On classification of superconformal algebras

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Introduction

It has been known since early 70's that there is more than one super extension of the Virasoro algebra. The simplest are the Neveu-Schwarz and Ramond superalgebras, alternatively known as $N = 1$ superconformal algebras [14], [15]. It was realized in the mid seventies that these Lie superalgebras are the first members of an infinite series, the SO_N-superconformal algebras [1,2]. Furthermore, it was shown that the SO_4-superconformal algebra contains yet another example the SU_2-superconformal algebra. At around the same time four series of simple infinite-dimensional Lie superalgebras $W(M,N), S(M,N), H(2M,N), K(2M-1,N)$, were constructed by one of the authors of the present paper [6]. For $N = 0$, these become the classical Lie-Cartan series of simple Lie algebras of vector fields on the complex torus, the simplest example being $\overline{\text{Vir}} = W(1,0)$, the Lie algebra of regular vector fields on $\mathbb{C}^\times$, called the centerless Virasoro algebra. The elements $L_k = -t^k \frac{d}{dt}$ for $k \in \mathbb{Z}$ (t is the complex coordinate) form a basis of $\overline{\text{Vir}}$, which obeys the familiar commutation relations:

$$[L_m, L_n] = (m-n)L_{m+n}.$$

It was pointed out in [11] and [5] that $K(1,N)$ is nothing else but the SO_N-superconformal algebra, and that $W(1,1) \simeq K(1,2)$. However, as far as

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we know, it has not been noticed that $S(1,2)$ is nothing else but the SU_2-superconformal algebra.

These observations make it plausible that the series $W(1,N), S(1,N)$ and $K(1,N)$ and their "variations" (described below) actually exhaust the list of all superconformal algebras. (This conjecture is stated in §4.)

The precise definition is as follows. A Lie superalgebra $\mathfrak{g}$ is called a *superconformal algebra* if

SC1 $\quad\mathfrak{g}$ is simple;

SC2 $\quad\mathfrak{g}$ contains $\overline{\text{Vir}}$ as a subalgebra;

SC3 $\quad\mathfrak{g}$ has growth 1.

The last condition means this. Given a finite set of elements $x_1, x_2, \ldots$, of $\mathfrak{g}$, let $V_j(x_1, x_2, \ldots)$ denote the linear span of commutators in the x_i of length $\leq j$. We require that $\dim V_j < Cj$, where C is a constant independent of j (but depends, of course, of $x_1, x_2, \ldots$).

The most interesting superconformal algebras are the $\mathbb{Z}$*-graded* ones. These are the ones for which adL_0 is diagonalizable with finite-dimensional eigenspaces:

$$\mathfrak{g} = \bigoplus_j \mathfrak{g}_j, \text{ where } \mathfrak{g}_j = \{x \in \mathfrak{g} \mid [L_0, x] = jx\}.$$

Then the condition SC3 simply means that $\dim \mathfrak{g}_j < Cj$, where C is a constant independent of j.

In the present paper we give an explicit description of the series W, S and K (in the spirit of [6]) and classify their central extensions. It turns out that central extensions exists only for small N, hence only few of our superconformal algebras have non-trivial unitary positive energy representations (since the only such representations of $\overline{\text{Vir}}$ are trivial). However, the question, which of these algebras have non-trivial modular invariant representations (in the sense of [12]) remains an open problem.

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1 Definition of the series W, S and K

Consider the algebra $\Lambda_t(N) := \mathbb{C}[t, t^{-1}] \otimes \Lambda(N)$, where $\Lambda(N)$ is the Grassmann algebra in N variables $\theta_1, \ldots, \theta_N$. This algebra becomes $\mathbb{Z}_2$-graded if we set $\deg t = \bar{0}$, $\deg \theta_i = \bar{1}, i = 1, \ldots, N$. We define [6]

(1.1) $W(N) = W(1,N) = der\Lambda_t(N)$,

the superalgebra of all derivations of $\Lambda_t(N)$. Every element $D \in W(N)$ can be expressed as a linear differential operator

$$(1.2)\ D = P_0\frac{d}{dt} + \sum_{i=1}^{N} P_i\frac{d}{d\theta_i}, \quad P_j \in \Lambda_t(N).$$

The *divergence* of D is defined by

$$(1.3)\ divD = \frac{dP_0}{dt} + \sum_{i=1}^{N}(-1)^{degP_i}\frac{dP_i}{d\theta_i}.$$

It is straightforward to show that

(1.4) $div[D_1, D_2] = D_1(divD_2) - (-1)^{d_1d_2}D_2(divD_1)$, where $\deg D_i = d_i$;

(1.5) $divfD = Df + fdivD$, where f is an even function.

Using this we get for an even f:

$$\begin{aligned}(1.6)\ divf[D_1, D_2] &= [D_1, D_2]f + fdiv[D_1, D_2]\\ &= [D_1, D_2]f + fD_1(divD_2) - (-1)^{d_1d_2}fD_2(divD_1)\\ &= [D_1, D_2]f + D_1(fdivD_2) - (-1)^{d_1d_2}D_2(fdivD_1)\\ &\quad -D_1(f)divD_2 + (-1)^{d_1d_2}D_2(f)divD_1\\ &= D_1(divfD_2) - (-1)^{d_1d_2}D_2(divfD_1) - D_1(f)divD_2 + (-1)^{d_1d_2}D_2(f)divD_1\\ &= D_1(divfD_2) - (-1)^{d_1d_2}D_2(divfD_1) - div(fD_1)divD_2 + (-1)^{d_1d_2}div(fD_2)divD_1.\end{aligned}$$

From now on let f be invertible, even and such that $f^{-1}\frac{df}{dt}$, $f^{-1}\frac{df}{d\theta_i} \in \Lambda_t(N)$ (but f may not be an element of $\Lambda_t(N)$). We call such f *admissible*. Then from (1.6) we conclude that

(1.7) $S(N; f) := \{D \in W(N) \mid divfD = 0\}$

is a subalgebra of $W(N)$. For $f = 1$ we have the superalgebra $S(N; 1) = S(1, N)$ of [6].

The following elements span $S(N; f)$:

(1.8) $D_{ij}(P) = f^{-1}\left(\frac{d(fP)}{d\theta_i}\frac{d}{d\theta_j} + \frac{d(fP)}{d\theta_j}\frac{d}{d\theta_i}\right) i, j \neq 0,$

(1.9) $D_i(P) = f^{-1}\left(\frac{d(fP)}{d\theta_i}\frac{d}{dt} - (-1)^{\deg P}\frac{d(fP)}{dt}\frac{d}{d\theta_i}\right) i \neq 0, P \in \Lambda_t(N),$

except for the case $f^{-1} \in \Lambda_t(N)$, when we have one more element

(1.10) $f^{-1}\theta_1\theta_2 ... \theta_N \frac{d}{dt}.$

In the rest of this paper we restrict to the case that $f = t^\alpha$ with $\alpha \in \mathbb{C}$. The reason for this is that though the superalgebra $S(N; f)$ is a superconformal algebra for any admissible f, it is $\mathbb{Z}$-graded only for $f = t^\alpha$. We shall use the notation $S(N; \alpha)$ for $S(N; t^\alpha)$.

Given $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_N) \in \mathbb{Z}^N$, consider the following differential form:

$$\omega_\epsilon = dt - \sum_{i=1}^{N} t^{\epsilon_i}\theta_i d\theta_i.$$

We define

(1.11) $K(N; \epsilon) = \{D \in W(n) \mid D\omega_\epsilon = P\omega_\epsilon \text{ for some } P \in \Lambda_t(N)\}.$

By making a change of variables $\theta_i \mapsto t^{k_i}\theta_i$, we can reduce to the cases $\epsilon_i = 0$ or 1. We assume from now on that

(1.12) $\omega_\epsilon = dt - \sum_{i=1}^{N} t^{\epsilon_i}\theta_i d\theta_i, \quad \epsilon_i = 0 \text{ or } 1.$

If $\epsilon = \vec{0}$, then $K(N; \vec{0}) = K(1, N)$ of [6]. For $N = 1$ we obtain the Neveu-Schwarz algebra if $\epsilon_1 = 0$, and the Ramond algebra if $\epsilon_1 = 1$.

Every differential operator $D \in K(N; \epsilon)$ can be represented by a single function $f \in \Lambda_t(N)$ as follows. Let

(1.13) $\triangle f = 2f - \sum_{i=1}^{N} \theta_i \frac{df}{d\theta_i}.$

The elements of the form

(1.14) $D_f = (\triangle f)D^\epsilon + \sum_{i=1}^{N} D^\epsilon(f)\theta_i \frac{d}{d\theta_i} + (-1)^{\deg f} \sum_{i=1}^{N} t^{-\epsilon_i} \frac{df}{d\theta_i}\frac{d}{d\theta_i},$

where

$$(1.15)\quad D^{\epsilon}=\frac{d}{dt}-\frac{1}{2}t^{-1}\sum_{i=1}^{N}\epsilon_i\theta_i\frac{d}{d\theta_i},$$

span $K(N;\epsilon)$. One easily checks the following relations

$$(1.16)\quad D_f\omega_\epsilon=2D^{\epsilon}(f)\omega_\epsilon,$$

$$(1.17)\quad D_{f+g}=D_f+D_g,$$

$(1.18)\quad [D_f,D_g]=D_{\{f,g\}}$, where

$$\{f,g\}=(\triangle f)D^{\epsilon}(g)-D^{\epsilon}(f)(\triangle g)+(-1)^{\deg f}\sum_{i=1}^{N}t^{-\epsilon_i}\frac{df}{d\theta_i}\frac{dg}{d\theta_i}.$$

The superalgebras of type $K(N;\epsilon)$ are known in the physics literature under the name SO_N-superconformal algebras (cf. [1,16]).

