A. Computational lemmas

Let $f(x) := (2x+1)^{2\log_3(2x+1)}$, $x \geq 0$, and $\tilde{f}(n) = f(n)$ if $n > 2$, $\tilde{f}(2) := 60$.

(A1) Lemma. If $x, y, t \in \mathbb{N}$ then

(a) $\tilde{f}(x)\tilde{f}(y) \leq \tilde{f}(xy)$ if $x \geq 2$, $y \geq 2$

(b) $t!\,(\tilde{f}(x))^t \leq \tilde{f}(x^t)$ if $x \geq 2$, $t \geq 1$

(c) $1.025 \cdot \tilde{f}(4)\tilde{f}(y) \leq \tilde{f}(4y)$ if $y \geq 2$

(d) $(1.025\tilde{f}(4))^t\, t! \leq \tilde{f}(4^t)$ if $t \geq 2$.

Proof. Consider $F(x,y) := \ln f(x) + \ln f(y) - \ln f(xy)$.

We have $\partial F/\partial x = (8/\ln 3)\ln(2x+1)/(2x+1) + 2/(2x+1)$

$-(8/\ln 3)\ln(2xy+1)/(2xy+1) - 2y/(2xy+1)$. Note

that $1/(2x+1) < y/(2xy+1) = 1/(2x+1/y)$ for $x \geq 0$,

$y > 1$, i.e., $\partial F(x,y)/\partial x < 0$ for $x \geq 0, y > 1$. Hence

$F(x,y) \leq F(2,y)$ for $y > 1$.

Then $dF(2,y)/dy = (8/\ln 3)\ln(2y+1)/(2y+1) + 2/(2y+1)$ $-(16/\ln 3)\ln(4y+1)/(4y+1) - 4/(4y+1)$ and the inequality $2/(2y+1) < 4/(4y+1)$ for $y > 1$ whence $F(2,y) < F(2,4)$. Now one checks directly that $F(2,4) < -0.87$ thus proving (a) if x or $y \geq 4$. One checks directly that $\tilde{f}(2)\tilde{f}(2) = 3600 < f(4) = 59{,}049$, $\tilde{f}(2)\tilde{f}(3) = 414000$ $< f(6) = 2{,}067{,}423$, $\tilde{f}(3)\tilde{f}(3) = 47{,}610{,}000 < f(9) = 1.35 \cdot 10^8$.

This proves (a).

To prove (b) it is sufficient to assume that $t \geq 2$. We consider as above

$F(t, x) := \ln t! + t \ln f(x) - \ln f(x^t)$.

We have $\partial F(t,x)/\partial x = t f'(x)/f(x) - t x^{t-1} f'(x^t)/f(x^t)$

< 0 as above. Thus $F(t,x) < F(t,3)$ for $x > 3$, $t \geq 1$. We

consider t now as a real variable.

$F(t,3) = \ln \Gamma(1+t) + t\left((2/\ln 3)\ln^2 7 + \ln 7\right) - (2/\ln 3)\ln^2(2\cdot 3^t + 1)$

$- \ln(2\cdot 3^t + 1)$.

Write $2 \cdot 3^t = 3^{t+\log_3 2} \leq 3^{t+0.63}$ and then

$\ln(2\cdot 3^t + 1) \leq \ln(2\cdot 3^t) \leq \ln 3\,(t + 0.63)$. Then

using the Stirling inequality for the Γ-function

(see) we get

$F(t,3) \leq (t + 1/2)\ln t - t + 0.92 + 1/12t + 8.84t$

$- 2\ln 3\,(t+0.63)^2 - \ln 3\,(t + 0.63)$

$\leq (t + 1/2)\ln t - 2.19t^2 + 4t - 0.6 =: h(t)$

(we replaced $1/12t$ by $1/24$ since $t \geq 2$).

Now $h'(t) = \ln t + (t+1/2)/t - 4.38t + 4 \leq \ln t - 4.38t + 5.25$

$h''(t) = 1/t - 4.38$. Since $h''(t) < 0$ for $t > 2$, $h'(t) < h'(2)$ $= -2.8 < 0$, for $t>2$, i.e. $h(t) < h(3) \doteq -4.46$ for $t \geq 3$, i.e., (b) holds if $x \geq 3$, $t \geq 3$. One checks: $2 \cdot \tilde{f}(3) \cdot f(3) = 9.52 \cdot 10^7 < f(9) = 1.36 \cdot 10^8$ whence (b) holds for $x = 3$.

