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A basis construction for free Lie algebras

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Abstract. We construct a new type of basis for the free Lie algebra. The basis elements are linear combinations of $(n-1)!$ normed Lie monomials with n-th roots of unity as coefficients (where n is the number of free generators).

1. Introduction

A well-known problem in combinatorial Lie theory is that of determining bases of free Lie algebras. All constructions we know obtain bases along combinatoric lines and result in a system of monomials with a rather complicated bracket structure. At first, a freely generating set Z of the free Lie algebra is ordered. Then certain words in the free monoid Z^* over Z are subjected to a suitable process of bracketing so that their interpretation as monomials in the free Lie algebra L^Z over Z yields a basis. Of course, Z may be assumed to be finite, say $Z = \{z_1, \ldots, z_l\}$. Then it suffices to construct a basis of the space $L^Z(k_1, \ldots, k_l)$ of the homogeneous Lie elements of an arbitrarily given multidegree $(k_1, \ldots, k_l)$. By a classical result of Witt ([11, Satz 3]), the dimension of $L^Z(k_1, \ldots, k_l)$ is the so-called "necklace number" with respect to the tuple $(k_1, \ldots, k_l)$:

$$\dim L^Z(k_1, \ldots, k_l) = \frac{1}{n} \sum_{d | k_1, \ldots, k_l} \mu(d) \frac{\frac{n}{d}!}{\frac{k_1}{d}! \ldots \frac{k_l}{d}!}$$

where $n = k_1 + \cdots + k_l$ [6,2.1]. A famous basis construction by Chen, Fox and Lyndon ([5], [9,5.3]) is, historically, not the first one but may be viewed as a paradigm of all constructions hitherto known. They put $z_1 < \cdots < z_l$ and start from the usual lexicographic order on Z^*. A "necklace" with respect to the tuple $(k_1, \ldots, k_l)$ is a word $w \in Z^*$ of length $n = k_1 + \cdots + k_l$ in which any letter z_i occurs exactly k_i times. The "necklace number" is the cardinality of the set $\mathbb{B}$ of those of these words $w \in \mathbb{Z}^*$ which have the following property: If $w = w_1 w_2$ is a nontrivial decomposition of w, then $w < w_2 w_1$. Trivially, $\mathbb{B}$ contains the particular word

$$z_{a_1} \ldots z_{a_n} := z_1^{k_1} \ldots z_l^{k_l}.$$

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(That is, $a_1 = \cdots = a_{k_1} := 1$, $\alpha_{k_1+1} = \cdots = \alpha_{k_1+k_2} := 2, \ldots$) By bracketing appropriately and replacing the associative multiplication by the Lie multiplication, the elements of $\mathbb{B}$ are turned into a basis of $L^Z(k_1,\ldots,k_l)$ ([5], [10, I, 1.]) . The bracketing rule, however, can only be given inductively, by a process of reduction to smaller homogeneous subspaces for which, by assumption, the rule has already been defined.

Exploiting thoroughly the algebraic properties of L^Z, we construct *directly* (that is, without any inductive recurrence) a basis of a completely different character of $L^Z(k_1,\ldots,k_l)$: It consists of linear combinations of left-normed Lie monomials.

We write $\circ$ for the Lie multiplication and call the product $z_{i_1} \circ z_{i_2} \circ \cdots \circ z_{i_n} := (\cdots(z_{i_1} \circ z_{i_2})\cdots) \circ z_{i_n}$ *left-normed*. We suppose that the field R of coefficients has characteristic 0 and that it contains a primitive n-th root of unity ε. The symmetric group on $\{1,\ldots,n\}$ will be denoted by S_n, and we set $T := \mathrm{Stab}_{S_n}(1)$. Every $\alpha \in S_n$ has a unique representation of the form

$$\alpha = (21)^{i_1}(321)^{i_2}\cdots(n\ldots1)^{i_{n-1}} \qquad (0 \le i_s \le s).$$

Following Klyachko [7,2.], we set $\operatorname{ind}\alpha := i_1 + \cdots + i_{n-1}$. A special case of our main result is the following

Theorem. *For each $w \in \mathbb{B}$ choose $\varrho \in S_n$ such that $w = z_{a_{1\varrho}} \cdots z_{a_{n\varrho}}$. Then the set of the Lie elements*

$$\sum_{\sigma\in T} \varepsilon^{\operatorname{ind}\sigma\varrho^{-1}} z_1 \circ z_{a_{2\sigma}} \circ \cdots \circ z_{a_{n\sigma}}$$

is a basis of $L^Z(k_1,\ldots,k_l)$.

