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Generalized Derivations of Nonassociative Algebras

by

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§1 Introduction

Throughout $\mathfrak{a}$ will be a nonassociative algebra over a field k with $\operatorname{char} k \neq 2$. If $k = \mathbb{R}$ and $\mathfrak{a}$ is commutative, the quadratic differential equation $\frac{dX}{dt} = X^2$ where $X \in \mathfrak{a}$ and X^2 is the square of X in $\mathfrak{a}$ has been extensively studied (both [8] and [11] have extensive bibliographies). These and other authors have shown that the algebraic properties of $\mathfrak{a}$ give a great deal of information about the solutions. In particular the derivations of $\mathfrak{a}$ are useful in understanding the differential equation [8] where $D \in \operatorname{End}_k \mathfrak{a}$ is a *derivation* if $D(xy) = D(x)y + xD(y)$ for all $x, y \in \mathfrak{a}$. If there is a derivation which is diagonal, writing the differential equation in terms of the eigenspaces of such a derivation gives a nice decomposition of the differential equation. Unfortunately, many interesting quadratic differential equations occur in simple algebras having no derivations and this seems to be connected to the origin being a stable equilibrium point. Therefore, it is natural to try to find linear operators on $\mathfrak{a}$ which can exist when the origin is stable and which have some of the properties of derivations.

Example 1.1: For the quadratic differential equation

$$\frac{dx}{dt} = y^2 + xy \qquad (1.1.1)$$
$$\frac{dy}{dt} = -x^2 - xy$$

$\mathfrak{a} = \mathbb{R}^2$ with multiplication defined by

$$\begin{bmatrix} x \\ y \end{bmatrix} \cdot \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} yd + \frac{1}{2}xd + \frac{1}{2}yc \\ -xc - \frac{1}{2}xd - \frac{1}{2}yc \end{bmatrix}.$$

It is easy to check that $\mathfrak{a}$ has no derivations. But $D = \frac{1}{2}\begin{bmatrix} -1 & 3 \\ 3 & -1 \end{bmatrix}$ does satisfy

$$D(XY) = -D(X)\cdot Y - X\cdot D(Y) \tag{1.1.2}$$

for all $X,Y \in \mathfrak{a}$. $\begin{bmatrix}1\\1\end{bmatrix}$ is an eigenvector of D of eigenvalue 1 and $\begin{bmatrix}1\\-1\end{bmatrix}$ is an eigenvector of D of eigenvalue -2. Relative to this new basis (1.1.1) has the nice form

$$\begin{aligned}\frac{dz}{dt} &= -zw\\ \frac{dw}{dt} &= z^2\end{aligned} \tag{1.1.3}$$

where $z = x+y$ and $w = x-y$. From this we get x^2+y^2 is constant for solutions of (1.1.1) so the origin is stable.

It is clear that a diagonal linear operator on $\mathfrak{a}$ satisfying (1.1.2) will always give a nice decomposition of $\frac{dX}{dt} = X^2$ in terms of its eigenspaces. Hence we say $D \in \operatorname{End}_k\mathfrak{a}$ is an *antiderivation* of the algebra $\mathfrak{a}$ if $D(xy) = -D(x)\cdot y - x\cdot D(y)$ for all $x,y \in \mathfrak{a}$. The set of all antiderivations has no natural structure as an algebra, but the linear span of the set of derivations and antiderivations is a Lie algebra, leading to the following definition.

Definition 1.2: $(D_1, D_2) \in \operatorname{End}_k\mathfrak{a} \times \operatorname{End}_k\mathfrak{a}$ is a *generalized derivation* of the algebra $\mathfrak{a}$ if for all $x,y \in \mathfrak{a}$, $i,j = 1,2$, $i \neq j$

