^7(\mathrm{Mat}(3,4))$$

of weight $\Lambda = 3\lambda_1 + 2\lambda_2 + 2\lambda_3 - \mu_2 - 3\mu_3 - 3\mu_4$.

As is known (e.g., see [H]), to every partition $d = \sum_{1\leq i\leq r} a_i$, where $a_1 \geq \cdots \geq a_r \geq 0$ an irreducible representation ρ_1^d corresponds; we denote it $\rho_1(a_1, \ldots, a_r)$.

Similarly, let $\rho_2(b_1, \ldots, b_s)$, where $b_1 \geq \cdots \geq b_s \geq 0$, denote the irreducible representation ρ_2^d corresponding to the partition $d = \sum_{1\leq j\leq s} b_j$.

Then, the weight (2) of a normal monomial is the highest weight of an irreducible subrepresentation $\rho_1(a_1, \ldots, a_r) \otimes \rho_2(b_1, \ldots, b_s)^*$ of the representation $(\rho_1)^d \otimes (\rho_2^*)^d$ of $\mathfrak{gl}(r) \oplus \mathfrak{gl}(s)$ in the space $T^d(V) \otimes T^d(W^*)$. Let us prove that the irreducibles of $\Lambda^d(\rho)$ are isomorphic to these representations.

Lemma. *The element $v \in \Lambda^d(\mathrm{Mat}(r,s))$ is a highest weight vector of the representation $\Lambda^d(\rho)$ if and only if v is normal.*

Proof. The element v is a highest weight vector if and only if

$$d\rho(E_{ij})v = d\rho(F_{lk})v = 0 \text{ for any } i < j \text{ and } k < l.$$

Then,

$$d\rho(E_{ij})e_{jk} = E_{ij}e_{jk} = e_{ik}; \quad d\rho(F_{lk})e_{tk} = -e_{tk}F_{lk} = -e_{tl},$$

whereas the images of the other basis vectors vanish. Cleary, this shows that every normal monomial is a highest weight vector.

To prove the converse statement, let us show that every highest weight vector v contains a normal monomial v_0 (as a factor). This would imply that $v = v_0$.

Indeed, if $v - v_0 \neq 0$, then $v - v_0$ is also a highest weight vector, and hence contains a normal monomial v_1 of the same weight as v_0, and hence is proportional to v_0. This is a contradiction.

Let $v \in \Lambda^d(\mathrm{Mat}(r,s))$ be a highest weight vector. Then, v contains factors e_{1j}. Indeed, assume that v contains factors e_{tj}, but does not contain factors e_{ij} with $i < t$. Then,

$$v = \sum_{j_1 < \cdots < j_k} e_{tj_1} \wedge \cdots \wedge e_{tj_k} \wedge w_{j_1 \ldots j_k},$$

where the $w_{j_1 \ldots j_k}$ are polynomials in e_{ij}, not all of which vanish. If $t > 1$, then

$$0 = d\rho(E_{1t})v = \sum_{j_1 < \cdots < j_k} \sum_{\alpha} e_{tj_1} \wedge \cdots \wedge e_{tj_\alpha} \wedge \cdots \wedge e_{tj_k} \wedge w_{j_1 \ldots j_k},$$

which is a contradiction.

Let now α_1 be such that v contains $e_{1\alpha_1}$, but does not contain e_{1j} with $j < \alpha_1$. Then, $v = e_{1\alpha_1} \wedge v_1 + w$, where v_1 and w do not contain e_{1j} with $j \leq \alpha_1$. For any $j > \alpha_1$, we have

$$0 = d\rho(F_{\alpha_1,j})v = -e_{1j} \wedge v_1 - e_{1\alpha_1} \wedge d\rho(F_{\alpha_1,j})v_1 + d\rho(F_{\alpha_1,j})w,$$

where $d\rho(F_{\alpha_1,j})v_1$ and $d\rho(F_{\alpha_1,j})w$ do not contain e_{1j}.

Hence, $e_{1j} \wedge v_1 = 0$ for all $j > \alpha_1$, and $v_1 = e_{1,\alpha_1+1} \wedge \cdots \wedge e_{1,s} \wedge v_2$, where v_2 does not contain the elements e_{1j}.

If v contains elements e_{ij} with $i > 1$, then, as at the beginning of the proof, we show that v contains e_{2j}. Let v_2 contain elements e_{2,α_2}, but does not contain $e_{2,j}$ with $j < \alpha_2$. Then, $\alpha_2 \geq \alpha_1$.

Indeed, assume that $\alpha_2 < \alpha_1$ and consider the part u of v of the form

$$u = e_{1,\alpha_1} \wedge \cdots \wedge e_{1,s} \wedge e_{2,\alpha_2} \wedge e_{2,l_1} \wedge \cdots \wedge e_{2,l_m} \wedge \ldots,$$

where $\alpha_2 < l_1, \cdots < l_m$. Then,

$$d\rho(E_{12})u = e_{1,\alpha_1} \wedge \cdots \wedge e_{1,s} \wedge$$
$$(e_{1,\alpha_2} \wedge e_{2,l_1} \wedge \cdots \wedge e_{2,l_m} + e_{2,\alpha_2} \wedge \textstyle\sum_l e_{2,l_1} \wedge \cdots \wedge e_{1,l_t} \wedge \cdots \wedge e_{2,l_m}) \wedge \ldots$$

since $d\rho(E_{12})v = 0$, the element v must contain a factor containing e_{1,α_2}; this is impossible.

The same arguments as above easily show that

$$v_2 = e_{2,\alpha_2} \wedge \cdots \wedge e_{2,s} \wedge v_3,$$

where v_3 does not contain $e_{1,j}$ and $e_{2,j}$. Repeate this argument several times to construct a normal monomial entering v. □

This lemma immediately implies the following

Theorem. *For any positive integer d, there is the following decomposition into the direct sum of irreducibles of multiplicity 1:*

$$\Lambda^d(\rho_1 \otimes \rho_2^*) \simeq \oplus \rho_1(p_1, \dots, p_r) \otimes \rho_2(q_s, \dots, q_1)^*,$$

where the sum runs over all partitions $d = \sum p_i$ for $p_1 \geq \cdots \geq p_r \geq 0$ and q_j is the tally of cells in the jth column of the normal polynomial serving as the highest weight vector of the corresponding irreducible component.

The decomposition into irreducibles of $\Lambda^d(\rho_1 \otimes \rho_2)$ is similarly described. For the analog of this result for the representation $S^d(\rho_1 \otimes \rho_2)$, see [Zh, Th.3 of Ch.8].

References

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