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Automorphisms of Modular Lie Algebras

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Abstract: We give a short argument that certain modular Lie algebras have surprisingly large automorphism groups.

1. Introduction and statement of results

A *classical* Lie algebra is one that has a Chevalley basis associated with an irreducible root system. If $\mathbf{L}$ is a classical Lie algebra over a field K of characteristic 0 then $\mathbf{L}$ has the following properties: (1) The automorphism group of $\mathbf{L}$ contains the Chevalley group associated with the root system as a normal subgroup with torsion quotient group; (2) $\mathbf{L}$ is simple. It has been known for some time that these properties do not always hold when K has positive characteristic, even when (2) is relaxed to condition (2′) $\mathbf{L}$ is quasi-simple, (that is, $\mathbf{L}/\mathbf{Z}(\mathbf{L})$ is simple, where $\mathbf{Z}(\mathbf{L})$ is the center of $\mathbf{L}$), but the proofs have involved explicit computations with elements of the algebras. See [Stein] (whose introduction surveys the early results in this area), [Hog].

Our first result is an easy demonstration of the instances of failure for (1) or (2′) by use of graph automorphisms for certain Dynkin diagrams; see (2.4), (3.2) and Table 1. Only characteristics 2 and 3 are involved here.

We also determine the automorphism groups of algebras of the form $\mathbf{L}/\mathbf{Z}$, where $\mathbf{Z}$ is a central ideal of $\mathbf{L}$ and $\mathbf{L}$ is one of the above classical quasisimple Lie algebras failing to satisfy (1) or (2′); see (3.8) and (3.9). As far as we know, existing literature deals only with the cases $\mathbf{Z} = \mathbf{Z}(\mathbf{L})$ and full arguments are not published. Our proofs use the classification of simple algebraic groups with elementary arguments from the theories of Lie algebras and finite groups, plus a few fairly well-known facts about finite subgroups of algebraic groups. Our arguments are relatively noncomputational, more nearly self-contained and shorter than earlier treatments. To keep this article brief, we do not consider the nonexceptional cases, which are dealt with in [Stein] and earlier literature, though our methods would work. We also give easy proofs of some results of [Hiss] on the action of a Chevalley group on its classical Lie algebra. It is possible that our graph automorphism technique is useful on modules other than the adjoint module.

2. Preliminary Results

2.1 Definition *We call a Lie algebra of* classical type *if it has a Chevalley basis over the field K.*

Thus, the automorphism group of a classical type Lie algebra contains the Chevalley group of that type.

2.2 Lemma *Let σ be an automorphism of an algebra. Suppose that C is the fixed point subalgebra and $\phi(t) \in K[t]$. Then $N = Im(\phi(\sigma))$ is stable under multiplication by C. In particular, $C \cap N$ is an ideal of C.*

Proof Straightforward. QED

The following is an immediate consequence.

2.3 Corollary *Let A be an algebra over the field K, and let σ be an automorphism of A with fixed point subalgebra C. Suppose that σ has minimal polynomial $(1-t)^k$, and let $N = Im(1-\sigma)^{k-1}$.*

(i) *If $C \neq N$ then C is not simple.*

(ii) *If $N \not\subseteq Z(C)$, $Z(C)$ is an ideal and $C \neq N + Z(C)$, then C is not quasi-simple.*

2.4 Corollary The *following Lie algebras are not quasi-simple:*

Types $B_n(K)$, $C_n(K)$, $F_4(K)$ when char$(K) = 2$.

Type $G_2(K)$ when char$(K) = 3$.