$$D_i(xy) = D_j(x)\cdot y + x\cdot D_j(y). \tag{1.2.1}$$

Thus if D is a derivation and E is an antiderivation of $\mathfrak{a}$, then $(D+E, D-E)$ is a generalized derivation of $\mathfrak{a}$ and conversely if (D_1,D_2) is a generalized derivation of $\mathfrak{a}$ then D_1+D_2 is a derivation and D_1-D_2 is an antiderivation of $\mathfrak{a}$. Gender $\mathfrak{a}$ will denote the set of all generalized derivations of $\mathfrak{a}$. Gender $\mathfrak{a}$ is a Lie algebra with the bracket defined by

$$[(D_1,D_2),(E_1,E_2)] := ([D_1,E_1],[D_2,E_2]). \tag{1.3}$$

Then θ is an automorphism of Gender $\mathfrak{a}$ where

$$(D_1,D_2)\theta := (D_2,D_1). \tag{1.4}$$

Thus the derivation algebra $\operatorname{Der}\mathfrak{a}$ is isomorphic to the $+1$ eigenspace of θ and the -1 eigenspace of θ is isomorphic to the set Antider $\mathfrak{a}$ of all antiderivations of $\mathfrak{a}$, Thus Antider $\mathfrak{a}$ has the natural structure of a Lie triple system [3].

Perhaps more interesting to an algebraist is the fact that generalized derivations arise in the study of some noncommutative Jordan algebras [1] and that at least one simple Lie algebra has antiderivations as the following example shows.

Example 1.5: $\mathrm{sl}(2,k) \cong k^3$ with the product defined by $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} := \begin{bmatrix} bz - cy \\ 2ay - 2bx \\ 2cx - 2az \end{bmatrix}$ where $\begin{bmatrix} a & b \\ c & -a \end{bmatrix} \mapsto \begin{bmatrix} a \\ b \\ c \end{bmatrix}$. Then it is easy to show that

Antider $\mathrm{sl}(2,k) = \left\{ \begin{bmatrix} -2a & b & c \\ 2c & a & d \\ 2b & e & a \end{bmatrix} \middle| a,b,c,d,e \in k \right\}$ and

Der $\mathrm{sl}(2,k) = \left\{ \begin{bmatrix} 0 & a & b \\ -2b & 2c & 0 \\ 2a & 0 & -2c \end{bmatrix} \middle| a,b,c,d,e \in k \right\}$ so Gender $\mathrm{sl}(2,k) \cong \mathrm{sl}(3,k)$.

Example 7 of [5] is another instance of a Lie algebra with antiderivations. However as will be shown in Theorem 5.1 below, $\mathrm{sl}(2,k)$ is the only simple Lie algebra with nonsingular trace form having antiderivations if char $k \neq 3,5,7$.

Example 3.6 of [1] gives a whole class of anticommutative algebras having antiderivations by Lemma 4.4 of [1]. In fact, it is easy to generate new examples of algebras having antiderivations by the following construction. If $\mathfrak{a}$ is an algebra and $A \in \mathrm{Aut}\,\mathfrak{a}$, the group of automorphisms of $\mathfrak{a}$, define a new algebra $\mathfrak{a}_A$ on the vector space $\mathfrak{a}$ with product $\cdot_A$ by setting

$$x \cdot_A y := A(xy). \tag{1.6}$$

Then if $A^2 = id$ and $D \in \mathrm{Der}\,\mathfrak{a}$, $(D, ADA) \in$ Gender $\mathfrak{a}_A$ so if $ADA \neq D$, $D - ADA$ is an antiderivation of $\mathfrak{a}$. More generally, if $(D_1,D_2) \in$ Gender $\mathfrak{a}_A$ then $(D_1,AD_2A), (D_2,AD_1A) \in$ Gender $\mathfrak{a}_A$. Since $\mathfrak{a} = (\mathfrak{a}_A)_A$, Gender $\mathfrak{a} \cong$ Gender $\mathfrak{a}_A$.