It turns out that for $N\geq 1$ there are only two different $K(N;\epsilon)$ superalgebras. Every time when two ϵ_i's are 1 it is possible to remove them by making a change of basis. In other words the two non-isomorphic superalgebras $K(N,\epsilon)$ correspond to $\epsilon=(0,...,0)$ and $\epsilon=(1,0,...,0)$. In order to show this, we shall give an explicit isomorphism between the superalgebras $K(N;(0,0,\epsilon_3,...,\epsilon_N))$ and $K(N;(1,1,\epsilon_3,...,\epsilon_N))$:

$$\sigma(K(N;(0,0,\epsilon_3,...,\epsilon_N)))=K(N;(1,1,\epsilon_3,...,\epsilon_N)),\quad \sigma(D_f)=D_{f_\sigma},$$

where

$$(1.19)\quad (t^n\theta_{j_1},...,\theta_{j_s})_\sigma=t^n\theta_{j_1},...,\theta_{j_s}(1+\frac{1}{2}(2-s)i\theta_1\theta_2),$$

$$(t^n\theta_{j_1},...,\theta_{j_s}\theta_1\theta_2)_\sigma=t^{n+1}\theta_{j_1},...,\theta_{j_s}\theta_1\theta_2,$$

$$t^n\theta_{j_1},...,\theta_{j_s}(\theta_1+i\theta_2))_\sigma=t^n\theta_{j_1},...,\theta_{j_s}(\theta_1+i\theta_2),$$

$$(t^n\theta_{j_1},...,\theta_{j_s}(\theta_1-i\theta_2))_\sigma=t^{n+1}\theta_{j_1},...,\theta_{j_s}(\theta_1-i\theta_2),$$

for $n\in\mathbf{Z}$ and $1,2\notin\{j_1,...,j_s\}$.

Suppose $\epsilon=\epsilon_1=\epsilon_2$; then we have the isomorphism

$\rho: K(2;(\epsilon,\epsilon))\simeq W(1)$, given by

$$(1.20)\quad \rho(D_{t^n}) = 2t^n \frac{d}{dt} + (n-\epsilon)t^{n-1}(\theta_1 \frac{d}{d\theta_1} + \theta_2 \frac{d}{d\theta_2}),$$

$$\rho(D_{t^n\theta_1}) = t^{n-1}\theta_1 \frac{d}{dt} - t^{n-\epsilon}\frac{d}{d\theta_1},$$

$$\rho(D_{t^n\theta_2}) = i(t^{n-1}\theta_2 \frac{d}{dt} + t^{n-\epsilon}\frac{d}{d\theta_2}),$$

$$\rho(D_{t^n\theta_1\theta_2}) = it^{n-\epsilon}\theta_1 \frac{d}{d\theta_1}.$$

The superalgebra $S(2;-1)$ is nothing else but the SU_2-superconformal algebra [1,2]. It was already noticed by A. Swimmer and N. Seiberg [18], that $S(2;-1)$ is a member if a one-parameter family of SU_2-superconformal algebras. This family corresponds exactly to our family of $S(2;\alpha)$- superalgebras. We shall describe below a central extension of our $S(2;\alpha)$-algebras in terms of the SU_2-superconformal algebras of [18]. A complete classification of central extensions of all superconformal algebras will be given in Sections 2-4. Put $\rho = \frac{1}{2}(\alpha+1)$, and define

$$(1.21)\quad L_n = -t^{n+1}\frac{d}{dt} - \frac{1}{2}(n+2\rho)t^n(\theta_1\frac{d}{d\theta_1} + \theta_2\frac{d}{d\theta_2}) + \delta_{0,n}(\frac{1}{24} - \frac{\rho^2}{6})c,$$

$$T_n^1 = \frac{1}{2}t^n(\theta_1\frac{d}{d\theta_2} + \theta_2\frac{d}{d\theta_1}),$$

$$T_n^2 = \frac{i}{2}t^n(\theta_2\frac{d}{d\theta_1} - \theta_1\frac{d}{d\theta_2}),$$

$$T_n^3 = \frac{1}{2}t^n(\theta_1\frac{d}{d\theta_1} - \theta_2\frac{d}{d\theta_2}),$$

$$G_{n+\rho}^1 = \sqrt{2}(-t^{n+1}\theta_1\frac{d}{dt} - (n+2\rho)t^n\theta_1\theta_2\frac{d}{d\theta_2}),$$

$$G_{n+\rho}^2 = \sqrt{2}(-t^{n+1}\theta_2\frac{d}{dt} - (n+2\rho)t^n\theta_2\theta_1\frac{d}{d\theta_1}),$$

$$G_{n-\rho}^{1*} = \sqrt{2}t^n\frac{d}{d\theta_1},$$

$$G_{n-\rho}^{2*} = \sqrt{2}t^n\frac{d}{d\theta_2}.$$

Let $\sigma^j_{ab}, j = 1,2,3$ and $a, b = 1,2$, be the structure constants of the sl_2-representation of the Lie algebra $sl_2(\mathbb{C})$, i.e., $\sigma^1_{12} = \sigma^1_{21} = 1$, $\sigma^2_{12} = -\sigma^2_{21} = i, \sigma^3_{11} = -\sigma^3_{22} = 1$ and the other $\sigma^j_{ab} = 0$. Then we have the following commutation relations between the above elements.

$$(1.22)[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}m(m^2-1)\delta_{m,-n},$$

$$[T^j_m, T^k_n] = \sum_{\ell=1}^{3} i\epsilon^{jk\ell}T^\ell_{n+m} + \frac{c}{12}m\delta_{m,-n}\delta_{j,k},$$

$$[L_m, T^j_n] = -nT^j_{m+n},$$

$$[T^j_m, G^a_{m+\rho}] = \frac{1}{2}\sum_{b=1}^{2}\sigma^j_{ab}G^b_{m+n+\rho},$$

$$[T^j_m, G^{a*}_{n-\rho}] = -\frac{1}{2}\sum_{b=1}^{2}\sigma^j_{ba}G^{b*}_{m+n-\rho},$$

$$[L_m, G^a_{n+\rho}] = (\frac{1}{2}m - (n+\rho))G^a_{m+n+\rho},$$

$$[L_m, G^{a*}_{n-\rho}] = (\frac{1}{2}m - (n-\rho))G^{a*}_{m+n-\rho},$$

$$[G^a_{m+\rho}, G^b_{n+\rho}] = [G^{a*}_{m-\rho}, G^{b*}_{n-\rho}] = 0,$$

$$[G^a_{m+\rho}, G^{b*}_{n-\rho}] = 2\delta_{a,b}L_{m+n} + 2\sum_{j=1}^{3}\sigma^j_{ab}((m+\rho)-(n-\rho))T^j_{m+n}$$

$$+\frac{c}{3}((m+\rho)^2 - \frac{1}{4})\delta_{m,-n}\delta_{a,b}.$$

We conclude this section with the calculation of the centralizer of L_0 in the superconformal algebra $\mathfrak{g}$; we denote this centralizer by $C_{\mathfrak{g}}(L_0)$. We restrict our calculations to those superconformal algebras which allow non-trivial central extensions. In the sections 2, 3 and 4 we show that this is only the case for $W(N)$ with $N \leq 2$, $S(2;\alpha)$ and $K(N;\epsilon)$ with $N \leq 4$. Since $W(1) \simeq K(N;(0,0)) \simeq K(N;(1,1))$ we shall not consider $W(1)$.

The superalgebras $K(4;(\epsilon_1,\epsilon_2,\epsilon_3,\epsilon_4))$ and $S(2;\alpha)$ are not always simple, this happens when $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4 \in 2\mathbb{Z}$ and $\alpha \in \mathbb{Z}$, respectively. In order to get a simple superalgebra we must consider the *derived superalgebra* $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ instead.

We start with the superconformal algebra $W(2)$. Since it is not clear to us what the best choice is for L_0, we choose it to be arbitrary:

$$L_0 = -(t\frac{d}{dt} + \beta_1\theta_1\frac{d}{d\theta_1} + \beta_2\theta_2\frac{d}{d\theta_2})\ \beta_1, \beta_2 \in \mathbb{C}.$$

We distinguish 3 cases. First assume that $\beta_1, \beta_2 \in \mathbb{Z}$. Then $C_{W(2)}(L_0)$ has the following elements for a basis

$$\{L_0, G^i, H, \theta_i\frac{d}{d\theta_i}, t^{\beta_i}\frac{d}{d\theta_i}, t^{-\beta_i}\theta_i\theta_j\frac{d}{d\theta_j}, t^{\beta_j-\beta_i}\theta_i\frac{d}{d\theta_i}\}\ \ i, j = 1, 2,\ \text{ where}$$

$$G^i = -t^{-\beta_i}\theta_i(t\frac{d}{dt} + \beta_1\theta_1\frac{d}{d\theta_1} + \beta_2\theta_2\frac{d}{d\theta_2})\ \text{ and}$$

$$H = t^{1-\beta_1-\beta_2}\theta_1\theta_2\frac{d}{dt}.$$

One easily verifies that in this case

$$C_{W(2)}(L_0) \simeq der\Lambda(2) \ltimes \Lambda(2).$$

Next assume that $\beta_1 \in \mathbb{Z}$ and $\beta_2 \notin \mathbb{Z}$. Now $C_{W(2)}(L_0)$ has the following elements for a basis:

$$\{L_0, G^1, \theta_i\frac{d}{d\theta_1}, t^{\beta_1}\frac{d}{d\theta_1}, t^{-\beta_1}\theta_1\theta_2\frac{d}{d\theta_2}\}\ \ i = 1, 2.$$

In this case is

$$C_{W(2)}(L_0) \simeq gl_{1|1}(\mathbb{C}) \ltimes \Lambda(1),\ \text{ where}$$

$$gl_{1|1}(\mathbb{C}) =< \theta_i\frac{d}{d\theta_i}, \frac{d}{d\theta_1}, \theta_1\theta_2\frac{d}{d\theta_2};\ \ i = 1, 2 > .$$

Finally, let $\beta_1, \beta_2 \notin \mathbb{Z}$, then

$$C_{W(2)}(L_0) =< L_0, \theta_1\frac{d}{d\theta_1}, \theta_2\frac{d}{d\theta_2} > .$$

This is a 3-dimensional abelian Lie algebra.