Proof Apply the previous result to the standard graph automorphisms of D_{n+1}, A_{2n-1}, E_6, and D_4. QED

3. The Main Results

To obtain our main results, we apply the observations above to the following situation:

3.1 Definition *A* special quadruple *is a 4-tuple $(\mathbf{L}, \sigma, M, K)$ where $\mathbf{L}$ is a classical Lie algebra over K, K is a field of characteristic $p > 0$, σ is a standard graph automorphism of $\mathbf{L}$, σ has order p, and M is the fixed point subgroup for the action of σ on the Chevalley group associated to $\mathbf{L}$; M is itself a Chevalley group, associated with the fixed points of σ on the root lattice; thus, M is a group of automorphisms of the fixed point subalgebra of σ.*

Notice that we are identifying σ with an automorphism of the root system, say, Φ and that there is understood to be a set Π of fundamental roots and a Chevalley basis $\mathcal{B} = \{h_\alpha, e_\beta \mid \alpha \in \Pi, \beta \in \Phi\}$ permuted by σ as σ permutes the subscripts.

3.2 Lemma *Let $(\mathbf{L}, \sigma, M, K)$ be a special quadruple, $\mathbf{C} := Ker(1-\sigma)$ and let $\mathbf{N} = Im(1-\sigma)^{p-1}$. Then M acts on the Lie algebra $\mathbf{C}/\mathbf{N}$. Furthermore $\mathbf{C}$ contains a subalgebra $\mathbf{S}$ of classical type such that $\mathbf{C} = \mathbf{N} + \mathbf{S}$ and $\mathbf{S} \cap \mathbf{N} \subseteq \mathbf{Z}(\mathbf{S})$ where $\mathbf{Z}(\mathbf{S})$ is the center of $\mathbf{S}$. If $\mathbf{L}$ is of type D_n or E_6 then the root system associated to $\mathbf{S}$ is irreducible. For any central subalgebra $\mathbf{Z}$ of S, M acts faithfully on $\mathbf{S}/\mathbf{Z}$.*

3.3 Definition *We call* $\mathbf{S}$ a covering subalgebra *for the quadruple $(\mathbf{L}, \sigma, M, K)$.*

3.4 Notation *Define $\mathbf{Q} := \mathbf{C}/\mathbf{N}$. Also, when K is algebraically closed, let $A := Aut(\mathbf{S}/\mathbf{Z}(\mathbf{S}))^0$; this is an algebraic group which contains an isomorphic copy of M as* *an algebraic subgroup. Let R be the Chevalley group associated to* $\mathbf{S}$*; by definition, it is a subgroup of* $Aut(\mathbf{S})$.

Proof [of (3.2)] It is clear that $\mathbf{N} \subseteq \mathbf{C}$, so that $\mathbf{N}$ is an ideal in $\mathbf{C}$ and $M \subseteq Aut(\mathbf{C})$ acts on $\mathbf{C}/\mathbf{N}$. The action is faithful, as we can see by passing to the algebraic closure where M becomes a direct product of simple groups. For convenience, let $H := \langle h_\alpha | \alpha \in \Pi\rangle$ and let E_0 be the subspace spanned by $\{e_\beta \mid \beta \text{ is fixed by } \sigma\}$.

Let $\mathbf{S}$ be the subalgebra generated by E_0. Then $\mathbf{S} = \mathbf{S} \cap H \oplus E_0 \subseteq \mathbf{C}$, and $\mathbf{S} \cap \mathbf{N} \subseteq \mathbf{Z}(\mathbf{S}) \cap H$. In fact, $\{\sum_{\alpha \in \mathcal{O}} h_\alpha \mid \mathcal{O} \text{ is a regular } \sigma\text{-orbit on } \Pi\}$ is a basis for $\mathbf{S} \cap \mathbf{N}$.