(In particular, $\dim L^Z(k_1,\ldots,k_l) = |\mathbb{B}|$, in agreement with the result by Witt mentioned above.)

At the end of [8], G.P. Kukin claims to have constructed a basis of the free Lie algebra consisting of normed Lie monomials. Trying to follow his line of reasoning, however, we do neither understand his argument for the generating property nor that for uniqueness (in case of correctness one of the two arguments would have sufficed as the number of Kukin's Lie elements of a certain multidegree is equal to the corresponding Witt dimension); moreover, in the case of a Lie algebra with two free generators, Kukin's construction seems to omit (at least) the 1-dimensional subspace of the homogeneous elements of multidegree (2,2) as his Lie element is 0 in this case.

2. Proofs and Details

Let $X = \{x_1, \ldots, x_n\}$ and A^X be the free associative R-algebra over X. Let L^X be the subalgebra of the Lie algebra $(A^X, +, \circ)$ generated by X (where $\circ$ is the usual Lie multiplication, $a \circ b = ab - ba$). The subspace $\mathscr{A}$ of the multilinear homogeneous elements of degree n in A^X is an RS_n-left module with respect to the antihomomorphic action

$$\beta x_{1\alpha} \cdots x_{n\alpha} := x_{1\beta\alpha} \cdots x_{n\beta\alpha} \qquad (\alpha, \beta \in S_n),$$

and an RS_n-right module with respect to the homomorphic action

$$x_{1\alpha} \cdots x_{n\alpha}\beta := x_{1\alpha\beta} \cdots x_{n\alpha\beta} \qquad (\alpha, \beta \in S_n).$$

By linearization, the space $L^Z(k_1, \ldots, k_l)$ is embedded in the space $\mathscr{L} := L^X(\underbrace{1, \ldots, 1}_{n})$ of the multilinear Lie elements over X ([1,3.3], [4,1.]). We have

$$x_{1\alpha} \circ \cdots \circ x_{n\alpha}\beta = x_{1\alpha\beta} \circ \cdots \circ x_{n\alpha\beta} \qquad (\alpha, \beta \in S_n),$$

hence $\mathscr{L}$ is the submodule of the right module $\mathscr{A}$ which is generated by $x_1 \circ \cdots \circ x_n$. Let J be the kernel of the RS_n-right module epimorphism

$$RS_n \rightarrow \mathscr{L}$$

$$\sum_{\sigma \in S_n} c_\sigma \sigma \mapsto \sum_{\sigma \in S_n} c_\sigma x_{1\sigma} \circ \cdots \circ x_{n\sigma},$$

furthermore

$$\mathscr{X} := \{\pi \mid \pi \in S_n, \quad 1\pi > \cdots > 1 < \cdots < n\pi\},$$

and

$$\eta_j := -\sum_{\substack{\pi \in \mathscr{X} \\ 1\pi = j}} (-1)^{1\pi^{-1}} \pi \qquad \text{for } 1 \leq j \leq n.$$

Put

$$v_n := \frac{1}{n} \sum_{j=1}^{n} \eta_j.$$

Then we have

$$v_n = \frac{1}{n} \prod_{j=n}^{2} (id - (j \ldots 1)),$$

and by the Specht/Wever criterion ([2,2.5.5]),

$$v_n^2 = v_n\,. \tag{1}$$

We conclude that

$$x_1 \circ \cdots \circ x_n v_n = x_1 \circ \cdots \circ x_n\,, \tag{2}$$

as indeed

$$x_1 \circ \cdots \circ x_n v_n = n v_n x_1 \ldots x_n v_n = n v_n^2 x_1 \ldots x_n = n v_n x_1 \ldots x_n = x_1 \circ \cdots \circ x_n\,.$$

By [3, Lemma], $id - \eta_j \in J$ for $1 \leq j \leq n$, and therefore

$$x_1 \ldots x_n v_n(id - \eta_j) = v_n x_1 \ldots x_n(id - \eta_j) = \frac{1}{n} x_1 \circ \cdots \circ x_n(id - \eta_j) = 0\,,$$

that is,

$$v_n(id - \eta_j) = 0 \quad \text{for} \quad 1 \leq j \leq n\,. \tag{3}$$

Proposition 1. $J = (id - v_n)RS_n$.

