see top of p 15.8.

15. Maximality of some finite linear subgroups of $GL_n(\mathbb{C})$.

Let $\varphi_m : Alt_m \longrightarrow GL_{m-1}(\mathbb{C})$ be the non-trivial component of the transitive permutation representation of Alt_m on m letters and $\psi_m : Alt_m \longrightarrow GL_{\frac{m(m-3)}{2}}(\mathbb{C})$ the representation of Alt_m such that the permutation representation (of degree $m(m-1)/2$) of Alt_m on unordered pairs of m distinct letters is equivalent to

$id \oplus \varphi_m \oplus \psi_m$.

We shall call a finite subgroup G of $GL_n(\mathbb{C})$ <u>nearly</u> <u>maximal</u> if for any finite subgroup H of $GL_n(\mathbb{C})$ the inclusion $H \supseteq G$ implies that $H \subseteq N_{GL_n(\mathbb{C})}(G)$. In the cases we consider below near-maximality of G means that $N_{GL_n(\mathbb{C})}(G)$ is, modulo its center, a maximal finite subgroup of $GL_n(\mathbb{C})$. But this interpretation does not hold in other examples.

(15.1) Proposition. For any $r \in \mathbb{N}$ there exists $n_1 = n_1(r) \in \mathbb{N}$ such that if $G := \bigotimes_{i=1}^{r} \varphi_{m_i}(Alt_{m_i}) \subseteq GL_{\prod_{i=1}^{r}(m_i-1)}(\mathbb{C})$, and $m_i \geq n_1(r)$, $i = 1, \dots, r$, then G is nearly maximal in $GL_{\prod_{i=1}^{r}(m_i-1)}(\mathbb{C})$.

(15.2) Proposition. There exists $n_2 \in \mathbb{N}$ such that if $G := \psi_m(Alt_m) \subseteq GL_{m(m-3)/2}(\mathbb{C})$ and $m \geq n_2$ then G is nearly maximal in $GL_{m(m-3)/2}(\mathbb{C})$

(15.3) Proposition. If $G := \bigoplus_{i=1}^{r} \varphi_{m_i}(\mathrm{Alt}_{m_i}) \leq GL_{\sum_{i=1}^{r}(m_i-1)}(\mathbb{C})$ and $m_i \geq 10$, $i = 1, \ldots, r$, then G is nearly maximal in $GL_{\sum_{i=1}^{r}(m_i-1)}(\mathbb{C})$.

(15.4) Proposition. There exists $n_4 \in \mathbb{N}$ such that if $G := \varphi_m(\mathrm{Alt}_m) \oplus \psi_n(\mathrm{Alt}_n) \leq GL_{(m-1)+n(n-3)/2}(\mathbb{C})$ and $m \geq n_4$, $n \geq n_4$ then G is nearly maximal in $GL_{(m-1)+n(n-3)/2}(\mathbb{C})$.

(15.5) Remark. (15.3) and (15.4) give examples of nearly maximal reducible finite linear groups.

(15.6) Proof of (15.1). First, if $r = 1$, then $\tilde{G} := \varphi_{m_1}(\mathrm{Sym}_{m_1})$ is a group generated by reflections. This G is known not to be maximal for $m_1 = 9$ (for Sym$_9$ is contained in the Weyl group of type E_8). When $m_1 > 9$ consider $H \supseteq G$, $H \subseteq Gl_m(\mathbb{C})$, H finite and then replace H by $\tilde{H} := \langle H, \tilde{G} \rangle$. H contains then a normal subgroup generated by reflections and from the classification of finite groups generated by reflections we see that G is nearly maximal if $m_1 \geq 10$ (so that $n_1(1) = 10$).

Let us now chose $n_1 = n_1(r)$ so that $n_1(r) \geq 49$ and $(n_1!/2)^r > (2(n_1-1)^r+1)^{2\log_3(2(n_1-1)^r+1)}$. Clearly, such $n_1(r)$ exists and it is easy to see that $n_1(r) > n_1(a)$ if $a < r$.