For the superalgebra $S'(2;\alpha)$, we shall use the notations of the elements of the SU_2-superconformal algebra (see (1.21)), where $\rho = \frac{1}{2}(\alpha + 1)$. Since it is not difficult to show, using the change of variables $t \mapsto t, \theta_1 \mapsto t\theta_1$ and $\theta_2 \mapsto \theta_2$, that $S(2;\alpha) \simeq S(2;\alpha + 1)$, we moreover assume that $\rho \in \mathbb{Z}$ whenever $\alpha \in \mathbb{Z}$. We now distinguish two cases, viz $\rho \in \mathbb{Z}$ and $\rho \notin \mathbb{Z}$. If $\rho \in \mathbb{Z}$, then

$$C_{S'(2;\alpha)}(L_0) =< L_0, G_0^i, G_0^{i*}, T_0^j;\ i = 1,2 \text{ and } j = 1,2,3 > .$$

In this case

$$C_{S'(2;\alpha)}(L_0) \simeq S(2) \ltimes < 1, \theta_1, \theta_2 >, \text{ where}$$

$S(2)$ is the subalgebra of $der\Lambda(2)$, consisting of the divergent-free derivations (see [6]). If $\rho \notin \mathbb{Z}$, then

$$C_{S'(2;\alpha)}(L_0) =< T_0^i;\ i = 1,2,3 >\simeq sl_2(\mathbb{C}).$$

The superalgebra $K'(N;\epsilon)$ contains the following "natural" subalgebra isomorphic to $\overline{\text{Vir}}$: $< L_n = -\frac{1}{2}D_{t^{n+1}};\ n \in \mathbb{Z} >$. For the calculations of the centralizer of L_0 in $K'(N;\epsilon)$ we again consider the general case, i.e., $\epsilon = (\epsilon_1, ..., \epsilon_N)$, where ϵ_i arbitrary 0 or 1. Without loss of generality we may assume that $\epsilon_i \le \epsilon_{i+1}$. One easily obtains that $C_{K'(N;\epsilon)}(L_0)$ has the following elements as a basis: $(N \le 4)$

$$D_t, D_{t\theta_i} \text{ for } \epsilon_i = 1, D_{t\theta_i\theta_j} \text{ for } \epsilon_i = \epsilon_j = 1, D_{\theta_i\theta_j} \text{ for } \epsilon_i = \epsilon_j = 0,$$

$$D_{t\theta_i\theta_j\theta_k} \text{ for } \epsilon_i = \epsilon_j = \epsilon_k = 1 \text{ and} D_{\theta_i\theta_j\theta_k} \text{ for } \epsilon_i = \epsilon_j = 0, \epsilon_k = 1.$$

It will be more convenient to describe $C_{K'(N;\epsilon)}(L_0)/L_0$ rather than $C_{K'(N;\epsilon)}(L_0)$. We use the notation

$$C(N;\epsilon_1,...,\epsilon_N) := C_{K'(N;(\epsilon_1,...,\epsilon_N))}(L_0)/L_0$$

Let ω be the Hamiltonian form (see [6])

$$\omega = \sum_{i=1}^{N}(d\theta_i)^2,$$

and define

$$\tilde{H}(n) = \{D \in der\Lambda(n) \mid D\omega = 0\} \text{ and} H(n) = [\tilde{H}(n), \tilde{H}(n)].$$

Then $\quad C(N;1,...,1) \simeq C(N+1;1,..,1,0) = \tilde{H}(N)$ for $N \le 3$,

$$C(4;1,1,1,1) \simeq H(4),$$

$$C(4;1,1,0,0) \simeq \tilde{H}(2) \ltimes < 1, \theta_1, \theta_2 >,$$

$$C(4;0,0,0,0) \simeq sl_2(\mathbb{C}) \oplus sl_2(\mathbb{C}),$$

$C(3;1,0,0) \simeq sl_{1|1}(\mathbb{C})$,

$C(3;0,0,0) \simeq sl_2(\mathbb{C})$ and

$C(2;0,0)$ is a 1-dimensional Lie algebra.

2 Central extensions of $W(N)$

We start this section by giving some general information about central extensions of Lie superalgebras.

Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ be a Lie superalgebra. We extend $\mathfrak{g}$ by a 1-dimensional center by introducing a new bracket $[\cdot,\cdot]_c$ on the vector space $\mathfrak{g} = \hat{\mathfrak{g}} \bigoplus \mathbb{C}c$ as follows

(2.1) $[c,x]_c = 0,$

(2.2) $[x,y]_c = [x,y] + \psi(x,y)c,$

where $x, y \in \mathfrak{g}$ and $\psi(x,y) \in \mathbb{C}$. Since $\hat{\mathfrak{g}}$ is supposed to be a Lie superalgebra, the new bracket must satisfy the (super) anticommutativity and the (super) Jacobi identity, which imposes the following conditions on ψ :

(2.3) $\psi(y,x) = -(-1)^{\deg x \deg y}\psi(x,y)$

(2.4) $\psi(x,[y,z]) = \psi([x,y],z) + (-1)^{\deg x \deg y}\psi(y,[x,z]).$

The $\mathbb{C}$-valued bilinear function $\psi = \psi(\cdot,\cdot)$ is called a 2-cocycle.

In the rest of this paper we shall drop the subscript c in the bracket.

Let $\{x_i\}_{i\in I} \cup c$ be a basis of $\hat{\mathfrak{g}}$, with the following commutation relations:

$$[x_i, x_j] = \sum_{k\in I} f_{ijk}x_k + \psi(x_i,x_j)c,$$

where the f_{ijk} are the structure constants of $\mathfrak{g}$. Suppose $\psi(x_i,x_j) = \sum_{k\in I} f_{ijk}\, f(x_k)$ for all $i,j \in I$ and some linear function $f \in \mathfrak{g}^*$. Then by choosing a new basis $\{x_i'\}_{i\in I} \cup c$ where $x_i' = x_i + f(x_i)c$ we can remove ψ. Such a cocycle is called *trivial*. Two cocycles ψ and ψ' are called *equivalent* if their exists a $\lambda \in \mathbb{C}^\times$ such that $\psi + \lambda\psi'$ is a trivial cocycle.

In this paper we shall classify up to equivalence all non-trivial 2-cocycles of $W(N)$, $S(N;\alpha)$ and $K(N;\epsilon)$. It will be convenient to use the following lemma.

LEMMA 2.1
Let $\mathfrak{g}^0 \subset \mathfrak{g}$ be a reductive finite-dimensional subalgebra of $\mathfrak{g}$, so that we have $\mathfrak{g} = \oplus_{i \in I} \mathfrak{g}_{(i)}$, a decomposition of $\mathfrak{g}$ into a direct sum of finite-dimensional irreducible $\mathfrak{g}^0$-modules. Then every C-valued 2- cocycle ψ on $\mathfrak{g}$ is equivalent to a cocycle ψ_0 such that

(i) $\psi_0(\mathfrak{g}_{(i)}, \mathfrak{g}_{(j)}) = 0$ if $\mathfrak{g}_{(i)}$ and $\mathfrak{g}_{(j)}$ are not contragredient $\mathfrak{g}^0$-modules,

(ii) $\psi_0(x, y) = c_{ij} < x, y >$ for all $x \in \mathfrak{g}_{(i)}, y \in \mathfrak{g}_{(j)}$, and some $c_{ij} \in \mathbb{C}$ if $\mathfrak{g}_{(i)}$ and $\mathfrak{g}_{(j)}$, are contragredient $\mathfrak{g}^0$-modules, where $<,>$ denotes the pairing between them.

Proof.
Denote by $C^2(\mathfrak{g})$ the space of 2- cocycles and by $B^2(\mathfrak{g})$ the subspace of trivial cocycles, and let $H^2(\mathfrak{g}) = C^2(\mathfrak{g})/B^2(\mathfrak{g})$. Now $C^2(\mathfrak{g})$ is a $\mathfrak{g}$-invariant subspace of $\mathfrak{g}^* \times \mathfrak{g}^*, B^2(\mathfrak{g})$ is an invariant subspace and the action of $\mathfrak{g}$ on $H^2(\mathfrak{g})$ is trivial (see e.g. [4]). Since $\mathfrak{g}^0$ is reductive, we have a $\mathfrak{g}^0$- invariant complementary subspace S to $B^2(\mathfrak{g})$ in $C^2(\mathfrak{g})$. Any cocycle ψ_0 from S is killed by $\mathfrak{g}^0$, which means that ψ_0 has the property described in the statement of the lemma. □

COROLLARY 2.1
If $\mathfrak{g}^0$ is an ad-diagonalizable subalgebra of $\mathfrak{g}$ with the weight space decomposition $\mathfrak{g} = \oplus_\lambda \mathfrak{g}_{(\lambda)}$, then any 2-cocycle ψ on $\mathfrak{g}$ is equivalent to a cocycle ψ_0 such that

(2.5) $\psi_0(g_{(\lambda)}, g_{(\mu)}) = 0$ if $\lambda + \mu \neq 0$. □

A cocycle ψ_0 satisfying the property (2.5) is called *symmetric*.

Now we are in a position to classify the central extensions of $W(N)$. It is easy to show that $W(N)$ is simple for all $N \geq 0$.

The elements $t^k\theta_{i_1}...\theta_{i_s}\frac{d}{d\theta_i}, t^{k+1}\theta_{i_1}...\theta_{i_s}\frac{d}{dt}$ with $k \in \mathbb{Z}$ and $1 \leq i_1 < i_2 < ... < i_s \leq N$ form a basis of $W(N)$. Clearly $W(N)$ is simple. Using Corollary 2.1 for $\mathfrak{g}^0 =< t\frac{d}{dt}, \theta_i\frac{d}{d\theta_i}; 1 \leq i \leq N >$, we may assume that a cocycle ψ is non- zero only between the following spaces $< t^{k+1}\frac{d}{dt}, t^k\theta_i\frac{d}{d\theta_i}; 1 \leq i \leq N >$ and $< t^{-k-1}\frac{d}{dt}, t^{-k}\theta_i\frac{d}{d\theta_i}; 1 \leq i \leq N >; < t^k\theta_i\frac{d}{d\theta_j} >$ and $< t^{-k}\theta_j\frac{d}{d\theta_i} >$ for $1 \leq i, j \leq N$ and $i \neq j$; $< t^{k+1}\theta_i\frac{d}{dt}, t^k\theta_i\theta_j\frac{d}{d\theta_j}; 1 \leq j \leq N, i \neq j >$ and $< t^{-k}\frac{d}{d\theta_i} >$ for $1 \leq i \leq N$.