The next result shows that Gender $\mathfrak{a}$ is a well behaved algebra invariant. Here $C_L(\mathfrak{a}) := \{x \in \mathfrak{a} \mid xy = 0 \ \forall y \in \mathfrak{a}\}$ is the *left center* of $\mathfrak{a}$, $C_R(\mathfrak{a}) = \{x \in \mathfrak{a} \mid yx = 0 \ \forall y \in \mathfrak{a}\}$ is the *right center* of $\mathfrak{a}$, $C(\mathfrak{a}) := C_L(\mathfrak{a}) \cap C_R(\mathfrak{a})$ is the *center* of $\mathfrak{a}$, $Z(\mathfrak{a}) := C(\mathfrak{a}^-)$ is the *commutator* of $\mathfrak{a}$, and $W(\mathfrak{a}) := C(\mathfrak{a}^+)$ is the *anticommutator* of $\mathfrak{a}$ where $\mathfrak{a}^-$ is $\mathfrak{a}$ with the bracket product $[x,y] := xy - yx$ and $\mathfrak{a}^+$ is $\mathfrak{a}$ with the commutative product $\circ$ defined by $x \circ y := \frac{1}{2}(xy + yx)$.

Proposition 1.7: Suppose $\mathfrak{a}$ and $\mathfrak{b}$ are algebras and $(D_1, D_2) \in \text{Gender}\,\mathfrak{a}$.

(i) If $\mathfrak{a} \cong \mathfrak{b}$, then Gender $\mathfrak{a} \cong$ Gender $\mathfrak{b}$
(ii) If $I \trianglelefteq \mathfrak{a}$ with $I = I^2$, then $D_i(I) \subseteq I$ for $i = 1,2$.
(iii) If $x \in C_L(\mathfrak{a})$, then $D_i(x) \in C_L(\mathfrak{a})$ for $i = 1,2$.
(iv) If $x \in C_R(\mathfrak{a})$, then $D_i(x) \in C_R(\mathfrak{a})$ for $i = 1,2$.
(v) If $x \in C(\mathfrak{a})$, then $D_i(x) \in C(\mathfrak{a})$ for $i = 1,2$.
(vi) $(D_1,D_2) \in$ Gender $\mathfrak{a}^+$ and $(D_1,D_2) \in$ Gender $\mathfrak{a}^-$.
(vii) If $x \in Z(\mathfrak{a})$, then $D_i(x) \in Z(\mathfrak{a})$ for $i = 1,2$.
(viii) If $x \in W(\mathfrak{a})$, then $D_i(x) \in W(\mathfrak{a})$ for $i = 1,2$.

Proof: The proof is essentially the same as for derivations.

Example 1.8: Suppose char $k = 0$. For $\alpha \in k$ define a multiplication on $\mathfrak{a}_\alpha = k^2$ by $\begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} := \begin{bmatrix} yb + \frac{1}{2}\alpha xb + \frac{1}{2}\alpha ya \\ -xa - \frac{1}{2}xb - \frac{1}{2}ya \end{bmatrix}$. As in Example 1.1, Der $\mathfrak{a}_\alpha = 0$ for all $\alpha \in k$, dim Gender $\mathfrak{a}_1 = 1$, and Gender $\mathfrak{a}_\alpha = 0$ if $\alpha \neq 1$. Thus $\mathfrak{a}_1 \not\cong \mathfrak{a}_\alpha$ for $\alpha \neq 1$.

Hence Gender $\mathfrak{a}$ is a more discriminating invariant than Der $\mathfrak{a}$.

The next proposition is trivial to prove, but quite useful when doing computations.

Proposition 1.9: Suppose $\mathfrak{a}$ is commutative and $D \in$ End $\mathfrak{a}$. Then D is an antiderivation of $\mathfrak{a}$ iff $D(x^2) = -2x \cdot Dx$ for all $x \in \mathfrak{a}$.

In §2 we show for $\mathfrak{a}$ finite dimensional Gender $(\mathfrak{a})$ contains the semisimple and nilpotent parts of its elements when k is algebraically closed (Theorem 2.4). In §3 we consider algebras with identity and show (Proposition 3.1) that if char $k \neq 3$, such an algebra has no antiderivations. In charactistic 3 we determine the antiderivations of algebras with identity (Proposition 3.2). §4 deals with commutative associative algebras (without identity) where char $k \neq 3$. We show in Theorem 4.1 that such an algebra has no antiderivations if $C(\mathfrak{a}) = 0$ and $\mathfrak{a} = \mathfrak{a}^2$ and in Theorem 4.2 that every antiderivation is nilpotent if $C(\mathfrak{a}) = 0$ and $\mathfrak{a}$ is finite dimensional. In §5 we show that a central simple finite dimensional Lie algebra of dimension ≥ 4 having a nonsingular trace form on some representation has no antiderivations if char $k \neq 3,5$, or 7.

