The 31st Annual Vojtěch Jarník
International Mathematical Competition
Ostrava, 13th April 2024
Category I

Competitor No. 3

Problem 2 *Let n be a positive integer and let A, B be two complex nonsingular $n \times n$ matrices such that*

$$A^2B - 2ABA + BA^2 = 0\,.$$

Prove that the matrix $AB^{-1}A^{-1}B - I_n$ is nilpotent. (Here I_n denotes the $n \times n$ identity matrix. A matrix X is called nilpotent if there exists a positive integer k such that $X^k = 0$.) [10 points]

We are given that A and B are nonsingular, meaning A^{-1} and B^{-1} exist and $AA^{-1} = A^{-1}A = I_n$ and $BB^{-1} = B^{-1}B = I_n$. Now let's manipulate the given equation:

$$A^2B - 2ABA + BA^2 = 0$$

We post-multiply the entire equation by $B^{-1}A^{-1}$:

$$A^2 - 2AB + BA = 0$$

Since matrix multiplication is associative, we can rewrite the above equation as

$$(AB)(AB^{-1}A^{-1}B) = AB$$

This implies $AB^{-1}A^{-1}B = I_n$

Now let $N = AB^{-1}A^{-1}B - I_n$, we can see that

$$N = AB^{-1}A^{-1}B - I_n = I_n - I_n = 0.$$

Hence N raised to the first power itself is the zero matrix, which implies that N is nilpotent with $k=1$.

wrong