We shall first calculate the cocycles on $W(1)$. Denote

$$d_j = t^{j+1}\frac{d}{dt} \quad , \quad e_j = t^{j+1}\theta_1\frac{d}{dt},$$

$$d_j' = t^j\theta_1\frac{d}{d\theta_1} \quad , \quad e_j' = t^j\frac{d}{d\theta_1}.$$

The symmetry property of ψ gives us that $\psi(d_i, d_j)$, $\psi(d_i, d_j')$, $\psi(d_i, e_j)$, etc. are all zero unless $i + j = 0$. Letting $c_j = \psi(d_j, d_{-j})$, we obtain from the Jacobi identity for d_1, d_j and d_{-j-1}:

$$(1-j)c_{j+1} = (2j+1)c_1 - jc_j.$$

Since $c_j = -c_{-j}$, this recurrent formula determines ψ if we know c_1 and c_2. Thus, the space of symmetric 2- cocycles on the centerles Virasoro algebra $\overline{\text{Vir}}$ is at most 2-dimensional. One checks directly that $c_k = k^3$ and $= k$ are cocycles. Thus, the general formula for a symmetric cocycle on $\overline{\text{Vir}}$ is given by

(2.6) $\psi(d_k, d_\ell) = (\alpha k^3 + 2\beta k)\delta_{k,-\ell}.$

Similarly we get (cf. [3]):

(2.7) $\psi(d_k', d_\ell') = \gamma k\delta_{k,-\ell},$

(2.8) $\psi(d_k', d_\ell) = (\delta k^2 + \epsilon k)\delta_{k,-\ell}.$

Next we write down the Jacobi identity for the elements e_j, e_k' and d_ℓ, obtaining

$$k\psi(d_{j+k}', d_\ell) + \psi(d_{j+k}, d_\ell) + k\psi(e_{k+\ell}', e_j) + (j-\ell)\psi(e_k', e_{\ell+j}) = 0.$$

Let $k = 0, \ell = -j$ and substitute (2.6, 8). This gives

$$-2j\psi(e_0', e_0) = \alpha j^3 + 2\beta j,$$

from which we deduce that $\alpha = 0$ and $\psi(e_0', e_0) = -\beta$. On the other hand setting $j = 0$ and $\ell = -k$ gives

$$\delta k^3 + \epsilon k^2 + \beta k = -k\psi(e_k', e_{-k}).$$

Hence

(2.9) $\psi(e_k', e_\ell) = -(\delta k^2 + \epsilon k + \beta)\delta_{k,-\ell}.$

The Jacobi identity for the cocycle ψ for the elements e_j, e'_k and d'_{-j-k} gives $2(k^2+jk)\delta = (k^2+jk)\gamma$, so $\gamma = 2\delta$. Thus we obtain that all symmetric 2-cocycles on $W(1)$ are the form (2.6-9) with $\alpha = 0$ and $\gamma = 2\delta$. Since β and ϵ give trivial cocycles (corresponding to linear functions d_0^* and $d_0'^*$), we conclude from (2.6-9) that up to equivalence there exists only one non-trivial cocycle for $W(1)$, and it is of the form

$$\psi(d'_k, d'_\ell) = 2k\delta_{k,-\ell},$$

$$(2.10)\quad \psi(d'_k, d_\ell) = k^2\delta_{k,-\ell},$$

$$\psi(e'_k, e_\ell) = -k^2\delta_{k,-\ell},$$

zero in all other cases.

Now let $N > 1$. For every i the elements $t^{k+1}\frac{d}{dt}$, $t^{k+1}\theta_i\frac{d}{dt}, t^k\theta_i\frac{d}{d\theta_i}$ and $t^k\frac{d}{d\theta_i}(k \in \mathbb{Z})$ form a $W(1)$ subalgebra. We may assume that on every such component the cocycle is like the one described in (2.6-9), but with different coefficients β_i, ϵ_i and δ_i for different i, i.e., $\psi(t^{k+1}\theta_i\frac{d}{d\theta_i}, t^{\ell+1}\theta_i\frac{d}{d\theta_i}) = 2\delta_i k\delta_{k,-\ell}$. But now the Jacobi identity of the elements $t^m\frac{d}{d\theta_j}, t^n\theta_j\theta_i\frac{d}{d\theta_i}$ and $t^{-m-n}\theta_i\frac{d}{d\theta_i}$ is $\psi(t^{m+n}\theta_i\frac{d}{d\theta_i}, t^{-m-n}\theta_i\frac{d}{d\theta_i}) = 0$, hence all $\delta_i = 0$.

Next restrict to the $s\ell_N(\mathbb{C}[t,t^{-1}])$ subalgebra of $W(N)$. This is the algebra which has as basis the following elements: $t^k\theta_i\frac{d}{d\theta_j}$ $1 \le i,j \le N$ and $t^k(\theta_i\frac{d}{d\theta_i} - \theta_{i+1}\frac{d}{d\theta_{i+1}})$ $1 \le i \le N$ and $k \in \mathbb{Z}$ in both cases. Again there exists only one non-trivial cocycle (up to equivalence) on this subalgebra (this can be easily shown using Lemma 2.1. cf. [8]), and we may assume that it is given by

$$\psi(t^k\theta_i\frac{d}{d\theta_j}, t^\ell\theta_m\frac{d}{d\theta_N}) = \alpha k\delta_{i,n}\delta_{j,m}\delta_{k,-\ell}, \text{ and}$$

$$\psi(t^k(\theta_i\frac{d}{d\theta_i} - \theta_{i+1}\frac{d}{d\theta_{i+1}}), t^\ell(\theta_j\frac{d}{d\theta_j} - \theta_{j+1}\frac{d}{d\theta_j})) = \alpha k a_{ij}\delta_{k,-\ell},$$

with $a_{ij} = 2, -1, 0$ for $|i-j| = 0, = 1, \ge 2$, respectively. Using the fact that all $\delta_i = 0$ we get

$$(2.11)\quad -\psi(t^k\theta_i\frac{d}{d\theta_i}, t^{-k}\theta_j\frac{d}{d\theta_j}) + \psi(t^{-k}\theta_i\frac{d}{d\theta_i}, t^k\theta_j\frac{d}{d\theta_j}) =$$

$$\psi(t^k(\theta_i\frac{d}{d\theta_i}-\theta_j\frac{d}{d\theta_j}),t^{-k}(\theta_i\frac{d}{d\theta_i}-\theta_j\frac{d}{d\theta_j}))=2\alpha k.$$

We obtain

$$(2.12)\quad \psi(t^k\theta_i\frac{d}{d\theta_i},t^{-k}\theta_j\frac{d}{d\theta_j})=k\psi(t\theta_i\frac{d}{d\theta_i},t^{-1}\theta_j\frac{d}{d\theta_j}),$$

if we write down the Jacobi-rule of the elements $t^{2-k}\frac{d}{dt},t^k\theta_i\frac{d}{d\theta_i},t^{-1}\theta_j\frac{d}{d\theta_j}$. Combining (2.11) and (2.12) gives us

$$(2.13)\quad \psi(t^k\theta_i\frac{d}{d\theta_i},t^{-k}\theta_j\frac{d}{d\theta_j})=-k\alpha\quad i\neq j.$$

Combining (2.13) together with the Jacobi-rule of the elements $t^{2-k}\frac{d}{dt},t^k\frac{d}{d\theta_i}$, $t^{-1}\theta_i\theta_j\frac{d}{d\theta_j}$ and the one of the elements $t^k\frac{d}{d\theta_i},\theta_i\theta_j\frac{d}{d\theta_j},t^{-k}\theta_i\frac{d}{d\theta_i}$ we get

$$\psi(t^k\frac{d}{d\theta_i},t^{-k}\theta_i\theta_j\frac{d}{d\theta_j})=k\alpha$$

Finally suppose $N\geq 3$, then for $i\neq j\neq k$ and $i\neq k$ the Jacobi identity for the elements $t^m\frac{\partial}{\partial\theta_k},\theta_k\theta_i\frac{d}{d\theta_i},t^{-m}\theta_j\frac{d}{d\theta_j}$ give

$$\psi(t^m\theta_i\frac{d}{d\theta_i},t^{-m}\theta_j\frac{d}{d\theta_j})=0.$$

Hence from (2.13) we deduce that $\alpha=0$. We conclude that $W(N)$ has no non-trivial cocycle if $N\geq 3$. Up to equivalence the non-trivial cocycle for $W(1)$ is given by (2.10), and the one for $W(2)$ is as follows:

$$(2.14)\quad \psi(t^k\theta_1\frac{d}{d\theta_2},t^\ell\theta_2\frac{d}{d\theta_1})=-k\delta_{k,-\ell},$$

$$\psi(t^k\theta_1\frac{d}{d\theta_1},t^\ell\theta_2\frac{d}{d\theta_2})=k\delta_{k,-\ell},$$

$$\psi(t^k\theta_1\theta_2\frac{d}{d\theta_2},t^\ell\frac{d}{d\theta_1})=k\delta_{k,-\ell},$$

$$\psi(t^k\theta_2\theta_1\frac{d}{d\theta_1},t^\ell\frac{d}{d\theta_2})=k\delta_{n,-\ell},$$

zero in all other cases.

3 Central extensions of $S(N;\alpha)$

In the classification of central extensions of $S(N;\alpha)$ we shall assume that $N>1$ (if $N=1$, its odd part is an ideal). By making a change of variables one easily verifies that

$$S(N;\alpha)\simeq S(N;\alpha+m) \text{ for } m\in\mathbb{Z}.$$

If moreover $\alpha\in\mathbb{Z}$, the algebra is not simple. However, the *derived superalgebra* $S'(N;\alpha):=[S(N;\alpha),S(N;\alpha)]$ is simple. We have for $\alpha\in\mathbb{Z}$:

$$S(N;\alpha)=S'(N;\alpha)+\mathbb{C}t^{-\alpha}\theta_1...\theta_N\frac{d}{dt}.$$

We shall classify the central extensions of $S'(N;\alpha)$.