For each σ-orbit $\mathcal{O}$ on Π, let $\bar{\mathcal{O}}$ be the smallest connected subset of nodes containing $\mathcal{O}$. Also let $\gamma_{\mathcal{O}} = \sum_{\beta \in \bar{\mathcal{O}}} \beta$. Then $\{\gamma_{\mathcal{O}} \mid \mathcal{O} \text{ is a } \sigma\text{-orbit on } \Pi\}$ is a fundamental system of roots for the fixed points of σ on Φ. The table below gives the type of $\mathbf{S}$ for each of the special quadruples. (The following observation allows the type of $\mathbf{S}$ to be read off the Dynkin diagram for $\mathbf{L}$. If $p = 2$ and $\bar{\mathcal{O}} \cap \bar{\mathcal{O}}' \neq \emptyset$ for distinct orbits $\mathcal{O}$ and $\mathcal{O}'$, then $\bar{\mathcal{O}} \subset \bar{\mathcal{O}}'$ or vice versa. Without loss, assume the former. Then all but two of the summands in $\gamma_{\mathcal{O}'} - \gamma_{\mathcal{O}}$ are orthogonal to all of the summands in $\gamma_{\mathcal{O}}$, so that $(\gamma_{\mathcal{O}}, \gamma_{\mathcal{O}'}) = (\gamma_{\mathcal{O}}, \gamma_{\mathcal{O}}) + (\gamma_{\mathcal{O}}, \gamma_{\mathcal{O}'} - \gamma_{\mathcal{O}}) = 2 - 2 = 0$.)

Suppose $\mathbf{L}$ has type d_4 and $p = 2$. Let θ be the standard graph automorphism of $\mathbf{L}$ of order 3 (it is inverted by σ under conjugation), and let $\mathbf{S}_\theta$ be the fixed points of θ on $\mathbf{L}$. The θ-orbits on Φ consist of fixed points, which are also fixed by σ, and the regular orbits, each of which is a union of a fixed point of σ and a regular σ-orbit. This implies that $\mathbf{S}_\theta \subseteq \mathbf{C}$ and that $\mathbf{S}_\theta \cap \mathbf{N} = 0$. Thus there is a second possibility for $\mathbf{S}$ in this case, as indicated in Table 1, below.

The last statement is clear since M contains a copy of the Chevalley group associated to $\mathbf{S}$ and M becomes simple when the field is sufficiently large.

Notice that a further example of type $\mathbf{a}$ occurs here since $\mathbf{a}_3 \cong \mathbf{d}_3$. QED

Table 1: Special quadruples and covering algebras
($p = 3$ in the first case, $p = 2$ in all others)

$\mathbf{L}$	M	dim $\mathbf{C}$	dim $\mathbf{N}$	$\mathbf{S}$	dim $\mathbf{S}$	dim $\mathbf{Z}(\mathbf{S})$	dim $\mathbf{N} \cap \mathbf{S}$
$\mathbf{d}_4$	G_2	14	7	$\mathbf{a}_2$	8	1	1
$\mathbf{e}_6$	F_4	52	26	$\mathbf{d}_4$	28	2	2
$\mathbf{d}_{n+1}, n \geq 3$	B_n	$2n^2 + n$	$2n + 1$	$\mathbf{d}_n$	$n(2n-1)$	$\begin{cases} 1, n \text{ odd} \\ 2, n \text{ even} \end{cases}$	1
$\mathbf{a}_{2n-1}, n \geq 2$	C_n	$2n^2 + n$	$2n^2 - n - 1$	$\mathbf{a}_1 \oplus \ldots \oplus \mathbf{a}_1$	$3n$	n	$n - 1$
$\mathbf{d}_4$	B_3	21	7	$\mathbf{g}_2$	14	0	0

3.5 Corollary *For each of the following triples* $(\mathbf{S}, X, p)$*, the Chevalley group of type* X *acts faithfully on the central quotient of the classical Lie algebra of type* $\mathbf{S}$ *in characteristic* p.

(i) $\mathbf{S} = \mathbf{a}_2$, $X = G_2$, $p = 3$.

(ii) $\mathbf{S} = \mathbf{d}_4$, $X = F_4$, $p = 2$.

(iii) $\mathbf{S} = \mathbf{d}_n$, $X = B_n$, $p = 2$.

(iv) $\mathbf{S} = \mathbf{g}_2$, $X = B_3$, $p = 2$.