As before we have $F(t,x) \leq F(t,4)$ for $x \geq 4$, $t \geq 1$. So to prove (b) for $x \geq 4$ it suffices to prove it for $x = 4$. Instead we will prove the stronger claim (d). We have

$$\ln\left((1.025 f(4))^t \cdot \Gamma(1+t) / f(4^t) \right) \leq (t+1/2)\ln t - t + 0.92$$

$$+ 1/12t + 11.011t - (2/\ln 3)\ln^2(2^{2t+1}+1) - \ln(2^{2t+1}+1)$$

$$\leq (t+1/2)\ln t - t + 0.96 + 11.011t - \underbrace{(2/\ln 3)\ln^2 2}_{0.87}(2t+1)^2 - \underbrace{(\ln 2)}_{0.69}(2t+1)$$

$$\leq (t+1/2)\ln t - 3.48t^2 + 5.15t - 0.6 =: h_1(t).$$

-1.72

Now the same argument as above gives $h_1(t) < h_1(2)$ $= -1.72 < 0$ whence (d) holds for $t \geq 2$ whence

(b) holds for $x \geqslant 4$. It remains to check that (b) holds for $x=2$. We have

$$\ln\left(\Gamma(1+t)(60)^t/f(2^t)\right) \leqslant (t+1/2)\ln t - 0.87t^2 + 0.67t - 0.6 =: h_2(t)$$

with $h_2(2) = -0.24 < 0$ and $h_2'(t) < 0$ for $t \geqslant 2$ whence as above (b) for $x=2$.

We skip the proof of (c) as it is the same as that of (a).

Let $g(x) := \Gamma(x+3)$. Recall that $\Gamma(n+3) = (n+2)!$ if $n \in \mathbb{N}$.

(A2) Lemma. Let $x, y, t \in \mathbb{N}$

(a) $g(x)\, g(y) \leq g(xy)$ if $x, y \geq 2$;

(b) $t!\,(g(x))^t \leq g(x^t)$ if $x \geq 3$, $t \geq 1$.

Proof. We have $g(x)\, g(y) \leq (x+2)! \prod_{x+3 \leq i \leq x+y+4} i$
$= (x+y+4)!$ One easily checks
that $xy+2 \geq x+y+2$ if $x \geq 3$ and $y \geq 3$.
If $x = 2$ then $g(x)\, g(y) = 2(y+2)! \leq$
$(2y+2)!$. Thus (a) holds if $x \geq 2$, $y \geq 2$
as claimed.

To prove (b) take logarithms and
consider $F(x,t) := \ln t! + t \ln g(x) - \ln g(x^t)$.
Thus $F(x,t) = \ln\Gamma(1+t) + t \ln\Gamma(x+3) - \ln\Gamma(x^t+3)$
Then $\partial F / \partial x = t(\ln\Gamma)'(x+3) - t x^{t-1} (\ln\Gamma)'(x^t+3)$

A2

Since $(\ln\Gamma)'(x)$ is strictly increasing

√ for $x \geq 2$ (by [ ]) we see that $\partial F/\partial x < 0$

for any t. Thus $F(x,t) < F(\ ,t) =$

$\ln\Gamma(1+t) + t(\ln 120) - \ln\Gamma(3^t+3)$

and $dF(3,t)/dt = (\ln\Gamma)'(1+t) -$

$(\ln 3)\, 3^t (\ln\Gamma)'(3^t+3)$ which is < 0

√ for $t \geq 1$ as before (by). Hence

$F(x,t) < F(3,t) < F(3,2) = -7.234$

(A3) Lemma. (a) $f(x) \le g(x)$ if $x \geq 10$

(b) $x^2 f(x) \le g(x)$ if $x \geq 14$

(c) $x f(x) \le g(x)$ if $x \geq 13$

Proof. Taking logarithms of both sides we see that the left ones produce a concave (for $x \geq 2$) function and the right one a convex one. One checks that the reverse inequalities hold for $x = 2$. Therefore there is just one intersection point of curves representing two sides. Since (a) (resp. (b)) can be checked to hold for $x = 10$ (resp $x = 14$) the claim follows from the above remarks.