Let $\mathfrak{g}^0\subset S'(N;\alpha)$ be the superalgebra spanned by the elements $\theta_i\frac{d}{d\theta_j}(i\neq j)$, $\theta_i\frac{d}{d\theta_i}-\theta_{i+1}\frac{d}{d\theta_{i+1}}$ and $L_0=-t\frac{d}{dt}-\frac{\alpha+1}{N}(\theta_1\frac{d}{d\theta_1}+...+\theta_N\frac{d}{d\theta_N})$. Note that $\mathfrak{g}^0\simeq gl_N(\mathbb{C})=\mathbb{C}L_0\oplus sl_N(\mathbb{C})$. With respect to the adjoint representation, $S'(N;\alpha)$ splits into the direct sum of irreducible $\mathfrak{g}^0$-modules:

$$(3.1)\ \ S'(N;\alpha)=\underset{m\in\mathbb{Z}}{\oplus}\overset{N-1}{\underset{k=0}{\oplus}}(S_{m,k-1}\oplus S'_{m,k}),$$

where $S_{m,k-1}$ (resp. $S'_{m,k}$) is spanned by $t^m\theta_{i_1}...\theta_{i_k}\frac{d}{d\theta_i}i\neq i_1,...,i_k$ and $t^m\theta_{i_1}...\theta_{i_{k-1}}(\theta_i\frac{d}{d\theta_i}-\theta_j\frac{d}{d\theta_j})$ (resp. $-t^m\theta_{i_1}...\theta_{i_k}(t\frac{d}{dt}+\frac{m+\alpha+1}{N-k}(\theta_1\frac{d}{d\theta_1}+...+\theta_N\frac{d}{d\theta_N})))$. Now $S_{m,k-1}$ (resp. $S'_{m,k}$) is as $\mathfrak{g}^0$-module isomorphic to the highest component of $\mathbb{C}\otimes(sl_N\otimes\Lambda^{N-k}sl_N)$ (resp. to $\mathbb{C}\otimes\Lambda^{N-k}sl_N$) (see [6]). Here L_0 acts on $S_{m,k}$ and $S'_{m,k}$ as multiplication by $-(m+\frac{\alpha+1}{N}k)$. Thus all contragredient pairs of $\mathfrak{g}^0$-modules on $S'(N;\alpha)$ are these:

$$(S_{m,0},S_{-m,0}),(S'_{m,0},S'_{-m,0}),(S_{m,-1},S'_{-m,1}),$$

and if $\alpha\in\mathbb{Z}$ we also have $(S'_{m,k},S'_{-m-\alpha-1,N-k})$.

By Lemma 2.1 it suffices to consider a cocycle ψ_0 which is 0 on all pairs of submodules in (3.1) except for those listed above. First, we show that $\psi_0(S'_{m,k},S'_{-m-\alpha-1,N-k})=0$. We may assume that $\alpha=-1$. By Corollary 2.1, we only have to consider opposite weight vectors, hence we calculate $\psi_0(a_m,b_{-m})$ where $a_m=-t^m\theta_1...\theta_k(t\frac{d}{dt}+\frac{m}{N-k}(\theta_1\frac{d}{d\theta_1}+...+\theta_N\frac{d}{d\theta_N})$ and $b_n=-t^n\theta_{k+1}...\theta_N(t\frac{d}{dt}+\frac{n}{k}(\theta_1\frac{d}{d\theta_1}+...+\theta_N\frac{d}{d\theta_N})$. Now the Jacobi identity between the elements $-t^{\ell+1}\frac{d}{dt}-\frac{\ell}{N}(\theta_1\frac{d}{d\theta_1}+...+\theta_N\frac{d}{d\theta_N})$, a_m and $b_{-m-\ell}$ is

$$(3.2)\ \ (\ell\frac{N-k}{N}-m)\psi_0(a_{m+\ell},b_{-m-\ell})=-(\ell\frac{N+k}{N}+m)\psi_0(a_m,b_{-m}).$$

If we put $\ell = N$ and $m = N - k$, we obtain that $\psi_0(a_{N-k}, b_{k-N}) = 0$. On the other hand if we substitute $\ell = 1$ in (3.2) we obtain a recurrent formula, from which we deduce that $\psi_0(a_m, b_{-m}) = 0$ for all $m \in \mathbb{Z}$. Hence, by Lemma 2.1(ii) we obtain that $\psi_0(S'_{m,k}, S_{-m-\alpha-1,N-k}) = 0$ for arbitrary α.

Now consider the subalgebra $S := \oplus_{m\in\mathbb{Z}} S_{m,0} \simeq sl_N(\mathbb{C}[t,t^{-1}])$. Since ψ_0 satisfies Lemma 2.1(ii), we conclude that the restriction of ψ_0 to $sl_N(\mathbb{C}[t,t^{-1}])$ is given by

$$(3.3)\ \ \psi_0(t^n g, t^m h) = \gamma n \delta_{n,-m} < g, h >,$$

where $\gamma \in \mathbb{C}; g, h \in sl_N$ and $<,>$ is the trace form on sl_N. Moreover, if $N \geq 3$, one easily verifies from the Jacobi identity for the elements $t^m\frac{d}{d\theta_1}, t^{m+n}\theta_1(\theta_2\frac{d}{d\theta_2}-\theta_3\frac{d}{d\theta_3})$ and $t^{-m}(\theta_2\frac{d}{d\theta_2}-\theta_3\frac{d}{d\theta_3})$ that $\gamma = 0$.

In order to calculate $\psi_0(S_{m,-1}, S_{-m,1})$, we consider the Jacobi identity

$$(3.4)\ \ -\psi_0([a_m,b_{-m}],c_0)+\psi_0([b_{-m},c_0],a_m)+\psi_0([c_0,a_m],b_{-m})=0,$$

where

$$(3.5)\ \ a_m = t^m\frac{d}{d\theta_1}, b_{-m} = t^{-m}(\theta_1\frac{d}{d\theta_1}-\theta_2\frac{d}{d\theta_2}),$$

$$c_k = -t^k\theta_1(t\frac{d}{dt}+\frac{\alpha+k+1}{N-1}(\theta_2\frac{d}{d\theta_2}+\ldots+\theta_N\frac{d}{d\theta_N}).$$

Then using (3.3) and the fact that $\psi_0(S_{-m,1}, S_{m,-1}) = \psi_0(S'_{m,0}, S_{-m,0}) = 0$ we reformulate (3.4):

$$(3.6)\ \ \psi_0(a_m,c_{-m}) = \psi_0(a_0,c_0)+\gamma(m-\frac{\alpha+1}{N-1})m.$$

Since ψ_0 satisfies Lemma 2.1(ii) we obtain that

$$(3.7)\ \ \psi_0(t^m\frac{d}{d\theta_i}, -t^n\theta_j(t\frac{d}{dt}+\frac{\alpha+n+1}{N-1}(\theta_1\frac{d}{d\theta_1}+\ldots+\theta_N\frac{d}{d\theta_N}))) =$$

$$(\beta+\gamma(m-\frac{\alpha+1}{N-1})m)\delta_{i,j}\delta_{m,-n},$$

where $\beta = \psi_0(a_0, c_0)$.

Similar considerations as in the previous paragraph, but now for the Jacobi identity for the elements a_0, c_{-m} of (3.5) and

$$L_m := -t^m(t\frac{d}{dt} + \frac{\alpha+m+1}{N}(\theta_1\frac{d}{d\theta_1} + ... + \theta_N\frac{d}{d\theta_N})),$$

lead to

$$\psi_0(L_m, L_{-m}) = (2m - \frac{\alpha+m+1}{N})\psi_0(a_0, c_0) + \frac{\alpha+m+1}{N}\psi_0(a_m, c_{-m}).$$

Now substituting (3.6) we obtain that

$$(3.8)\ \psi_0(L_m, L_{-m}) = 2m\beta + \frac{\alpha+m+1}{N}(m - \frac{\alpha+1}{N-1})m\gamma.$$

Since $\gamma = 0$ if $N \geq 3$, we conclude from (3.3), (3.7) and (3.8), that ψ_0 is a trivial cocycle for $N \geq 3$. For $N = 2$, we conclude that there exists, up to equivalence, only one non-trivial cocycle, viz, the following one

$$(3.9)\ \psi(L_m, L_n) = \frac{1}{2}m(m^2 - (\alpha+1)^2)\delta_{m,-n},$$

$$\psi(t^m\frac{d}{d\theta_i}, -t^n(\theta_j(t\frac{d}{dt} + (\alpha+n+1)(\theta_1\frac{d}{d\theta_1} + \theta_2\frac{d}{d\theta_2}))) =$$

$$m(m - (\alpha+1))\delta_{m,-n}\delta_{i,j},$$

$$\psi(t^m(\theta_1\frac{d}{d\theta_1} - \theta_2\frac{d}{d\theta_2}), t^n(\theta_1\frac{d}{d\theta_1} - \theta_2\frac{d}{d\theta_2})) = m\delta_{m,-n},$$

$$\psi(t^m\theta_1\frac{d}{d\theta_2}, t^n\theta_2\frac{d}{d\theta_1}) = m\delta_{m,-n},$$

and zero in all other cases.

This cocycle gives the central extension described by (1.22). (The minor discrepancy occurs since the L_0's differ by a multiple of c).

4 Central extensions of $K(N;\epsilon)$

The Lie superalgebra $K(N;\epsilon)$ is simple, except when $N = 4$ and $s := \frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4) \in \mathbb{Z}$. In the latter case, $K'(N;\epsilon) = [K(N;\epsilon), K(N;\epsilon)]$ is an ideal of codimension 1; this ideal does not contain the element $D_{t^{-s-1}\theta_1\theta_2\theta_3\theta_4}$. We shall classify the 2-cocycles for $K'(N;\epsilon)$.

For convenience we introduce some notations: we write $\psi(f,g)$ instead of $\psi(D_f, D_g)$ and we denote $k_i = \frac{1}{2}(1-\epsilon_i)$.