It is worth mentioning that the ideal $\mathbf{N} \cap \mathbf{C}$ of $\mathbf{C}$ is associated to Chevalley basis elements for short roots in the root system inherited by $\mathbf{C}$ from Φ and σ. This is immediate from our procedure since short roots are associated with sums over the regular orbits for σ. Similarly, it is immediate that $\mathbf{N} \cap \mathbf{C}$ is stable under the Chevalley group R (and even M). This seems much easier than an argument within $\mathbf{C}$ since a direct verification that the above subspace is an ideal and is stable under the Chevalley group, starting from the definitions of the Chevalley basis and the Chevalley generators $x_r(t)$, would require a study of chains in the particular root systems and calculations of binomial coefficients.

We have shown that each of the algebras $\mathbf{S}/\mathbf{Z}(\mathbf{S})$ from the previous corollary admits, as automorphisms, a Chevalley group M properly containing the Chevalley group associated to the classical Lie algebra $\mathbf{S}$. We next proceed to determine the full automorphism group of $\mathbf{S}/\mathbf{Z}(\mathbf{S})$. We use algebraic group techniques to prove that the group is not bigger than the group M when the field is algebraically closed. The situation is then fairly clear for general fields since we are dealing with Chevalley groups; the occurrence of outer diagonal automorphisms here depends on the particular field.

3.6 Lemma *Suppose $char(K) = 2$ and that $\mathbf{L}$ is a classical Lie algebra of type d_n, $n \geq 3$. Let h_{ij}, h'_{ij} denote the elements h_r of the Cartan subalgebra $\mathbf{h}$ of $\mathbf{L}$ given by roots $r = e_i + e_j, e_i - e_j$, in the usual notation [Bour]. If n is odd, $\mathbf{Z}(\mathbf{L})$ has basis $h_{ij} + h'_{ij}$, for any pair $i \neq j$. If n is even, $\mathbf{Z}(\mathbf{L})$ has basis consisting of an $h_{ij} + h'_{ij}$ as above, and $h_{12} + h_{34} + \ldots + h_{n-1,n}$. The usual graph automorphism of order 2 (associated to a determinant -1 diagonal transformation) commutates the second basis vector above to the first. As a module for the Weyl group ($\cong 2^{n-1} : Sym_n$), the maximal trivial quotient of the Cartan subalgebra is zero. In fact, the module structure for the Weyl group on $\mathbf{h}$ has ascending factors of dimensions $1, n-1$ for n odd and $1, 1, n-2$ for n even.*

Proof Exercise. QED

3.7 Lemma *Suppose that $\mathbf{Z}(\mathbf{S}) \neq 0$ and that $\mathbf{S}$ does not have type $\mathbf{a}_1 \oplus \ldots \oplus \mathbf{a}_1$ (so that $n \geq 3$ when $\mathbf{S}$ has type $\mathbf{d}_n$). Then $\mathbf{Z}(\mathbf{S})$ is not complemented by an R-submodule.*

Proof Suppose we are in the $\mathbf{a}_2$ case. Let u be a unipotent element of R with a single Jordan block of size 3 on the natural module V. It has the same Jordan canonical form on the dual module, and on the tensor of these two representations, which contains the representation on $\mathbf{S}$ as a constituent, it has three Jordan blocks of size 3. On $\mathbf{S}/\mathbf{Z}(\mathbf{S})$, of dimension 7, it has at least 3 dimensions of fixed points. If there were a complement invariant under u, u would have fixed point subspace on $\mathbf{S}$ of dimension at least 4, a contradiction.

All remaining cases satisfy $char(K) = 2$ and $\mathbf{S}$ has type $\mathbf{d}_n$, for some $n \geq 3$. The previous result deals with this case. QED

It is worth observing that the previous result gives a trivial proof of the F_4 and G_2 cases of the Hauptsatz of [Hiss], which determines the submodule structure of a classical Lie algebra with respect to its Chevalley group; for, if there were a submodule complementing the nontrivial ideal (of dimensions 26 and 7, respectively), we could descend to a covering subalgebra and its Chevalley group and contradict the previous lemma.