Other authors have considered generalizations of algebra derivations, see for example [10] and [2], and the terms "antiderivation" and "generalized derivation" have appeared in the literature with meanings other than the ones given here.

§2 Gender ($\mathfrak{a}$) Is Almost Algebraic

Recall from [7] that a Lie algebra $\mathcal{L} \subseteq \mathrm{gl}(V)$ over an algebraically closed field is *almost algebraic* if $\mathcal{L}$ contains the semisimple and nilpotent parts of all of its elements. To show that Gender ($\mathfrak{a}$) is almost algebraic, we need the following construction. Given an algebra $\mathfrak{a}$ we define an algebra structure on $\mathcal{T}(\mathfrak{a}) := \mathfrak{a} \oplus \mathfrak{a}$ by defining

$$(x,y)(z,w) := (yw,xz) \tag{2.1}$$

for $(x,y),(z,w) \in \mathcal{T}(\mathfrak{a})$. If $(D_1,D_2) \in$ Gender $\mathfrak{a}$, then $(D_1,D_2) \in \mathrm{Der}\,\mathcal{T}(\mathfrak{a})$ where for $(x,y) \in \mathcal{T}(\mathfrak{a})$

$$(D_1,D_2)(x,y) := (D_1(x),D_2(y)). \tag{2.2}$$

Define $\sigma \in GL(\mathcal{T}(\mathfrak{a}))$ by $\sigma(x,y) = (x,-y)$. The next lemma is straightforward.

Lemma 2.3: If $D \in \mathrm{Der}\,\mathcal{T}(\mathfrak{a})$ and $\sigma D = D\sigma$, then $D = (D_1,D_2)$ for some $(D_1,D_2) \in$ Gender $\mathfrak{a}$.

Theorem 2.4: Suppose k is algebraically closed and $\mathfrak{a}$ is finite dimensional. (i) Gender $\mathfrak{a}$ is almost algebraic.
(ii) If D is an antiderivation of $\mathfrak{a}$, then the semisimple and nilpotent parts of D are also antiderivations.

Proof: (i) $\mathrm{Der}\,\mathcal{T}(\mathfrak{a})$ is almost algebraic by Lemma 4.2b of [6] so if E is the semisimple part of (D_1,D_2) as an operator on $\mathcal{T}(\mathfrak{a}), E \in \mathrm{Der}\,\mathcal{T}(\mathfrak{a})$. Since (D_1,D_2) commutes with σ, E commutes with σ by Proposition 4.2b of [6] so $E \in$ Gender $\mathfrak{a}$ by Lemma 2.3. The nilpotent part of (D_1,D_2) is $(D_1,D_2) - E$, so it is also in Gender $\mathfrak{a}$. Hence Gender $\mathfrak{a}$ is almost algebraic.

(ii) Let $E = (E_1,E_2)$ be the semisimple part of the antiderivation $D = (D_1,D_2) \in$ Gender $\mathfrak{a}$ so $D_2 = -D_1$. For $A \in \mathrm{End}\,(V)$ and $\lambda \in k$ define $V_\lambda(A) := \{v \in V \mid (A - \lambda I)^n v = 0$ for some $n \in \mathbb{Z}^+\}$. Then if B is the semisimple part of A and $v \in V_\lambda(A)$, $B(v) = \lambda v$ and this property defines B. Thus, since $\mathcal{T}(\mathfrak{a})_\lambda(D) = \mathfrak{a}_\lambda(D_1) \oplus \mathfrak{a}_\lambda(D_2) = \mathfrak{a}_\lambda(D_1) \oplus \mathfrak{a}_{-\lambda}(D_1)$ we see $E_2(x) = -\lambda x = -E_1(x)$ for $x \in \mathfrak{a}_\lambda(D_1)$. Hence $E_2 = -E_1$ so E is an antiderivation of $\mathfrak{a}$. Then the nilpotent part $D - E$ of D is also an antiderivation.