(A5) Lemma. $2 \cdot 2^{2x^2+x} f(y) \leq f(2^x y)$

if $x \geq 2$, $y \geq 2^{\beta x}$ where $\beta = (\log 3 - 1)/2 = 0.29248125$

Proof. As usual set (with $y = 2^{\alpha x}$, α variable)

$$F(x) := \ln 2 + (2x^2+x)\ln 2 + (2/\ln 3)\left(\ln(2^{\alpha x+1}+1)\right)^2$$

$$+ \ln(2^{(\alpha+1)x+1}+1) - (2/\ln 3)\left(\ln(2^{(\alpha+1)x+1}+1)\right)^2$$

$$- \ln(2^{(\alpha+1)x+1}+1)$$

$$\leq \ln 2 + (2x^2+x)\ln 2 + (2/\ln 3)\Big(\ln 2^{\alpha x+1} +$$

$$+ \ln(1+2^{-\alpha x-1})\Big)^2 + \ln 2^{\alpha x+1} + \ln(1+2^{-\alpha x-1})$$

$$- (2/\ln 3)\left(\ln 2^{(\alpha+1)x+1}\right)^2 - \ln 2^{(\alpha+1)x+1}$$

$$\leq \ln 2 + (2x^2+x)\ln 2 + (2/\ln 3)\Big((\ln 2)\cdot(\alpha x+1) +$$

$$2^{-\alpha x-1}\Big)^2 + (\ln 2)(\alpha x+1) + 2^{-\alpha x-1}$$

$$- (2/\ln 3)(\ln 2)^2((\alpha+1)x+1)^2 - (\ln 2)((\alpha+1)x+1)$$

(we used first term approximation to

A5.

$\ln(1+\varepsilon)$, i.e. $\ln(1+\varepsilon) < \varepsilon$ for $\varepsilon > 0$ small).

Collecting the terms with like powers of x we have

$$F(x) \leq (2\ln 2)(1 - 2(\ln 2/\ln 3)\alpha - \ln 2/\ln 3)x^2$$

$$+ \ln 2\,((4/\ln 3)\alpha \cdot 2^{-\alpha x - 1} - \alpha - 4\ln 2/\ln 3)x$$

$$+ (4(\ln 2)\,2^{-\alpha x - 1} + 2^{-2\alpha x - 1})/\ln 3$$

The coefficient of x^2 is non-positive if $\alpha \geqslant (\log 3 - 1)/2$. Assuming $\alpha \geqslant (\log 3 - 1)/2$ and recalling that $x \geqslant 2$ we get

$$F(x) \leq \ln 2\,((4/\ln 3)\alpha \cdot 2^{-1.58} - \alpha - 4\ln 2/\ln 3)x$$

$$+ (4(\ln 2)\,2^{-1.58} + 2^{-2.16})/\ln 3$$

$$\leq -1.697\,x + 1.05$$

Thus $F(x) < 0$ for $x \geqslant 2$ and $y = 2^{\alpha x} \geqslant 2^{\beta x}$ as claimed.

(A5.1) Remark. Our argument also shows that $y \geq 2^{\beta x}$ is the best one can get.

A6

For a simple group G isomorphic to Suz, $\cdot 1$, $\cdot 2$, ${}^2\bar{A}_3(9)$, or $D_4(2)$, take the numbers a_1, a_2 from Table T7.2 (or from Table TA6 below). Set, as in (7.1),

$$F(G,n) := \begin{cases} 1 & \text{if } n < a_1 \\ |\mathrm{Aut}\,G| & \text{if } a_1 \leq n < a_2 \\ f(n) & \text{if } n \geq a_2 \end{cases}$$

Set

$$F(H,n) := \begin{cases} 1 & \text{if } n \leq 3 \\ f(n) & \text{if } n \geq 4 \end{cases}$$

for other sporadic simple groups.

(A6) Lemma. Let G_1 and G_2 be two sporadic simple groups or isomorphic to ${}^2\bar{A}_3(9)$. Let $F_i(n) := F(G_i, n)$, $i=1,2$. Then

(a) $F_1(n)\,F_2(m) \leq f(nm)$ for $n, m \geq 2$, $F_1(n) \neq 1$, $F_2(n) \neq 1$,

(b) $t!\,(F_1(n))^t \leq f(n^t)$ for $n \geq 2$, $t \geq 2$,

(c) $f(m)\,F_1(n) \leq f(nm)$ for $m \geq 4$, $n \geq 2$

(d) $\tilde{f}(m)\,F_1(n) \leq f(nm)$ for $m \geq 2$, $n \geq 2$ unless

$G_1 \simeq \cdot 1$ and $n \leq 37$.