Let now $m := \prod_{i=1}^{r}(m_i - 1)$ and let $H \subseteq Gl_m(\mathbb{C})$ be a finite group such that $H \supseteq G$. H is primitive since so is G. Let $\overline{S}$ be the socle of H/center and S its preimage in H. Then, as in Section 11, S is a central product of centrally simple groups $G_1, \ldots, G_t$, extraspecial groups $E_1, \ldots, E_s$ and of the center C of H. We have, of course, need to be proven

$G \subseteq S$ whence at once $s = 0$. We show now that each $\varphi_{m_i}(\mathrm{Alt}_{m_i}) \otimes \mathrm{id}$ is contained in some G_j. Suppose that is not so. Assume, for definiteness, that G_1 is such that the projections of $\varphi_{m_1}(\mathrm{Alt}_{m_1}) \otimes \mathrm{id}$ both on G_1 and on $G_2 \cdots G_t$ are non-trivial. The representation of the central product of G_1 and $G_2 \cdots G_t$ on k^m is equivalent to the tensor product of representations $\pi \otimes \omega$ of the two factors. Restricted to $\varphi_{m_1}(\mathrm{Alt}_{m_1}) \otimes \mathrm{id}$ it implies that φ_{m_1} is a tensor product of two representations of Alt_{m_1}. But, since φ_{m_1} is a non-trivial representation of smallest dimension, it can not be a tensor product. Thus each $\varphi_{m_i}(\mathrm{Alt}_{m_i}) \otimes \mathrm{id}$ is contained in some G_i.

Thus $\{1,\dots,r\} = \bigcup_{i=1}^{t} I_i$ with $I_i \cap I_j = \emptyset$ if $i \neq j$ and $G_j \supseteq \bigotimes_{i\in I_j} \varphi_{m_i}(Alt_{m_i}) \otimes id$.

Let us now argue by induction on r. The case $r=1$ was dealt with before. (Make the inductive assumption.) Since $n_1(r) \geqslant n_1(|I_j|)$ for $j=1,\dots,t$ it follows that either $|I_j| = 1$ for $j=1,\dots,t$ (and then $t=r$, whence $H \leq N_{GL_m(\mathbb{C})}(G)$) or $t=1$. Our choice of $n_1(r)$, together with (6.1), (7.1) implies then that $G_1 \simeq Alt_d$ for some $d \leq m+1$.

Let φ_i denote the imbedding of Alt_{m_i} in Alt_d. Let $\Omega_1,\dots,\Omega_b$ be different orbits of $\varphi_1(\mathrm{Alt}_{m_1})$ on $\{1,\dots,d\}$. Let $\{1,\dots,b\} = \bigcup_{i=1}^{c} J_i$ so that the orbits Ω_α and Ω_β are equivalent if and only if $\alpha, \beta \in J_i$ for some i. Then $Z_{\mathrm{Alt}_d}(\varphi_1(\mathrm{Alt}_{m_1})) \simeq \prod_{i=1}^{c} \mathrm{Sym}\, J_i$ where each $\mathrm{Sym}\, J_i$ permutes the orbits Ω_α, $\alpha \in J_i$. Since $\varphi_i(\mathrm{Alt}_{m_i}) \subseteq Z_{\mathrm{Alt}_d}(\varphi_1(\mathrm{Alt}_{m_1}))$ for $i = 2, \dots, r$ and by the inductive assumption we must have $\varphi_i(\mathrm{Alt}_{m_i}) = \mathrm{Sym}\, J_{\alpha(i)}$ for $i = 2, \dots, r$ and an appropriate $\alpha(i) = 1, \dots, c$

Note that $\mathrm{Alt}\, J_i$ acts trivially on $\bigcup_{\alpha \notin J_i} \Omega_\alpha$. This implies that each $\mathrm{Alt}\, J_i$ has at most two types of orbits on $\{1, \dots, d\}$; and if exactly two then one type is trivial. By symmetry this, therefore, holds for all $\varphi_i(\mathrm{Alt}_{m_i})$, and, in particular, for $i = 1$. Thus $c \leq 2$, $r \leq 3$.