Hence if there are any nonnilpotent antiderivations of the commutative algebra $\mathfrak{a}$ over $\mathbb{R}$, it is possible to get the kind of nice decomposition of the quadratic differential equation $\frac{dX}{dt} = X^2$ found in Example 1.1 (after passing to $\mathbb{C}$).

In passing it is interesting to note that it is possible for Gender $\mathfrak{a} \neq \operatorname{Der} \mathcal{T}(\mathfrak{a})$ as the next example shows.

Example 2.5: Suppose char $k \neq 3$. Let $\mathfrak{a} = k^2$ with algebra product defined by $\begin{bmatrix} x \\ y \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} := \begin{bmatrix} xa \\ 0 \end{bmatrix}$. Then $\mathcal{T}(\mathfrak{a}) = k^4$ with algebra product defined by $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} := \begin{bmatrix} zc \\ 0 \\ xa \\ 0 \end{bmatrix}$. Then Gender $\mathfrak{a} = \left\{ \left(\begin{bmatrix} 0 & 0 \\ 0 & a \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & b \end{bmatrix} \right) \middle| a,b \in k \right\}$ and

$$\operatorname{Der} \mathcal{T}(\mathfrak{a}) = \left\{ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & a & 0 & b \\ 0 & 0 & 0 & 0 \\ 0 & c & 0 & d \end{bmatrix} \middle| a,b,c,d \in k \right\}$$ so $\dim \text{Gender}\, \mathfrak{a} = 2, \dim \operatorname{Der} \mathcal{T}(\mathfrak{a}) = 4$, and Gender $\mathfrak{a} \neq \operatorname{Der} \mathcal{T}(\mathfrak{a})$.

The following proposition is an easy exercise.

Proposition 2.6: If $\mathfrak{a} = \mathfrak{a}^2$ or $C(\mathfrak{a}) = 0$, then $\operatorname{Der} \mathcal{T}(\mathfrak{a}) =$ Gender $\mathfrak{a}$.

§3 Algebras with Identity

Proposition 3.1 [10]: Suppose char $k \neq 3$ and $\mathfrak{a}$ has an identity. Then $\mathfrak{a}$ has no antiderivations so Gender $\mathfrak{a} = \operatorname{Der} \mathfrak{a}$.

Proof: Suppose D is an antiderivation of $\mathfrak{a}$. Then $D(1) = D(1^2) = -2 \cdot 1 \cdot D(1) = -2D(1)$ so $D(1) = 0$ since char $k \neq 3$. Now if $x \in \mathfrak{a}$, we get $D(x) = D(1 \cdot x) = -D(1) \cdot x - 1 \cdot D(x) = -D(x)$ so $D(x) = 0$ since char $k \neq 2$. Hence $D \equiv 0$.

For any algebra $\mathfrak{a}$ we define operators $L_x, R_y \in \operatorname{End} \mathfrak{a}$ for $x,y \in \mathfrak{a}$ by $L_x(y) := xy$ and $R_y(x) := xy$.

Proposition 3.2: Suppose char $k = 3$ and $\mathfrak{a}$ has an identity. Then
Antider $\mathfrak{a} = \{L_x |$ for all $y,z \in \mathfrak{a}$ $x(yz) = -(xy)z - y(xz)\}$.

Proof: Suppose $D \in$ Antider $\mathfrak{a}$ and $D(1) = x$. Then for all $y \in \mathfrak{a}$ $D(y) = -D(1)y - 1 \cdot D(y) = -xy - D(y)$ so $2D(y) = -xy = 2xy$ so $D(y) = xy$, i.e. $D = L_x$. Then $L_x(yz) = -(L_x y)z - y(L_x z)$. Conversely if $x \in \mathfrak{a}$ with $L_x(yz) = -(L_x y)z - y(L_x z)$ then $L_x \in$ Antider $\mathfrak{a}$.