(e) $1.025 f(4)\,F_1(n) \leq f(4n)$

Proof. The claims follow from (A1)(a), (b) if G_1 and G_2 are not isomorphic to Suz, $\cdot 1$, $\cdot 2$, $D_4(2)$, or ${}^2\bar{A}_3(9)$. If G_2 is not isomorphic to the 5 above groups then (a) follows from (c). Thus we may and shall assume that both G_1 and G_2 are isomorphic to Suz, $\cdot 1$, $\cdot 2$, $D_4(2)$, or ${}^2A_3(9)$.

Next, if in (a) $n \geq a_2$ where a_2 is for G_1 from Table T7.2 or TA6 then $F_1(n) = f(n)$ and (a) follows from (c). If in (a) we have further $m \geq$ (a_2 for G_2) then (a) follows from (A1)(a). Thus to prove (a) (modulo (c)) it is sufficient to check that $F_1(n)\, F_2(m) \leq f(nm)$ if (a_1 for G_1) $\leq n <$ (a_2 for G_1), (a_1 for G_2) $\leq m <$ (a_2 for G_2), i.e., that $|\mathrm{Aut}\, G_1| \cdot |\mathrm{Aut}\, G_2| \leq f((a_1 \text{ for } G_1)\cdot(a_1 \text{ for } G_2))$. This is verified directly (in the 25 cases). Thus it remains to prove (b), (c), and (d).

There again if $n \geq a_2$ (now there is only one group) then (b), (c), (d) follow from (A1)(a), (b). If $n < a_1$ they become trivial. Thus it remains to verify (b), (c), (d) for $a_1 \leq n < a_2$. In these cases $F_1(n) = |\mathrm{Aut}\, G_1|$ and the claims will follow if they hold for $n = a_1$.

For (b) we have (as in the proof of (A1)(b))

$$\ln\left(\Gamma(1+t)\,|G_1|^t/f(a_1^t)\right) \leq (t+1/2)\ln t - t + 0.96 + t\ln|G_1|$$
$$-(2/\ln 3)(t\ln a_1 + \ln 2)^2 - (t\ln a_1 + \ln 2) =$$
$$(t+1/2)\ln t - (2/\ln 3)(\ln^2 a_1)t^2 + (\ln|G_1| - 1 - (4/\ln 3)(\ln a_1)(\ln 2) - \ln a_1)t$$
$$-0.6.$$

The coefficients of the above expression are given explicitly in Table T A6. One easily establishes, as in the proof of (A1)(b) that the above function is negative for $t \geq 2$, whence (A6)(b)

To prove (c) and (d) consider

$r(x) := \ln\left(f(x)|\mathrm{Aut}\,G_1| / f(a_1 x)\right)$. One has $r'(x) =$

$(2/\ln 3)\ln(2x+1)/(2x+1) + 2/(2x+1) - (2/\ln 3)\ln(2a_1x+1)/(2a_1x+1)$

$- 2a_1/(2a_1x+1) < 0$ for $x \geq 2$. Thus

$r(x) \leq r(3)$ and from Table TA6 one

sees that $r(3) < 0$ if $G_1 \simeq Suz_1 \cdot 2$, $D_4(2)_{3/2}$ or $\bar{A}_3(9)$.

We also have $r(x) \leq r(4)$ and $r(4) < 0$ for

$G_1 \simeq \cdot 1$. This proves (c) and (d) except when $n=2$.

In this latter case one verifies

$60 \cdot |\mathrm{Aut}\,G_1| < f(2a_1)$ from Table TA6. Finally,

it is sufficient to check (e) for $n = a_1$ and this

is done directly.

Table TA6 (the last 2 rows to be used later).