3.8 Theorem *If K is algebraically closed then*

(i) $Aut(\mathbf{a}_2(K)/\mathbf{Z}(\mathbf{a}_2(K))) \cong G_2(K)$, *for char*$(K) = 3$.

(ii) $Aut(\mathbf{g}_2(K)) \cong B_3(K)$, *for char*$(K) = 2$. *Note that* $\mathbf{Z}(\mathbf{g}_2(K))) = 0$ *in this case.*

(iii) $Aut(\mathbf{d}_4(K)/\mathbf{Z}(\mathbf{d}_4(K))) \cong F_4(K)$, *for char*$(K) = 2$.

(iv) $Aut(\mathbf{d}_n(K)/\mathbf{Z}(\mathbf{d}_n(K))) \cong B_n(K)$, *for char*$(K) = 2$, $n = 3$ *or* $n > 4$.

Proof The approach to each case is similar, but the details vary. Let $(\mathbf{L}, \sigma, M, K)$ be a special quadruple with associated covering subalgebra $\mathbf{S}$; use the notations of (3.4). In each case, discussed below, we show that M acts irreducibly on $\mathbf{Q}/\mathbf{Z}(\mathbf{Q}) \cong \mathbf{S}/\mathbf{Z}(\mathbf{S})$. This implies that a normal unipotent subgroup of A is trivial, so any reductive subgroup centralizing M (identified with its image in A) must be both scalar and a group of automorphisms, hence trivial. Therefore, A is semisimple (and M projects nontrivially to each factor).

We also note that a maximal torus T of M acts on $\mathbf{Q}$ by pairwise distinct nontrivial linear characters at the root spaces and has fixed point subalgebra the Cartan subalgebra $\mathbf{h}$. It follows that each of these spaces is left invariant by $C_A(T)$, as are the 1-dimensional subspaces of $\mathbf{h}$ obtained by bracketing a root space and its negative. Since $C_A(T)$ effects a scalar on each invariant 1-space, indecomposability of the root system implies that $C_A(T)$ acts by a scalar-valued homomorphism on $\mathbf{h}$. Also, if $\mathbf{E}$ is any root space, the equation $[\mathbf{h}, \mathbf{E}] = \mathbf{E}$ implies that $C_A(T)$ centralizes $\mathbf{h}$. In particular, the Weyl group of A acts on $\mathbf{h}$.

An immediate corollary is that A and M have the same Lie rank; for otherwise, the kernel of the action of a torus of $C_A(T)$ on the above fundamental root spaces and their opposites would be positive dimensional. Finally, we deduce that A is simple.

We settle all cases now with the observation that, in each case, there is no embedding of M as a proper subgroup of a simple algebraic group with the same rank.

It remains to verify irreducibility of M on $\mathbf{Q}/\mathbf{Z}(\mathbf{Q}) \cong \mathbf{S}/\mathbf{Z}(\mathbf{S})$.

(i) $M \cong G_2(K)$ contains a subgroup of shape $2^3{\cdot}SL(3,2)$ [Gr 1990] which acts faithfully, hence irreducibly on the 7-dimensional module $\mathbf{Q}$ (51.7)[CuRe].

(ii) Here, $M \cong B_3(K)$ acts on $\mathbf{Q} \cong \mathbf{g}_2(K)$. To show that this action is irreducible, it suffices to show that a $G_2(2)$-subgroup X acts irreducibly. Let x be a 3-central element of X of order 3. Then $N_X(\langle x \rangle) \cong 3_+^{1+2} : [8:2]$ (the factor 8:2 is semidihedral), x is real in X, and x has fixed point subalgebra of dimension 8. If $P := O_3(N_X(\langle x \rangle))$, then any element of $P - \langle x \rangle$ is real and has fixed point subalgebra of dimension 4; such elements form a single $N_X(\langle x \rangle)$-conjugacy class. Orthogonality relations (done in characteristic 0 with Brauer characters) imply that P has 0 fixed point subalgebra on $\mathbf{Q}$. This also implies that $N_X(\langle x \rangle)$ has irreducible constituents of degrees 8 and 6, and x acts trivially on the 8-dimensional constituent. Since $\langle x^X \rangle$ has index 2 in X, and since elements from the other class of elements of order 3 have fixed point subalgebras of dimension 4, it follows that this 8-dimensional constituent does not represent a composition factor for X. Therefore X must act irreducibly on $\mathbf{Q}$.