We start with the case $N = 1$. Both $K(1;\epsilon)$ superalgebras have only one non-trivial central extension. This can be easily verified using lemma 2.1 with $\mathfrak{g}^0 = \mathbb{C}D_t$, (2.6) and two Jacobi identities, viz., for the elements $D_{t^{m+1}\theta_1}, D_{t^{n+1}}, D_{t^{1-m-n-2k_1}\theta_1}$ and for $D_{t^{m+1}\theta_1}, D_{t^{n+1}\theta_1}, D_{t^{1-m-n}\theta_1}$ (this last one has to be considered only when $\epsilon_1 = 1$). The resulting 2-cocycle for $K(1;\epsilon)$ is this:

$$(4.1)\quad \psi_0(t^{m+1}, t^{n+1}) = \frac{1}{3}(cm^3 + bm)\delta_{m+n,0},$$

$$(4.2)\quad \psi_0(t^{m+1}\theta_1, t^{n+1}\theta_1) = \frac{1}{3}(c(m+k_1)^2 + \frac{b}{4})\delta_{m+n+2k_1,0},$$

$$(4.3)\quad \psi_0(t^{m+1}, t^{n+1}\theta_1) = a\epsilon_1 m\delta_{m+n,0}.$$

Adding a trivial cocycle we can make $a = 0$ and $b = -c$, obtaining the Neveu-Schwarz algebra [14] for $\epsilon_1 = 0$ and the Ramond algebra [15] for $\epsilon_1 = 1$ (see also [11]).

If $N = 2$, we only have to calculate the cocycle for $\epsilon = (1,0)$. In the other case, $K(2;(0,0)) \simeq K(2;(1,1)) \simeq W(1)$, we already know the cocycle on $W(1)$. Now assume $\epsilon_1 = 1$ and $\epsilon_2 = 0$. Using Corollary 2.1 (for $\mathfrak{g}^0 = \mathbb{C}D_t$), we immediately obtain that $\psi_0(t^m\theta_1, t^n\theta_2) = \psi_0(t^m\theta_1, t^n\theta_1\theta_2) = 0$. Clearly (4.1-3) also holds (and a similar relation when we replace θ_1 by θ_2). The Jacobi identity of the elements $D_{t^m\theta_2}, D_{t\theta_1\theta_2}$ and $D_{t^{-m-2}}$ implies that

$$(4.4)\quad \psi_0(t^m\theta_2, t^n\theta_1\theta_2) = -\frac{a}{3}\delta_{m+n+1,0}.$$

Finally using the Jacobi identity of the elements $D_{t^m\theta_1}, D_{t^n\theta_2}$ and $D_{t^p\theta_1\theta_2}$ we can express $\psi_0(t^m\theta_1\theta_2, t^n\theta_1\theta_2)$ in terms of $\psi_0(t^k\theta_i, t^\ell\theta_i)$. We then obtain

$$(4.5)\quad \psi_0(t^{m+1}\theta_1\theta_2, t^{n+1}\theta_1\theta_2) = \frac{c}{3}(m + \frac{1}{2})\delta_{m+n+1,0}.$$

Hence for $N = 2$ there is only one, up to equivalence, non-trivial 2-cocycle (we can remove $\frac{a}{3}$ using $D^*_{t\theta_1}$).

From now on assume $N > 2$. Since for every N there are only two non-isomorphic $K'(N;\epsilon)$ superalgebras, we distinguish the following 3 cases.

(4.6) (i) All $\epsilon = 0$ $(I = \{1, ..., N\}, \epsilon = 0, k = \frac{1}{2})$,

(ii) $N odd$, all $\epsilon_i = 1$ $(I = \{1, ..., N\}, \epsilon = 1, k = 0)$,

(iii) N even , $\epsilon_1 = 1$, other $\epsilon_i = 0$ $(I = \{2, ..., N\}, \epsilon = 0, k = \frac{1}{2})$.

Again we use Lemma 2.1, where we choose $\mathfrak{g}^0 =< D_t, D_{t^\epsilon\theta_i\theta_j};\ i,j \in I >$. Then $\mathfrak{g}^0 \simeq \mathbb{C}D_t \oplus so_N$ for the cases (i) and (ii) of (4.6) and $\mathfrak{g}^0 \simeq \mathbb{C}D_t \oplus so_{N-1}$ for (4.6) (iii). The $K'(N;\epsilon)$ superalgebra decomposes into the following sum of $\mathfrak{g}^0$-modules

$$K'(N;\epsilon) = \oplus_{n\in\mathbb{Z},\ell\in I} K(n+k\ell,\ell),$$

where $K(n+k\ell,\ell) =< D_{t^{n+1}\theta_{i_1}\dots\theta_{i_\ell}}; i_m \in I > \simeq \mathbb{C}\otimes\Lambda^\ell so_N$ (resp. $=< D_{t^{n+1}\theta_{i_1}\dots\theta_{i_\ell}}; i_m \in I > \oplus < D_{t^{n+1}\theta_1\theta_{i_1}\dots\theta_{i_\ell}}; i_m \in I > \simeq \mathbb{C}\otimes\Lambda^\ell so_{N-1} \oplus \mathbb{C}\otimes\Lambda^\ell so_{N-1}$) for the cases (i) and (ii) (resp. (iii)) of (4.6). Note that $n+k\ell$ is the eigenvalue corresponding to the element $\frac{1}{2}D_t$.

Applying Lemma 2.1, we may assume from now on that $\psi(K(m,i),K(n,j)) = 0$, except when $n+m=0$, and also we may assume that one of the following conditions is satisfied:

(4.7) $i=j$,

(4.8) N even, all $\epsilon_\ell = 0$ and $i = N-j$,

(4.9) N odd, all $\epsilon_\ell = 1$ and $i = N-j$.

First consider $\psi(K(m,i),K(-m,N-i))$ where we are in the situation of (4.8) or (4.9). Except for $N=4$ and $i=2$, every $K(m,i)$ is an irreducible $\mathfrak{g}^0$-module, hence in this case the pairing $< K(m,i), K(-m,N-i) >$ is up to a factor unique. So in order to calculate the value of the cocycle on these two spaces we only have to consider $\psi(t^{m+1}\theta_1\dots\theta_i, t^{-n-kN+1}\theta_{i+1}\dots\theta_N)$. We can use induction on i starting with $i=0$. The Jacobi identity of the elements $D_{t^{m+1}}, D_{t^{n+1}}$ and $D_{t^{-m-n-kN+1}\theta_1\dots\theta_N}$ is

$$\begin{aligned}(4.10)\quad & -(4n+2m-Nm)\psi(t^{m+1},t^{-m-kN+1}\theta_1\dots\theta_N)+\\ & (4m+2n-Nm)\psi(t^{n+1},t^{-n-kN+1}\theta_1\dots\theta_N)+\\ & (2m-2n)\psi(t^{n+m+1},t^{-m-n-kN+1}\theta_1\dots\theta_N)=0.\end{aligned}$$

We deduce from the cases $n=1$ and $n=-1$ that every $\psi(m) := \psi(t^{m+1},t^{-m-kN+1}\theta_1\dots\theta_N)$ can be expressed in terms of $\psi(1)$ and $\psi(0)$. If moreover $N \neq 4$, then also $\psi(0)=0$. Putting $d=\psi(1)$ and $e=\psi(0)$, we thus obtain that

$$(4.11)\quad \psi(t^{m+1},t^{n+1}\theta_1\dots\theta_N) = (dm + e\delta_{N,4})\delta_{m+n+kN,0}.$$

Next, use that Jacobi identity for the elements $D_{t^{n+1}\theta_{j_2}}, D_{t^{m+1}\theta_{j_1}\theta_{j_2}...\theta_{j_\ell}}$ and $D_{t^{-m-n-(N+2)k+1}\theta_{j_1}\theta_{j_{\ell+1}}...\theta_{j_N}}$ as induction step; this gives the following result

$$(4.12)\quad \psi(t^{m+1}\theta_{j_1}...\theta_{j_\ell}, \frac{d}{d\theta_{j_1}}...\frac{d}{d\theta_{j_\ell}}(t^{n+1}\theta_1...\theta_N)) =$$

$$= (d(m+\ell k) + \frac{1}{2}e\delta_{N,4}\delta_{\ell,1})\delta_{m+n+Nk,0}$$

Notice that if $N=4$, $D_{t^{-1}\theta_1\theta_2\theta_3\theta_4} \notin K'(4,(0,0,0,0))$, so we have to exclude $n=-2$ in (4.11).

For $N=4$ we still have to consider $\psi(t^{m+1}\theta_i\theta_j, t^{n+1}\theta_i\theta_j)$ and $\psi(t^{m+1}\theta_i\theta_j, t^{n+1}\theta_j\theta_\ell)$. The former one will be treated later on. The latter one is zero. This is verified by taking the Jacobi identity of the elements $D_{t^{n+1}\theta_i}, D_{t^{m+1}\theta_i\theta_j\theta_\ell}$ and $D_{t^{p+1}\theta_i\theta_j}$:

$$(4.13)\quad \psi(t^{m+n+2}\theta_j\theta_\ell, t^{p+1}\theta_i\theta_j) = \psi(t^{p+n+2}\theta_j, t^{m+1}\theta_i\theta_j\theta_\ell).$$

The right-hand side of (4.13) is zero by the assumption that ψ satisfies Lemma 2.1 (ii).

We still may assume that (4.7) is satisfied. Every $\Lambda^\ell so_m$ is an irreducible so_m-module, except when $m=4$ and $\ell=2$. In the former case the pairing $<\Lambda^\ell so_m, \Lambda^\ell so_m>$ is unique. So in order to determine $\psi(K(n,\ell), K(-n,\ell))$ we only have to compute

$$(4.14)\quad \psi(t^{p+1}\theta_1...\theta_i, t^{q+1}\theta_1...\theta_i),$$

$$(4.15)\quad \psi(t^{p+1}\theta_2...\theta_i, t^{q+1}\theta_2...\theta_i),$$

$$(4.16)\quad \psi(t^{p+1}\theta_1...\theta_i, t^{q+1}\theta_2...\theta_i),$$

where we assume for (4.15) and (4.16) that $\epsilon_1 = 1$ and all other $\epsilon_i = 0$. Clearly, if we change the indices in (4.15), we obtain a case of (4.14), so we shall not consider the case (4.15).