$G \simeq$	${}^2\bar{A}_3(9)$	$D_4(2)$	Suz	$\cdot 1$	$\cdot 2$
a_1	6	8	12	24	20
a_2	7	9	18	49	24
$\lvert G\rvert$	$3.26\cdot 10^{6}$	$1.74\cdot 10^{8}$	$4.48\cdot 10^{11}$	$4.16\cdot 10^{18}$	$4.23\cdot 10^{13}$
$\lvert Aut\,G\rvert$	$8\cdot\lvert G\rvert$	$6\lvert G\rvert$	$2\lvert G\rvert$	$\lvert G\rvert$	$\lvert G\rvert$
$F(G, 2a_1)$	$3.89\cdot 10^{9}$	$1.53\cdot 10^{9}$	$4.63\cdot 10^{13}$	$4.16\cdot 10^{18}$	$1.5\cdot 10^{17}$
$F(G, 3a_1)$	$7.53\cdot 10^{11}$	$4.63\cdot 10^{13}$	$2.62\cdot 10^{16}$	$5.54\cdot 10^{21}$	$1.85\cdot 10^{20}$
$\ln(f(3)\cdot\lvert Aut G\rvert / f(3a_1))$	-1.43	-1.86	-1.44	1.65	-6.45
$(2/\ln 3)\ln^2 a_1$	5.84	7.87	11.24	18.38	16.34
$\left.\begin{array}{l}\ln\lvert Aut G\rvert - 1 - \ln a_1 \\ -(4/\ln 3)(\ln a_1)\ln 2\end{array}\right\}$	9.77	12.44	17.77	30.68	19.82
$24\lvert Aut\,G\rvert$	$6.26\cdot 10^{8}$	$2.57\cdot 10^{10}$	$2.15\cdot 10^{13}$	$1\cdot 10^{20}$	10^{15}
$64a_1\, f(a_1)$	$5.3\cdot 10^{8}$	$1.93\cdot 10^{10}$	$3\cdot 10^{12}$	$7.10\cdot 10^{16}$	$4.2\cdot 10^{15}$

(A 7) Lemma. (a) $f(n) \geqslant 4.796\, n^{2\log_3 n + 3.5}$ for $n \geqslant 2$,

(b) $n f(n) \leqslant n^{2\log n + 5}$ for $n \geqslant 4$,

(c) $n f(n) \leqslant 2 n^{2\log n + 1}$ for $n \geqslant 37$

(d) $2n^{2\log n + 1} \cdot 2m^{2\log m + 1} \leqslant 2(nm)^{2\log(nm)+1}$ for $n, m \geqslant 2$.

Proof.. We have

$$(2n+1)^{2\log_3(2n+1)} \geqslant (2n)^{2\log_3 2n + 1}$$

$$= 2^{(2/\log 3)(1+\log n)+1} \cdot n^{(2/\log 3)(1+\log n)+1}$$

$$= 2^{(2/\log 3)+1} \cdot n^{2\log_3 n + 4/\log 3 + 1} \geqslant 4.796\, n^{2\log_3 n + 3.5}$$

whence (a)

A 7

To prove (b) write

$$\ln f(x) = (2/\ln 3)(\ln(2x+1))^2 + \ln(2x+1)$$

$$= (2/\ln 3)(\ln 2x + \ln(1+1/2x))^2 + \ln 2x + \ln(1+1/2x)$$

$$\le (2/\ln 3)(\ln 2 + \ln x + 1/2x)^2 + \ln 2 + \ln x + 1/2x$$

$$=: h(x)$$

Then $h'(x) = (4/\ln 3)(1/x - 1/2x^2)(\ln 2 + \ln x + 1/2x)$

$+ 1/x - 1/2x^2 \le (4/\ln 3 + 1)/x + 4\ln x/x\ln 3 + 2/x^2\ln 3$

On the other hand, if $r(x) := x^{2\log x + 4}$ then

$$(\ln r)'(x) = 4\ln x/\ln 2 + 4/x$$

We have

$$4\ln x/x\ln 3 - 4\ln x/x\ln 2 = 4\ln x/x\,(1/\ln 3 - 1/\ln 2) \ge 2\ln x/x$$
$$\ge 4/x$$

for $x \ge 4$. Therefore for $x \ge 4$

$$(\ln r)'(x) - h'(x) \ge 8/x - (4/\ln 3 + 1)/x - 2/4x\ln 3 = 2.9/x$$

Thus $\ln r$ increases faster than $h(x)$ for $x \ge 4$.

Since $h(4) \le 11.06$ and $\ln r(4) \ge 11.09$ we get that $\ln r(x) > h(x)$ for $x \geq 4$. Since $h(x) \leq \ln f(x)$ this implies (b).

To prove (c) we set $s(x) := 2x^{2\log x}$.

Then $(\ln s)'(x) = (4/x\ln 2)\ln x$. Therefore

$$h'(x) - (\ln s)'(x) = 4(1/\ln 3 - 1/\ln 2)(\ln x)/x + (4/\ln 3 + 1)/x + 2/x^2\ln 3$$

$$\leq -2.1298 \ln x/x + 4.641/x + 1.8205/x^2$$

and, for $x \geq 37$,

$$\leq (-7.69 + 4.641 + 0.05)/x < 0.$$

Thus $\ln s(x)$ grows faster than $h(x)$ for $x \geq 37$. We have $\ln s(37) = 38.31$ and $h(37) = 38.25$ whence (c).