Corollary 3.3: Suppose char $k = 3$, $\mathfrak{a}$ has an identity, and L_x is an antiderivation of $\mathfrak{a}$. Then $x \in Z(\mathfrak{a})$. Hence if $Z(\mathfrak{a}) = \{1\}$, then $\mathfrak{a}$ has no antiderivations other than id and Gender $\mathfrak{a} \cong$ Der $\mathfrak{a} \oplus k$ id.

Proof: Since L_x is an antiderivation of $\mathfrak{a}$, for all $y \in \mathfrak{a}$, $xy = L_x(y) = L_x(y \cdot 1) = -L_x(y) \cdot 1 - y \cdot L_x(1) = -xy - yx$ so $2xy = -yx$, i.e. $xy = yx$ since char $k = 3$. Thus $x \in Z(\mathfrak{a})$. The rest of the corollary follows from Proposition 3.2.

Recall from [9] that $\mathfrak{a}$ is *alternative* if $(x,x,y) = 0 = (y,x,x)$ for all $x, y \in \mathfrak{a}$ where $(x,y,z) := x(yz) - (xy)z$.

Corollary 3.4: Suppose char $k = 3$ and $\mathfrak{a}$ is a commutative algebra. Then L_x is an antiderivation of $\mathfrak{a}$ for all $x \in \mathfrak{a}$ iff $\mathfrak{a}$ is alternative.

Proof: If $\mathfrak{a}$ is alternative, then $x(yz) - (xy)z = -y(xz) + (yx)z$, i.e.

$$\begin{aligned} x(yx) &= -y(xz) + 2(xy)z && \text{since } \mathfrak{a} \text{ is commutative} \\ &= -(xy)z - y(xz) && \text{since char } k = 3 \end{aligned}$$

so $L_x \in$ Antider $\mathfrak{a}$ for all $x \in \mathfrak{a}$. Conversely, if $L_x \in$ Antider $\mathfrak{a}$ for all $x \in \mathfrak{a}$, then $x(xz) = -x^2z - x(xz)$ implies $x(xz) = x^2z$ so $\mathfrak{a}$ is left alternative and hence alternative since it is commutative.

We see from Corollary 3.4 that Corollary 5 of [10] is false.

§4 Commutative Associative Algebras

We determined Gender $\mathfrak{a}$ for $\mathfrak{a}$ commutative associative with identity in characteristic 3 in Corollary 3.4. Example 2.5 shows there are commutative associative algebras in other characteristics which have antiderivations. The next theorem limits this behavior.

Theorem 4.1: Suppose k is infinite, char $k \neq 3$, and $\mathfrak{a}$ is a commutative associative algebra. If $C(\mathfrak{a}) = 0$ and $\mathfrak{a} = \mathfrak{a}^2$, then $\mathfrak{a}$ has no antiderivations.

Proof: Suppose D is an antiderivation of $\mathfrak{a}$. Then $D[x(yz)] = -Dx(yz) + x(Dy \cdot z) + x(y \cdot Dz) = D[(xy)z] = (Dx \cdot y)z + (x \cdot Dy)z - (xy) \cdot Dz$ so

$$x(y \cdot Dz) = Dx \cdot (yz) \tag{4.1.1}$$

for all $x, y, z \in \mathfrak{a}$. Now for all $x \in \mathfrak{a}$

$$\begin{aligned} D(x^4) &= D(x^2x^2) = -2x^2D(x^2) = -2x^2(-2x \cdot Dx) = 4x^3 \cdot Dx \\ &= D(x^4) = D(x \cdot x^3) = -Dx \cdot x^3 - x \cdot D(xx^2) \\ &= -x^3Dx - x[-Dx \cdot x^2 - xD(x^2)] \\ &= -x^3 \cdot Dx + x^3 \cdot Dx + x^2D(x^2) = x^2(-2x \cdot Dx) = -2x^3 \cdot Dx \end{aligned}$$