Proof. By (2), $id - v_n \in J$. On the other hand, the elements $id - \eta_j$ $(1 \leq j \leq n)$ generate the right ideal J ([3, Theorem]). Thus (3) implies that $v_n J = 0$, hence $J \subseteq (id - v_n)RS_n$.

In [3,3.], the following is shown:

Proposition 2. *RT is a complement of J in RS_n, and the mapping*

$$\tilde{}: \sum_{\sigma \in S_n} c_\sigma \sigma \mapsto \sum_{\sigma \in S_n} c_\sigma \eta_{1\sigma^{-1}} \sigma$$

is the projection of RS_n onto RT with kernel J.

Following Klyachko, we set $\zeta_n := \frac{1}{n} \sum_{j=0}^{n-1} \varepsilon^{-j} \tau^j$ where $\tau := (1 \ldots n)$, and $\lambda_n := \frac{1}{n} \sum_{\sigma \in S_n} \varepsilon^{\mathrm{ind}\, \sigma} \sigma$. Then ζ_n and λ_n are idempotent elements and we have

$$\lambda_n \zeta_n = \lambda_n, \quad \zeta_n \lambda_n = \zeta_n\,, \tag{4}$$

$$v_n \lambda_n = \lambda_n, \quad \lambda_n v_n = v_n\,, \tag{5}$$

(see [7] where products of permutations, however, have to be read from right to left, and [4, 3.4, 3.5]). It should be emphasized that the equations (5) are much deeper than (4). We compare coefficients to see that (5) is equivalent to

$$\begin{aligned} &\frac{1}{n}\sum_{\pi\in\mathscr{X}}(-1)^{1\pi^{-1}}\varepsilon^{\operatorname{ind}\pi^{-1}\sigma} = \varepsilon^{\operatorname{ind}\sigma} \quad \text{for all } \sigma\in S_n \\ &\frac{1}{n}\sum_{\pi\in\mathscr{X}}(-1)^{1\pi^{-1}}\varepsilon^{\operatorname{ind}\sigma\pi^{-1}} = \begin{cases}(-1)^{1\sigma^{-1}-1} & \text{for all } \sigma\in\mathscr{X}\\ 0 & \text{for all } \sigma\in S_n\setminus\mathscr{X}.\end{cases}\end{aligned} \tag{6}$$

(For a more general version of the second equation see [4, (10)]). The first equation in (6) is crucial for the proof of the following lemma:

Lemma 1. $\tilde{\lambda}_n = \sum_{\sigma\in T}\varepsilon^{\operatorname{ind}\sigma}\sigma$

Proof.

$$\begin{aligned}\tilde{\lambda}_n &= \frac{1}{n}\sum_{\sigma\in S_n}\varepsilon^{\operatorname{ind}\sigma}\eta_{1\sigma^{-1}}\sigma \qquad \text{by Proposition 2}\\ &= \frac{1}{n}\sum_{\sigma\in S_n}\sum_{\substack{\pi\in\mathscr{X}\\ 1\pi\sigma=1}}\varepsilon^{\operatorname{ind}\sigma}(-1)^{1\pi^{-1}-1}\pi\sigma\\ &= \frac{1}{n}\sum_{\sigma\in T}\sum_{\pi\in\mathscr{X}}(-1)^{1\pi^{-1}-1}\varepsilon^{\operatorname{ind}\pi^{-1}\sigma}\sigma\\ &= \sum_{\sigma\in T}\varepsilon^{\operatorname{ind}\sigma}\sigma, \qquad \text{by (6, upper part).}\end{aligned}$$

For all $Y\subseteq S_n$ we put $\overline{Y} := \sum_{\alpha\in Y}\alpha\in RS_n$. If Y is a subgroup of S_n, then the elements $\sigma\overline{Y}$ form a basis for the fixed space of Y in the right module RS_n whenever σ ranges over a left transversal of Y in S_n. In the following lemma, bases for the fixed space of Y in RS_n/J are obtained.