(iii) A subgroup of $M \cong F_4(K)$ of shape $3^3 : SL(3,3)$ acts irreducibly on $\mathbf{Q}$ due to the fact that its normal subgroup of order 3^3 acts with 26 distinct nontrivial linear characters, which form an orbit under the action of $SL(3,3)$.

(iv) If $n = 3$ then $\mathbf{Q} \cong \mathfrak{g}_2(K)$ by Lemma 3.2, and the result follows from case (ii). We therefore assume that $n > 4$. Let T be a maximal torus of $M \cong B_n(K)$. Then, $N_M(T)$ acts on the adjoint module $\mathbf{C}$ for M with irreducible constituents corresponding to the orbits of the Weyl group $W = N_M(T)/T$ on the short and long roots, plus the constituents for the action of W on the Cartan subalgebra $\mathbf{h}$ of $\mathbf{C}$. When we pass to $\mathbf{Q} = \mathbf{C}/\mathbf{N}$, we factor out the span of the short root spaces and a 1-dimensional central ideal. When we pass to the full central quotient of $\mathbf{Q}$ we get an irreducible action of W on the image of $\mathbf{h}$ (see (3.7). Since no Chevalley group element $x_r(t)$ of M, for r fixed by σ, leaves invariant the image of $\mathbf{h}$ in $\mathbf{Q}$, we deduce irreducibility of M on $\mathbf{Q}/\mathbf{Z}(\mathbf{Q})$, as required. QED

Finally, we determine automorphism groups for most remaining central quotients of the covering subalgebras from Table 1 (we exclude just line 4). The essential case is that of an algebraically closed field.

3.9 Theorem *Let* $\mathbf{S}$ *be a covering subalgebra from Table 1 with* $\mathbf{S}/\mathbf{Z}(\mathbf{S})$ *simple, and let* $\mathbf{Z}$ *be a central ideal properly contained in* $\mathbf{Z}(\mathbf{S})$. *Assume that* K *is algebraically closed. Then* $Aut(\mathbf{S}/\mathbf{Z})^0$ *is given by the natural action of* R, *where* R *is the Chevalley group associated to* $\mathbf{S}$, *except in the cases*

(i) $\mathbf{S}$ *has type* $\mathbf{d}_4$ *and* $\mathbf{Z}$ *is one of three particular one-dimensional ideals, in which case* $Aut(\mathbf{S}/\mathbf{Z})^0$ *corresponds to one of the three natural type* B_4 *subgroups between the images of* R *and* $M \cong F_4(K)$ *in* $Aut(\mathbf{S}/\mathbf{Z}(\mathbf{S}))^0$.

(ii) $\mathbf{S}$ *has type* $\mathbf{d}_n$, *for even* $n \geq 2$, $\dim \mathbf{Z} = 1$ *and* $Aut(\mathbf{S}/\mathbf{Z}) \cong B_n(K)$.

Proof Let $\mathbf{Z}$ be a central ideal proper in $\mathbf{Z}(\mathbf{S})$ and $A = Aut(\mathbf{S}/\mathbf{Z})^0$. We deal with the various cases of $\mathbf{Z}(\mathbf{S}) \neq 0$ indicated in Table 1. Note that if $\mathbf{Z}_1$ is any central ideal, the natural map from $\{\alpha \in Aut(\mathbf{S}) \mid \alpha \text{ leaves } \mathbf{Z}_1 \text{ invariant}\}$ to $Aut(\mathbf{S}/\mathbf{Z})$ is a monomorphism since $\mathbf{S}$ is perfect. Thus, identifying groups with their images in $Aut(\mathbf{S}/\mathbf{Z}(\mathbf{S}))$, we get containments $R \subseteq A \subsetneq M$. We determine the middle group for all relevant $\mathbf{Z}$.