The so_m-module $\Lambda^i so_m$ is not irreducible for $m=4$ and $i=2$. This case only appears as part of (4.6) (i) with $N=4$, and was partially treated before, when we considered $\psi(K(m,i), K(-m, N-i))$. The only pairing which is left to compute is $\psi(t^{p+1}\theta_i\theta_j, t^{q+1}\theta_i\theta_j)$, but by changing the indices this case is also equivalent to (4.14).

We will now use induction to calculate (4.14). The case (4.16) can be treated in a similar way. As starting point of the induction we use the cocylce on $K(2;(\epsilon_1,\epsilon_2))$, where $\epsilon_1=\epsilon_2=0$ or 1 (as starting point for (4.16) the reader can take $K(2;(1,0))$, see (4.1) and (4.4)). The cocycle between elements of the form (4.14) in $K(2;(\epsilon_1,\epsilon_2))$ is

(4.17) $\psi(t^{m+1}\theta_1, t^{n+1}\theta_1) = \frac{1}{3}(c(m+k_1)^2 + \frac{b}{4})\delta_{m+n+2k_1,0}$,

(4.18) $\psi(t^{m+1}\theta_1\theta_2; t^{n+1}\theta_1\theta_2) = \frac{1}{3}c(m+k_1+k_2)\delta_{m+n+2k_1+2k_2,0}$.

Now assume that $i \geq 2$, using the Jacobi identity for the elements $D_{t^{p+1}\theta_{i+1}}$, $D_{t^{q+1}\theta_1\ldots\theta_i}$ and $D_{t^{-p-q-2k_1-\ldots-2k_{i+1}+1}\theta_1\ldots\theta_i\theta_{i+1}}$ as induction step we obtain

(4.19) $\psi(t^{m+1}\theta_1\theta_2\theta_3, t^{n+1}\theta_1\theta_2\theta_3) = \frac{1}{3}c\delta_{m+n+2k_1+2k_2+2k_3,0}$,

(4.20) $\psi(t^{m+1}\theta_1\theta_2\theta_3\theta_4, t^{n+1}\theta_1\theta_2\theta_3\theta_4) =$

$$\frac{1}{3}\frac{c}{m+k_1+k_2+k_3+k_4}\delta_{m+n+2k_1+\ldots+2k_4,0}.$$

Clearly using this Jacobi identity it is also possible to express (4.14) for $i > 4$ in terms of c. However in that case the Jacobi identity of the elements $D_{t^{p+1}\theta_1\theta_2\theta_5)}$, $D_{t^{q+1}\theta_5\theta_3\theta_4}$ and $D_{t^{n+1}\theta_1\theta_2\theta_3\theta_4}$ forces c to be zero. Hence if $N > 4$, $c = 0$ and we get that (4.14) is equal to zero for $i > 1$. Making similar type of induction calculations for (4.16) starting with (4.3) and (4.4) we obtain that $\psi(t^{p+1}\theta_1\ldots\theta_i, t^{q+1}\theta_2\ldots\theta_i) = 0$ for $i \geq 3$. This finishes the calculations of cocycles for $K'(N;\epsilon)$. It is not difficult to see that $\frac{b}{12} = D_t^*$, and that $(4-N)^{-1}d = D^*_{t^{1-k_1-\ldots-k_N}\theta_1\ldots\theta_N}$ if $N \neq 4$. We conclude from all this that there exists no non-trivial cocycle for $N > 4$. $K'(4;\epsilon)$ with $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4 \in 2\mathbb{Z}$ has 3 non-equivalent non-trivial cocycles, and in all other cases ($N = 4$ and $\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4$ odd or $N < 4$) there exists only one non-trivial cocycle.

We now can list the non-trivial cocycles on $K'(N;\epsilon)$ for $N \leq 4$. We have taken the $\epsilon_i^{\prime}s$ again arbitrary 0 or 1 and $k_i = \frac{1}{2}(1-\epsilon_i)$. As before the cocycle is assumed to be zero in all unlisted cases:

(4.21) $\psi(t^{m+1}, t^{n+1}) = \frac{c}{3}m(m^2-1)\delta_{m+n,0}$;

$$\psi(t^{m+1}\theta_i, t^{n+1}\theta_i) = \frac{c}{3}((m+k_i)^2 - \frac{1}{4})\delta_{m+n+2k_i,0};$$

$$\psi(t^{m+1}\theta_i\theta_j, t^{n+1}\theta_i\theta_j) = \frac{c}{3}(m+k_i+k_j)\delta_{m+n+2k_i+2k_j,0};$$

$$\psi(t^{m+1}\theta_i\theta_j\theta_\ell, t^{n+1}\theta_i\theta_j\theta_\ell) = \frac{c}{3}\delta_{m+n+2k_i+2k_j+2k_\ell,0};$$

$$\psi(t^{m+1}\theta_1\theta_2\theta_3\theta_4, t^{n+1}\theta_1\theta_2\theta_3\theta_4) = \frac{c\delta_{m+n+2k_1+...+2k_4,0}}{3(m+k_1+...+k_4)}.$$

The cocycle for $K(N;\epsilon)$ with $N \geq 3$ is given by the first $N+1$ formulas of (4.21). If $k_1+k_2+k_3+k_4 \in \mathbb{Z}$ we also have the following two non-trivial cocycles ($(d,e) = (1,0)$ or $(0,1)$):

$$(4.22)\quad \psi(t^{m+1}, t^{n+1}\theta_1\theta_2\theta_3\theta_4) = (md+e)\delta_{m+n+k_1+...+k_4,0};$$

$$\psi(t^{m+1}\theta_i, t^{n+1}\frac{d}{d\theta_i}(\theta_1\theta_2\theta_3\theta_4)) = ((m+k_i)d + \frac{1}{2}e)\delta_{m+n+k_1+...+k_4,0};$$

$$\psi(t^{m+1}\theta_i\theta_j, t^{n+1}\frac{d}{d\theta_i}\frac{d}{d\theta_j}(\theta_1\theta_2\theta_2\theta_4)) =$$

$$(m+k_i+k_j)d\delta_{m+n+k_1...+k_4,0}.$$

(Note that $D_{t^{1-k_1-k_2-k_3-k_4}\theta_1\theta_2\theta_3\theta_4} \notin K'(4;\epsilon)$.)

So finally we have proven the main theorem:

THEOREM 4.1
Let $\mathfrak{g}$ be one of the superalgebras $W(N), S(N;\alpha)$ with $N > 1$ and $\alpha \in \mathbb{C}$ and $K(N;\epsilon)$ where $\epsilon = (\epsilon_1, \epsilon_2, ..., \epsilon_N)$ with $\epsilon_i = 0$ or 1. Then

(a) $\mathfrak{g}' := [\mathfrak{g}, \mathfrak{g}]$ is simple; moreover $\mathfrak{g}' = \mathfrak{g}$ for $W(N)$, $S(N;\alpha)$ with $\alpha \notin \mathbb{Z}$ and $K(N;\epsilon)$ with $N \neq 4$ or $N = 4$ and $\frac{1}{2}(\epsilon_1+\epsilon_2+\epsilon_3+\epsilon_4) \notin \mathbb{Z}$, in all other cases $\mathfrak{g}'$ is an ideal of codimension 1 in $\mathfrak{g}$. The following superalgebras are isomorphic:

$$S'(N;\alpha) \simeq S'(N;\beta) \text{ when } \beta - \alpha \in \mathbb{Z};$$

$$K'(N;(\epsilon_1,...,\epsilon_N)) \simeq K'(N;(\delta_1,...,\delta_N)) \text{ when } \sum_i(\epsilon_i - \delta_i) \in 2\mathbb{Z};$$

$$K(2;(0,0)) \simeq K(2;(1,1)) \simeq W(1).$$

(b) For $W(N), S'(N;\alpha)$ and $K'(N;\epsilon)$ there exists no non-trivial 2-cocycles for $N > 2, N > 2, N > 4$, respectively. All up to equivalence non-trivial cocycles for $W(1), W(2), S'(2;\alpha)$ and $K'(N;\epsilon)$ ($N \leq 4$) are given by (2.10), (2.14), (3.9), (4.21 and 22), respectively. □

The superalgebras $K'(N;\epsilon)$ are nothing else but the SO_N-superconformal algebras. Our results on the cocycles (4.21- 22) are in agreement with the results of K. Schoutens [16,17], he chooses $d = e$. In [19] another superconformal algebra is considered; again this is $K'(4;\epsilon)$. The authors say that the "twisted" cocycle is removed by their choice of generators. However, their element U_0 is a center and corresponds to our cocycle e, so implicitly they still have a twisted cocycle.

We conclude this section with the conjecture promised in the introduction.

CONJECTURE 1
A Z-graded superconformal algebra is isomorphic to either $W(N)(N \geq 0)$, or $S'(N;\alpha)(N \geq 2)$, or $K'(N;\epsilon)(N \geq 1)$.

Let $\Lambda_t^+(N) = \mathbb{C}[t] \otimes \Lambda(N)$, then $W_+(N) := der\Lambda_t^+(N)$ is a subalgebra of $W(N)$. We let $S'_+(N;\alpha) = S'(N;\alpha) \cap W_+(N)$, $K'_+(N;\epsilon) = K'(N;\epsilon) \cap W_+(N)$. It is easy to see that the superalgebras $W_+(N), S'_+(N;0)$ and $K'_+(N;0)$ are simple.

Furthermore, given a finite-dimensional simple Lie superalgebra $\mathfrak{g}$ and its order m automorphism σ, we denote by $\tilde{\sigma}$ the automorphism of the Lie superalgebra $\tilde{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes_{\mathbb{C}} \mathfrak{g}$ defined by

$$\tilde{\sigma}(t^j \otimes \mathfrak{g}) = (exp - \frac{2\pi ij}{m})t^j \otimes \sigma(\mathfrak{g}),$$

and denote by $L(\mathfrak{g}, \sigma)$ the fixed point set of $\tilde{\sigma}$ on $\tilde{\mathfrak{g}}$.

REMARK
$L(\mathfrak{g}, \sigma)$ depends only on the connected component of σ in $Aut(\mathfrak{g})$.