Hence $6x^3 \cdot Dx = 0$ so since $\operatorname{char} k \neq 2, 3$,

$$x^3 \cdot Dx = 0 \tag{4.1.2}$$

for all $x \in \mathfrak{a}$. Thus $(x+z)^3 \cdot (Dx + Dz) = 0$ implies

$$x^2z \cdot Dz = -xz^2 \cdot Dx \tag{4.1.3}$$

for all $x, z \in \mathfrak{a}$. Hence

$$\begin{aligned} -x^2 \cdot D(z^2) &= -x(x \cdot D(z^2)) = -Dx \cdot (xz^2) \quad \text{by (4.1.1)} \\ &= -xz^2 \cdot Dx = (x^2z) \cdot Dz \quad \text{by (4.1.3)} \\ &= -\frac{1}{2}x^2(-2z \cdot Dz) = -\frac{1}{2}x^2 \cdot D(z^2) \end{aligned}$$

so $x^2D(z^2) = 0$ for all $x, z \in \mathfrak{a}$. Thus $(x+y)^2D(z^2) = 0$ implies $xy \cdot D(z^2) = 0$ for all $x, y, z \in \mathfrak{a}$. Since $\mathfrak{a} = \mathfrak{a}^2$, this gives $D(z^2) \in C(\mathfrak{a})$ so $D(z^2) = 0$. Thus $D((z+w)^2) = 0$ implies $D(zw) = 0$ so $D \equiv 0$ since $\mathfrak{a} = \mathfrak{a}^2$.

Theorem 4.2: Suppose k is an algebraically closed field of characteristic zero and $\mathfrak{a}$ is finite dimensional, commutive, and associative with $C(\mathfrak{a}) = 0$. Then every antiderivation is a nilpotent endomorphism of $\mathfrak{a}$ and the subalgebra of Gender $\mathfrak{a}$ generated by antiderivations is nilpotent.

Proof: By Theorem 2.4(i) Gender $\mathfrak{a}$ is almost algebraic so if any antiderivation is not nilpotent, Gender $\mathfrak{a}$ has a semisimple antiderivation. Suppose BWOC that D is a semisimple antiderivation of $\mathfrak{a}$ and for $\lambda \in k$ define $\mathfrak{a}_\lambda = \{x \in \mathfrak{a} | Dx = \lambda x\}$ so $\mathfrak{a} = \sum_{\lambda \in k} \mathfrak{a}_\lambda$. Since D is an antiderivation, $\mathfrak{a}_\lambda \mathfrak{a}_\mu \subseteq \mathfrak{a}_{-(\lambda+\mu)}$. As in the proof of Theorem 4.1, $x(y \cdot Dz) = Dx(yz) = x(Dy \cdot z)$ since $\mathfrak{a}$ is commutative. Hence if $y \in \mathfrak{a}_\lambda$ and $z \in \mathfrak{a}_\mu$ for $\lambda \neq \mu$, we get $x(yz) = 0$ for all $x \in \mathfrak{a}$, i.e. $yz \in C(\mathfrak{a})$. Thus for $\lambda \neq \mu$ $\mathfrak{a}_\lambda \mathfrak{a}_\mu = 0$ since $C(\mathfrak{a}) = 0$. Hence if there is a $\lambda \in k$ such that $\mathfrak{a}_\lambda \neq 0$ and $\mathfrak{a}_{-2\lambda} = 0$, then $\mathfrak{a}_\lambda \subseteq C(\mathfrak{a})$. Hence $\mathfrak{a}_\lambda \neq 0$ implies $\mathfrak{a}_{-2\lambda} \neq 0$, but this means $\mathfrak{a}_\lambda = 0$ for $\lambda \neq 0$ since $\{\lambda \in k | \mathfrak{a}_\lambda \neq 0\}$ must be finite. Thus $D = 0$ so $\mathfrak{a}$ has no semisimple antiderivations, i.e. all antiderivations are nilpotent. The other conclusion follows from Corollary 5.8 and Lemma 5.4 of [4].