Case 1. $\mathbf{S} = \mathbf{a}_2(K)$, for $char(K) = 3$.

Here, $\dim \mathbf{Z} = 0$. Suppose that $R < A$. Then, $A = M \cong G_2(K)$. Let B be a subgroup of A isomorphic to $2^3 \cdot SL(3,2)$. Then, $O_2(B)$ operates fixed point freely on $\mathbf{S}/\mathbf{Z}(\mathbf{S})$ and so leaves invariant a unique complement V to $\mathbf{Z}(\mathbf{S})$ in $\mathbf{S}$. The subgroup A_V of A leaving V invariant is an algebraic subgroup of A. If A_V is positive dimensional, then an argument similar to that of (3.8) shows that A_V must be simple since it contains B and has rank at most 2. Table 1 of (1.8)[Gr 1990] implies that $A_V = M$, a contradiction to (3.7). We conclude that A_V is finite. The action of the 14-dimensional group A on a 6-dimensional space of complements in $\mathbf{S}$ to $\mathbf{Z}$ (this is equivalent to an action on a 6-dimensional affine space of vectors in the dual space of $\mathbf{S}$) therefore has 0-dimensional fiber, a contradiction.

Case 2. $\mathbf{S} = \mathbf{d}_n(K)$, for $\mathrm{char}(K) = 2$.

Identify M with $Aut(\mathbf{S}/\mathbf{Z}(\mathbf{S}))$. The image X of $Aut(\mathbf{S}/\mathbf{Z})^0$ in M is an algebraic group between R and M, so is either R or a natural B_n subgroup, or possibly $M \cong F_4(K)$ when $n = 4$. The latter possibility for X is quickly eliminated by arguing as in Case 1, with a subgroup $3^3 : SL(3,3)$ in the role of B, using Table 2 of (1.8)[Gr 1990].

Consider the possibility that $X \cong B_n(K)$. Then, X and R share a maximal torus and so the Weyl group of X acts on the image in $\mathbf{S}/\mathbf{Z}$ of a Cartan subalgebra of $\mathbf{S}$. We assume that this action extends that of the Weyl group of R by a determinant -1 diagonal transformation in the usual description of the root system of type $\mathbf{d}_n$. When triality is present (for us, this means $M \cong F_4(K)$), we may need to use triality to assume the above.

If $n \geq 2$ is even, dim $\mathbf{Z} = 0$ is impossible, by the way the graph automorphism of $\mathbf{d}_n$ acts on the elements of (3.6), since we would then have a nontrivial homomorphism of $B_n(K)$ to $GL(2,K)$. So, if n is even, dim $\mathbf{Z} = 1$. This case occurs, by Table 1. What is needed now is the result that such a $\mathbf{Z}$ must be the span of $h_{ij} + h'_{ij}$, in the notation of (3.6). This follows from the way the graph automorphism acts on $h_{12} + \ldots + h_{n-1,n}$ modulo $\mathbf{Z}$.

If n is odd, $n \geq 3$ and the only possibility here is $\mathbf{Z} = 0$. We show that $X \cong B_n(K)$ is impossible. Let T be the 1-dimensional torus in X whose fixed point subgroup has semisimple part Y isomorphic to $B_{n-1}(K)$. Without loss, Y corresponds to a subset of $n-1$ nodes of the Dynkin diagram for X. We may assume that the subgroup generated by root groups associated to the long roots in the root system for Y is in R. These roots therefore correspond to a subset of the given Chevalley basis and we have an associated classical subalgebra $C_{\mathbf{Q}}(T)$ of type $\mathbf{d}_{n-1}$. Since $n-1$ is even and since Y acts on $C_{\mathbf{Q}}(T)$, we have a contradiction to the previous paragraph, the case dim $\mathbf{Z} = 0$. QED

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