Lemma 2. *Let $Y\leq S_n$, and let $\mathscr{R}$ be a set of representatives for the double cosets $\langle\tau\rangle\varrho Y$ of length $n|Y|$ in S_n. Then the elements $\lambda_n\varrho\overline{Y}+J$ ($\varrho\in\mathscr{R}$) form a basis of the fixed space $C_{RS_n/J}(Y)$.*

Proof. The fixed space of Y in RS_n (that is, $RS_n\overline{Y}$) is generated by the elements $\sigma\overline{Y}$ ($\sigma\in S_n$). Hence the fixed space of Y in the right ideal $\zeta_n RS_n$ is generated by the elements $\zeta_n\sigma\overline{Y}$. A glance at the definition of ζ_n ensures us that this holds already if σ ranges only over a set of representatives of the double cosets $\langle\tau\rangle\gamma Y$ ($\gamma\in S_n$) by virtue of

$$\zeta_n\tau^j\gamma\overline{Y} = \varepsilon^j\zeta_n\gamma\overline{Y} \quad \text{for all } j. \tag{7}$$

If $|\langle\tau\rangle\gamma Y| < n|Y|$, then there exists j such that $0 < j < n$ and $\tau^j\gamma Y = \gamma Y$. Then we have $\zeta_n\gamma\overline{Y} = 0$, by (7). The classes $\langle\tau\rangle\gamma Y$ of order $n|Y|$ are represented by $\mathscr{R}$, and the elements $\zeta_n\varrho\overline{Y}$ $(\varrho \in \mathscr{R})$ are a linearly independent system as double cosets are pairwise disjoint. Thus

$$\text{the elements } \zeta_n\varrho\overline{Y} \ (\varrho \in \mathscr{R}) \text{ form a basis of } C_{\zeta_n RS_n}(Y). \tag{8}$$

The equations (4) imply that the left multiplication by λ_n induces an RS_n-isomorphism of $\zeta_n RS_n$ onto $\lambda_n RS_n$. In particular,

$$\text{the elements } \lambda_n\varrho\overline{Y} \ (\varrho \in \mathscr{R}) \text{ form a basis of } C_{\lambda_n RS_n}(Y). \tag{9}$$

By Proposition 1, $\upsilon_n RS_n$ is a complement of J in RS_n, and, by (5), $\lambda_n RS_n = \upsilon_n RS_n$. Therefore (9) implies that the elements $\lambda_n\varrho\overline{Y} + J$ $(\varrho \in \mathscr{R})$ form a basis of $C_{RS_n/J}(Y)$.

In the case of $Y = 1$ we have $C_{RS_n/J}(Y) = RS_n/J$ which implies the first part of the following corollary:

Corollary 1.

(i) *The elements* $\lambda_n\varrho + J$ *where* $\varrho \in T$ *form a basis of* RS_n/J.
(ii) *The elements* $\tilde{\lambda}_n\varrho$ *where* $\varrho \in T$ *form a basis of* RT.
(iii) $\tilde{\lambda}_n$ *is a unit in* RT.

Proof (ii): The elements $\tilde{\lambda}_n\varrho$ where $\varrho \in T$ are linearly independent, by (i), and are contained in RT. Their number is $|T|$. Hence they form a basis of RT.

(iii): By (ii), the left multiplication by $\tilde{\lambda}_n$ induces an automorphism of RT. As $\tilde{\lambda}_n \in RT$, this implies (iii).

We may consider $\zeta_n R$ as a faithful irreducible $\langle\tau\rangle$-right module. The structure of the RS_n-module $\mathscr{L}$ is then described by

$$\mathscr{L} \cong_{RS_n} (\zeta_n R)^{S_n} \quad \text{(see [7], [4])}, \tag{10}$$

for $(\zeta_n R)^{S_n} \cong_{RS_n} \zeta_n RS_n \underset{RS_n}{\cong} RS_n/J \underset{RS_n}{\cong} \mathscr{L}$. We will now illustrate this statement by the following description of the action of S_n on RS_n/J, making use of the basis given in Corollary 1(i).

Proposition 3. *For all* $\alpha, \beta \in S_n$

$$\sum_{\sigma \in T} \varepsilon^{\operatorname{ind} \sigma\beta}\sigma\alpha \equiv \sum_{\sigma \in T} \varepsilon^{\operatorname{ind} \sigma\alpha^{-1}\beta}\sigma \quad \text{modulo } J$$