Note that Example 2.5 shows that a commutative associative algebra $\mathfrak{a}$ may have semisimple antiderivations if $C(\mathfrak{a}) \neq 0$.

§5 Lie Algebras and Lie-Admissible Algebras

Theorem 5.1: Suppose $\operatorname{char} k \neq 3, 5$ or 7 and $\mathcal{L}$ is a central simple finite dimensional Lie algebra with $\dim \mathcal{L} \geq 4$ and B is a nonsingular trace form of $\mathcal{L}$ on some representation. Then $\mathcal{L}$ has no antiderivations so $\operatorname{Gender} \mathcal{L} \cong \mathcal{L}$.

Proof: Thoerem 3 of [5] gives that $\mathcal{L}$ has no antiderivations which satisfy $B(Dx, y) = B(x, Dy)$ for all $x, y \in \mathcal{L}$. The rest of the proof here will be to show that the step in the proof of Theorem 3 of [5] using this hypothesis can in fact be done without resorting to it.

Hence suppose k is algebraically closed and $\mathcal{L} = H \oplus \sum_{\alpha \in \Phi} \mathcal{L}_\alpha$ is a Cartan decomposition of $\mathcal{L}$. Suppose further that D is an antiderivation of $\mathcal{L}$ such that

$$\alpha(h) D(e_\alpha) = -\frac{1}{2} \alpha(Dh) e_\alpha \tag{5.1.1}$$

for all $\alpha \in \Phi,\ h \in H,\ e_\alpha \in \mathcal{L}_\alpha$ (this is (3.3) of [5]). Thus since $\alpha \not\equiv 0,\ D(e_\alpha) = a_\alpha e_\alpha$ for some $a_\alpha \in k$. Now for some $e_\alpha \in \mathcal{L}_\alpha,\ e_{-\alpha} \in \mathcal{L}_{-\alpha},\ h_\alpha = [e_\alpha, e_{-\alpha}]$ so $Dh_\alpha = -[De_\alpha, e_{-\alpha}] - [e_\alpha, De_{-\alpha}] = -(a_\alpha + a_{-\alpha}) h_\alpha$ so by 5.1.1

$$\begin{aligned} \alpha(h_\alpha) a_\alpha e_\alpha = \alpha(h_\alpha) D(e_\alpha) &= -\frac{1}{2} \alpha(Dh_\alpha) e_\alpha \\ &= \frac{1}{2} (a_\alpha + a_{-\alpha}) \alpha(h_\alpha) e_\alpha. \end{aligned}$$

Since $\alpha(h_\alpha) \neq 0,\ a_\alpha = \frac{1}{2}(a_\alpha + a_{-\alpha})$ so $a_\alpha = a_{-\alpha}$. Thus $Dh_\alpha = -2a_\alpha h_\alpha$ for all $\alpha \in \Phi$. Now for $\alpha, \beta \in \Phi$, using (5.1.1) again gives

$$a_\alpha \alpha(h_\beta) e_\alpha = \alpha(h_\beta) D(e_\alpha) = -\frac{1}{2} \alpha(Dh_\beta) e_\alpha = a_\beta \alpha(h_\beta) e_\alpha.$$

Thus $a_\alpha = a_\beta$ if $\alpha(h_\beta) = \langle \alpha, \beta \rangle \neq 0$. Since the Dynkin diagram of $\mathcal{L}$ is connected, $a_\alpha = a_\beta = a$ for all $\alpha, \beta \in \Phi$.

Corollary 5.2: Suppose $\operatorname{char} k = 0$, $\mathfrak{a}$ is a finite dimensional algebra with $\dim \mathfrak{a} \geq 4$ and $\mathfrak{a}^-$ is a semisimple Lie algebra. Then $\mathfrak{a}$ has no antiderivations.

Proof: This follows from Theorem 5.2 and Proposition 1.7(vi) and (ii).

An algebra with the property that $\mathfrak{a}^-$ is a Lie algebra is a *Lie admissible algebra*. Obviously associative algebras are Lie admissible.

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