¶ Suppose there is a trivial orbit, say Ω_1, of $\varphi_1(\mathrm{Alt}_{m_1})$. Let $\Omega_1 \in J_1$. Then $Z_{\mathrm{Alt}_d}(\mathrm{Alt}\, J_1) = \mathrm{Alt}(\{1, \dots, d\} - J_1)$ whence again by inductive assumption we must have $Z_{\mathrm{Alt}_d}(\mathrm{Alt}\, J_1)$, is one of the $\varphi_i(\mathrm{Alt}_{m_i})$. Thus we can assume (when there is a trivial orbit) in this case that $c = r = 2$ and $\varphi_i(\mathrm{Alt}_{m_i})$, $i = 1, 2$, acts through the natural representation on its non-trivial orbit. We have thus $d = m_1 + m_2$. Then

$m = (m_1-1)\cdot(m_2-1) \leq \left(\frac{d-2}{2}\right)^2$. Thus we are dealing with a representation of Alt_d of dimension $\leq \left(\frac{d-2}{2}\right)^2$. By R. Rasala [p 132, Result 2] this implies (since $d \geq 2n_1(2) \geq 20$) that $m \leq d-1$, i.e.

$(m_1-1)(m_2-1) \leq m_1+m_2-1$ or $(m_1-2)(m_2-2) \leq 2$.

This latter inequality is false for $m_1, m_2 \geq 10$. Thus our current assumption that there are trivial orbits is false as well.

Thus there are no trivial orbits and therefore $J_1 = \{1, \dots, d\}$. We assume $r > 1$, so that $r = 2$. Since $\varphi_2(\mathrm{Alt}_{m_2})$ acts as $\mathrm{Alt}\, J_1$ on the non-trivial orbits of $\varphi_1(\mathrm{Alt}_{m_1})$, we see that non-trivial orbits of $\varphi_i(\mathrm{Alt}_{m_i})$ are of length m_i for $i = 1, 2$. Thus $d = m_1 \cdot m_2$. Since $m = (m_1 - 1)(m_2 - 1) < d - 1$ it follows that Alt_d can have no representation of dimension m whence a contradiction in this case.

This concludes the proof of (15.1).

(15.7) Proof of (15.2). Again, take $n_2 \geq 49$ and such that $(n_2!)/2 > (n_2(n_2-3)+1)^{2\log_3(n_2(n_2-3)+1)+1}$.

Clearly, such n_2 exists. Set $n := m(m-3)/2$. Let $H \subseteq GL_n(\mathbb{C})$, H finite, $H \supseteq G$. Since G is primitive so is H. Let S be the preimage in H of the socle of H/center. Then S is a central product of centrally simple groups $G_1, \dots, G_t$, extraspecial groups $E_1, \dots, E_s$ ~~ant~~ and of the center C of H. We have again that $s = 0$. Clearly, $G \subseteq G_1 \cdots G_t$. Let $G_{(i)}$ be the projection of G on G_i, $i = 1, \dots, t$. The representation of $G_1 \cdots G_t$ on k^n is a tensor product $\otimes_{i=1}^{t} \pi_i$ of representation of the G_i. Therefore our representation φ_n

of $G \simeq Alt_m$ is the tensor product of

the $\pi_i/G_{(i)}$. But since by R. Rasala [ ] the smallest non-trivial representation of G has dimension $m-1$ and since $(m-1)^2 > m(m-1)/3$ it follows that φ_m is not a tensor product whence $t = 1$.

Then by (6.1), (7.1) and because of our choice of n_2 we see that $G_1 \simeq Alt_d$ for some d. We have since $Alt_d \supseteq Alt_m$ that $d \geq m$. If $d > m$ then $\frac{d(d-3)}{2} > n$ and hence by R. Rasala [, Result 2] we have

$n = d-1$, i.e. $d = m(m-3)/2 + 1$.

Consider the action of Alt_m on $\Omega := \{1, \dots, d\}$.

~~If Alt_m has a fixed point on Ω then $\mathrm{Alt}_m \subseteq \mathrm{Alt}_{d-1}$ and the restriction of our representation of Alt_d (of dimension $d-1$) to Alt_{d-1} would be irreducible which is not the case.~~ If Alt_m has an orbit $\Omega_1 \neq \Omega$, $\Omega_1 \neq \emptyset$, on Ω then $\mathrm{Alt}_m \subseteq \mathrm{Alt}\,\Omega_1 \times \mathrm{Alt}(\Omega - \Omega_1)$ and the restriction of our representation of Alt_d (of dimension $d-1$), to $\mathrm{Alt}\,\Omega_1 \times \mathrm{Alt}(\Omega - \Omega_1)$ would be irreducible. But this is not so if $\Omega_1 \neq \Omega$, $\Omega_1 \neq \emptyset$. Thus Alt_m is transitive on Ω.