Proof. For all $\gamma \in S_n$ and $0 \leq j \leq n-1$ we have ind $\gamma\tau^j =$ ind $\gamma - j$ by ([7, Lemma 1.3]). Therefore it suffices to assume that $\beta \in T$. Let $\alpha \in S_n$. Let $\beta' \in T$, $0 \leq j \leq n-1$ such that $\alpha = \tau^j \beta'$ and $\beta^* \in T$, $0 \leq k \leq n-1$ such that $\tau^j \beta^* = \beta\tau^k$. Then

$$\beta^{-1}\alpha = \beta^{-1}\tau^j\beta' = \tau^k\beta^{*^{-1}}\beta',$$

hence $\zeta_n\beta^{-1}\alpha = \zeta_n\tau^k\beta^{*^{-1}}\beta' = \varepsilon^k\zeta_n\beta^{*^{-1}}\beta'$. By (4), this implies that $\lambda_n\beta^{-1}\alpha = \varepsilon^k\lambda_n\beta^{*^{-1}}\beta'$. By means of Lemma 1 we conclude that $\sum_{\sigma\in T}\varepsilon^{\text{ind}\,\sigma\beta}\sigma\alpha$ is, modulo J, congruent to $\varepsilon^k\sum_{\sigma\in T}\varepsilon^{\text{ind}\,\sigma\beta'^{-1}\beta^*}\sigma = \sum_{\sigma\in T}\varepsilon^{\text{ind}\,\sigma\beta'^{-1}\beta^*\tau^{-k}}\sigma = \sum_{\sigma\in T}\varepsilon^{\text{ind}\,\sigma\alpha^{-1}\beta}\sigma$.

Putting $v_\varrho := \tilde{\lambda}_n\varrho + J$ for all $\varrho \in T$, we derive from this proof the following

Corollary 2. *Let $\varrho \in T$. Then we have*

(i) $v_\varrho\varrho' = v_{\varrho\varrho'}$ *for all* $\varrho' \in T$

(i) $v_\varrho\tau = \varepsilon^i v_{\varrho\sigma}$ *for suitable* $i \in \{0, \ldots, n-1\}$, $\sigma \in T$.

Thus, with respect to the basis elements $\underline{v}_\varrho$, the matrices corresponding to the elements of S_n in regard of their action on RS_n/J are monomial.

Now we will prove our main result a special version of which has been given in our introduction. Let ϕ be a mapping of X into Z such that $|z_j\phi^{-1}| = k_j$ for $1 \leq j \leq l$. We set

$$Y := \{\sigma \mid \sigma \in S_n,\ \sigma\phi = \phi\}.$$

Then Y is a Young subgroup of S_n of type $S_{k_1} \times \cdots \times S_{k_l}$. Let Λ_ϕ be the linearization mapping of $L^Z(k_1, \ldots, k_l)$ into $\mathscr{L}$ belonging to ϕ. Then (see [4,(4)]) we have

$$((x_{1\sigma}\phi) \circ \cdots \circ (x_{n\sigma}\phi))\Lambda_\phi = (x_{1\sigma} \circ \cdots \circ x_{n\sigma})\overline{Y} \quad \text{for all } \sigma \in S_n. \tag{11}$$

Theorem. *Let $\mathscr{R}$ be a set of representatives of the double cosets $\langle\tau\rangle\varrho Y$ of length $n|Y|$ in S_n. Then the set of the Lie elements*

$$\sum_{\sigma\in T}\varepsilon^{\text{ind}\,\sigma\varrho^{-1}}(x_{1\sigma}\phi) \circ \cdots \circ (x_{n\sigma}\phi) \quad (\varrho \in \mathscr{R})$$

is a basis of $L^Z(k_1, \ldots, k_l)$.

Proof. Our Lemmata 1 and 2, combined with Proposition 3, show that the elements $\sum_{\sigma\in T}\varepsilon^{\text{ind}\,\sigma\varrho^{-1}}\sigma\overline{Y} + J$ $(\varrho \in \mathscr{R})$ form a basis of $C_{RS_n/J}(Y)$. Therefore, the elements $\sum_{\sigma\in T}\varepsilon^{\text{ind}\,\sigma\varrho^{-1}}x_{1\sigma} \circ \cdots \circ x_{n\sigma}\overline{Y}$ $(\varrho \in \mathscr{R})$ form a basis of $C_{\mathscr{L}}(Y)$. By [4, (5)],

Λ_ϕ induces an isomorphism of $L(k_1, \ldots, k_l)$ onto $C_{\mathscr{L}}(Y)$. (Only here the fact is used that Y is a Young subgroup of S_n.) By virtue of (11), this implies the claim of the theorem.

Setting $x_i\phi := z_{a_i}$ for $1 \leq i \leq n$, we obtain the special version of our theorem as given in the introduction.

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