A7

For (d) dividing right-hand side by the left-hand side we have

$$2^{-1}\, n^{2\log n+2\log m+1}\cdot m^{2\log n+2\log m+1} \Big/ n^{2\log n+1}\cdot m^{2\log m+1}$$

$$=2^{-1}\, n^{2\log m}\cdot m^{2\log n}$$

which is, clearly, >1 if $n\geq 2$, $m\geq 2$, as claimed.

Set $s(x) := 2x^{2\log x + 1}$

(A8) Lemma. Let H be a sporadic group or centrally isomorphic to ${}^2A_3(9)$ and $F(n)$ the function associated to H in and (A6)(e). Then

$$s(m)\,F(n) \leq s(nm) \quad \text{for } n \geq 2,\ m \geq 128.$$

The proof is, essentially, the same as that of (A6)(c). We have to use only that $s(m)\,s(n) \leq s(mn)$ if $m \geq 128$, $n \geq 6$, $s(m)\,f(n) \leq s(nm)$ if $n \geq 6$, $m \geq 128$, and then to check the claim for $y_{a_1, a_2, |H|}$ for each of the groups in question.

(A 9). Lemma. (a) $t!((n+2)!)^t \le (nt+2)!$ for $t \ge 1$, $n > 12$

(b) $(n+2)!(m+2)! \le (n+m+2)!$ for $n, m > 12$.

Proof. Consider $F(t,x) := \ln\Gamma(t+1) + t\ln\Gamma(x+3) - \ln\Gamma(tx+3)$. We have $\partial F/\partial x = t(\ln\Gamma)'(x+3) - t(\ln\Gamma)'(tx+3) \le 0$ if $x \ge 2$, $t \ge 1$ since $(\ln\Gamma)'(s+1)$ increases as a function of s

vv for $s \ge$ by AO. We can assume $x \ge 13$, $t \ge 2$. Thus for these t and x

$$F(t,x) \ge F(t,13) = \ln\Gamma(t+1) + t\ln\Gamma(16) - \ln\Gamma(13t+3).$$

v We have $(\ln\Gamma)''(t+1) = \sum_{i\ge1}(t+i)^{-2}$ by AO.

Therefore

$$d^2F(t,13)/dt^2 = \sum_{i\ge1}(t+i)^{-2} - 169\sum_{i\ge1}(13t+2+i)^{-2} <$$

$$\sum_{i\ge1}(t+i)^{-2} - 169\sum_{i\ge1}(13t+2+13i)^{-2} - 169(13t+2+1)^{-2}$$

$$= \sum_{i\ge1}(t+i)^{-2} - \sum_{i\ge1}(t+i+2/13)^{-2} - (t+3/13)^{-2}$$

☆

$$< (t+1)^{-2} - (t+3/13)^{-2} < 0$$

Thus $dF(t,13)/dt$ strictly decreases for $t \geq 2$.

We have $dF/dt\,(2,13) = (\ln\Gamma)'(3) + \ln\Gamma(16) - 13(\ln\Gamma)'(29)$

$$= \left(-\gamma + \sum_{i=1}^{2} 1/i\right) + 27.9 + 13\left(\gamma - \sum_{i=1}^{28} 1/i\right)$$

$$= 12\gamma + 1.5 + 27.9 - 13 \cdot 3.927 = -14.7 < 0$$

(we used A0). Thus, since $dF(t,13)/dt$ decreases for $t \geq 2$ we have that $dF(t,13)/dt < 0$ for $t \geq 2$. Thus $F(t,13)$ decreases for $t \geq 2$. Since $F(2,13) = \ln(3!) + 2\ln(15!) - \ln(28!) = -11 < 0$ our claim (a) follows.

To prove (b) note that

$$(n+m+2)!/(n+2)!(m+2)! = (n+3)(n+4)\cdots(n+m+2)/(m+2)!$$

$$= 1 \cdot 2^{-1} \cdot ((n+3)/3)((n+4)/4) \cdots \geq 2^{-1}(13+3)/3 = 16/6 > 1$$

whence (b).

(A10) Lemma. If $a^m \cdot m!/(bm+2)! < 1$ for some m then it is < 1 for all larger m.