Since any primitive permutation representation of Alt_m of degree $> m$ has degree $\geq m(m-1)/2$ (see) it follows from $\frac{m(m-3)}{2}+1 < \frac{m(m-1)}{2}$ that the orbit of Alt_m on Ω has length divisible by m. Thus $m(m-3)/2+1 = rm$ for some $r \in \mathbb{N}$. Clearly $r > (m-3)/2$. Thus $2 = m(2r-(m-3))$ with $2r-(m-3) > 1$. This is clearly impossible for $m \geqslant 49$. This is a contradiction with the assumption $d > m$. Thus $d = m$ and (15.2) is proved.

(15.8) Proof of (15.3). For $r=1$ (15.3) reduces to (15.1). So assume $r \geq 2$.

Let $m := \sum_{i=1}^{r} (m_i - 1)$ and let $H \subseteq GL_m(\mathbb{C})$ be a finite group such that $H \supseteq G$. Replace H by $\langle H, \oplus_{i=1}^{r} \varphi_{m_i}(Sym_{m_i}) \rangle$ and let M be the normal subgroup of H generated by $\oplus_{i=1}^{r} \varphi_{m_i}(Sym_{m_i})$. Set $V := k^m$ and let $V = \oplus_{i=1}^{t} V_i$ be the decomposition of V into M-simple modules. Then $M_i := M | V_i$ is generated by reflections and is irreducible. Then by the classification of finite irreducible groups generated by reflections (see, e.g.,) and since $\dim V_i \geq \min(m_j - 1) \geq 9$ we have that M_i is a Weyl group of one of the types A_s, B_s, or D_s with $s = \dim V_i$. We can assume that $M_i = M$, $V_i = V$. Thus we obtain in case A_s that Alt_{m+1} contains a direct product of Alt_{m_i} with $\Sigma m_i = m + r$, and in the case of B_s and D_s, by taking quotient of M by its radical, that Alt_m contains a direct product of Alt_{m_i} with $\sum m_i = m + r$. This situation is easily seen to be impossible unless $r = 1$, M is of type A_{m+1}. Returning to our original M this means that each V_i is irreducible for exactly one $\varphi_{m_j}(Alt_{m_j})$ whence our claim.

(15.9) Proof of (15.4). Let us take $n_4 \geq \max(n_1(1), n_2)$ and such that $(n_4!/2)^2 > (2n_4 + n_4(n_4-3)+1)^{2\log_3(2n_4+n_4(n_4-3)+1)+1}$.

Let $d = m-1 + n(n-3)/2$ and let $H \subseteq GL_d(k)$, H finite, $H \supseteq G$. If H is reducible then the irreducible components clearly will have dimensions $m-1$ and $n(n-3)/2$ and then $H \subseteq N_{GL_d(k)}(G)$ by (15.1) and (15.2). If H is irreducible it is easily seen to be primitive. Let S be the preimage in H of the socle of H/centre. As before one shows that S must be centrally simple and then, by (6.1), (7.1), $S \simeq Alt_r$ for some r. One has at once, looking at lengths of orbits of Alt_m and Alt_n on $\Omega := \{1, \dots, r\}$ that $r \geq m + n$. Using again R. Rasala [, Result 2] and noting that

$$m - 1 + n(n-3)/2 < (m+n)(m+n-3)/2 \leq r(r-3)/2$$

we see that $r = d+1 = m + n(n-3)/2$.

Let $\Omega_1, \dots, \Omega_t$ be different orbits of Alt_m on Ω. Since the representation of Alt_r on k^d is the smallest one and since its restriction to Alt_m is again the smallest one taken once plus a number of trivial ones, we have that one orbit, say Ω_1, is of length 1 and the rest are of length 1. Then $Z_{Alt_r}(Alt_m) = Alt_{r-m}$ whence $Alt_n \leq Alt_{r-m}$ which contradicts near maximality of $\varphi_n(Alt_n)$ (i.e. contradicts (15.2)) unless $r - m = n$. But this latter variant is impossible as $n < n(n-3)/2$.