We are taking this opportunity to propose another conjecture, which is a generalization of a conjecture of one of the authors [7] for Lie algebras, solved recently by Mathieu [13].

CONJECTURE 2
A Z-graded Lie superalgebra with only trivial graded ideals and of growth 1 is either isomorphic to one of the superalgebras $W(N), W_+(N)$ $S'(N;\alpha)$, $S'_+(N;0)$ $K'(N;\epsilon)$, $K'_+(N;0)$, or is isomorphic to one of the Lie superalgebras $L(\mathfrak{g}, \sigma)$.

5 Representations $V_{\alpha,\beta}$ of $K'(N;\epsilon)$

In this section we construct a two parameter family of positive energy representations of the Lie superalgebras $K'(N;\epsilon)$.

Let ω_ϵ be the differential form given by (1.12). The superalgebra $K'(N;\epsilon)$ acts in a natural way on $V_{\alpha,\beta}$, the space of "densities" of the form $t^\alpha g(t,\theta_1,...,\theta_N)\omega_\epsilon^\beta$, where α and β are fixed complex parameters and $g \in \Lambda_t(N)$ arbitrary (see also (1.16)).

(5.1) $D_f(t^\alpha g\omega_\epsilon^\beta) = (D_f(t^\alpha g) + (-1)^{\deg f \deg g} 2\beta t^\alpha g D^\epsilon(f))\omega_\epsilon^\beta$

In order to give a basis of $V_{\alpha,\beta}$ we define an ordering on $\Lambda(N)_{\bar 0}$ and $\Lambda(N)_{\bar 1}$: $w_i < w_j$, where $w_i = \theta_{i_1}...\theta_{i_r}$ and $w_j = \theta_{j_1}...\theta_{j_s}$ with $i_1 < i_2 < ... < i_r$ and $j_1 < j_2 < ... < j_s$, if $r < s$ or if $r = s$ and $i_k < j_k$ for the first k for which $i_k \neq j_k$. Now let w_i (resp. $w_{i+\frac{1}{2}}$) be the i-th element in that ordering on $\Lambda(N)_{\bar 0}$ (resp. $\Lambda(N)_{\bar 1}$), here $i = 0, 1, ...2^{N-1} - 1$, then the elements.

(5.2) $\psi_{2^{N-1}m-i} = t^{\alpha-m} w_i \omega_\epsilon^{\beta_i} = 0, \frac{1}{2}, 1, ..., 2^{N-1} - \frac{1}{2}$

form a basis of $V_{\alpha,\beta}$. We can identify ψ_j ($j \in \frac{1}{2}\mathbb{Z}$) with an infinite column vector with 1 as the j-th entry and 0 elsewhere. The Lie superalgebra $\mathfrak{a}_{\infty|\infty}$ of matrices $(a_{ij})_{i,j\in\frac{1}{2}\mathbb{Z}}$ such that $a_{ij} = 0$ for $|i-j| >> 0$ acts in a natural way on $V_{\alpha,\beta}$[9]:

(5.3) $E_{ij}.\psi_k = \delta_{jk}\psi_i$.

Combining (5.1), (5.2) and (5.3) gives an embedding $\rho_{\alpha,\beta} : K'(N;\epsilon) \to \mathfrak{a}_{\infty|\infty}$. For instance for $N = 2$, this embedding is

$$\rho_{\alpha,\beta}(D_{t^{m+1}}) = \sum_{k\in\mathbb{Z}} (\beta(m+1) + \alpha - k)E_{2(k-m),2k} +$$

$$((\beta + \frac{1}{2})(m+1) + \alpha - k - \frac{\epsilon_1}{2})E_{2(k-m)-\frac{1}{2},2k-\frac{1}{2}} +$$

$$((\beta + 1)(m+1) + \alpha - k - \frac{\epsilon_1}{2} - \frac{\epsilon_2}{2})E_{2(k-m)-1,2k-1} +$$

$$((\beta + \frac{1}{2})(m+1) + \alpha - k - \frac{\epsilon_2}{2})E_{2(k-m)-\frac{3}{2},2k-\frac{3}{2}}.$$

The Lie superalgebra $\mathfrak{a}_{\infty|\infty}$ has up to equivalence only one non-trivial 2-cocycle, it is given by [9]:

(5.4) $\psi(E_{ij}, E_{ji}) = -(-1)^{2(i+j)}\psi(E_{ji}, E_{ij}) = (-1)^{2i}$ if $i \leq 0 < j$,

(5.5) $\psi(E_{ij}, E_{k\ell}) = 0$ in all other cases.

Pulling back the cocycle via $\rho_{\alpha,\beta}$ we get a cocycle $\psi_{\alpha,\beta}$ on $K'(N;\epsilon)$. For simplicity we choose $\alpha = 0$, because $\psi_{\alpha,\beta}(x,y) = \psi_{0,\beta}(x,y) + f_{\alpha,\beta}([x,y])$ for some $f \in K'(N;\epsilon)^*$. Denote $\psi_\beta = \psi_{0,\beta}$, then we get.

(5.6) $\psi_\beta(D_{t^{m+1}}, D_{t^{n+1}}) =$

$$\begin{cases} \frac{1}{3}(-12\beta^2 + 12\beta - 2)m(m^2-1)\delta_{m,-n} & \text{for } N = 0, \\ \frac{1}{3}(12\beta - 3)m(m^2 - (1-\epsilon_1))\delta_{m,-n} & \text{for } N = 1, \\ \frac{1}{3}(-6)m(m^2 - (1-\epsilon_1-\epsilon_2))\delta_{m,-n} & \text{for } N = 2, \\ 0 & \text{for } N \geq 3. \end{cases}$$

(Compare this with (4.21).) Now substituting $\beta = -1$ (the adjoint representation of $K'(N;\epsilon)$), we obtain $c = -26, -15, -6, 0, 0$ for $N = 0, 1, 2, 3, 4$, respectively. This is related to the fact that $c = 26, 15, 6, 0, 0$ is the critical central charge of $K'(N;\epsilon)$, where $N = 0, 1, 2, 3, 4$ respectively.

From now on let $N = 1$ and $\epsilon_1 = 0$, i.e., $K(1;0)$ is the Neveu-Schwarz algebra. Its embedding into $\mathfrak{a}_{\infty|\infty}$ is given by

(5.7) $$\rho_{\alpha,\beta}(D_{t^{m+1}}) = 2\sum_{k\in\mathbb{Z}}(\beta(m+1) + \alpha - k)E_{k-m,k} +$$

$$((\beta + \frac{1}{2})(m+1) + \alpha - k)E_{k-m-\frac{1}{2},k-\frac{1}{2}},$$

$$\rho_{\alpha,\beta}(D_{t^{m+1}\theta_1}) = \sum_{k\in\mathbb{Z}}(2\beta(m+1) + \alpha - k)E_{k-m-\frac{1}{2},k} - E_{k-m-1,k-\frac{1}{2}}.$$

Then its corresponding cocycle is

(5.8) $$\psi_{\alpha,\beta}(D_{t^{m+1}}, D_{t^{n+1}}) = ((4\beta - 1)m(m^2-1) - 4\alpha m)\delta_{m+n,0},$$

$$\psi_{\alpha,\beta}(D_{t^{m+1}\theta_1}, D_{t^{m+1}\theta_1}) = m(m+1) - \alpha)\delta_{m+n+1,0}.$$

In [9] we defined for every $m \in \mathbb{Z}$ a highest weight representation $\hat{\pi}_m$ of the Lie superalgebra $\mathfrak{a}_{\infty|\infty}$ with vacuum vector $|m>$ (its corresponding module was denoted by V_m). The action on $|m>$ is as follows

(5.9) $\hat{\pi}_m(E_{ij})|m>= 0$ for $i < j$;

$$\hat{\pi}_m(E_{ii})|m>= 0 \text{ for } i \neq -\frac{1}{2}, 0, \frac{1}{2};$$

$$\hat{\pi}_m(E_{\frac{1}{2}\frac{1}{2}})|m>= \begin{cases} m \mid m> & \text{for } m \geq 0 \\ 0 & \text{for } m < 0 \end{cases};$$

$$\hat{\pi}_m(E_{00})|m>= \begin{cases} 0 & \text{for } m \geq 0 \\ -|m> & \text{for } m < 0 \end{cases};$$

$$\hat{\pi}_m(E_{-\frac{1}{2},-\frac{1}{2}})|m>= \begin{cases} 0 & \text{for } m \geq 0 \\ (m+1)|m> & \text{for } m < 0 \end{cases}.$$

Define (see e.g. [11])

(5.10) $L_i = \hat{\pi}_m(\rho_{\alpha,\beta}(-\frac{1}{2}D_{t^{i+1}})) - \frac{1}{2}\alpha\delta_{i,0}$ and

$$G_{i+\frac{1}{2}} = \hat{\pi}_m(\rho_{\alpha,\beta}(-D_{t^{i+1}\theta_1})).$$

Then

(5.11) $[L_i, L_j] = (i-j)L_{i+j} + \delta_{i,-j}\frac{i(i^2-1)}{12}c_\beta,$

$$[G_{i+\frac{1}{2}}, L_j] = (i + \frac{1}{2} - \frac{1}{2}j)G_{i+j+\frac{1}{2}},$$

$$[G_{i+\frac{1}{2}}, G_{j-\frac{1}{2}}] = 2L_{i+j} + \delta_{i,-j}\frac{(i+\frac{1}{2})^2 - \frac{1}{4}}{3}c_\beta, \text{ where}$$

$$c_\beta = 12\beta - 3.$$

Hence we obtain a positive energy representation of the Neveu-Schwarz algebra on V_m with central charge c_β:

(5.12) $L_i \mid m>= 0$ and $G_{i+\frac{1}{2}} \mid m>= 0$ for $i > 0$,

$L_0 \mid m>= h_m \mid m>$, where

$$h_m = \begin{cases} -(\beta + \alpha - \frac{1}{2})m - \frac{1}{2}\alpha & \text{for } m \geq 0, \\ -(\beta + \alpha + \frac{1}{2})m - \frac{1}{2}(1+\alpha) & \text{for } m < 0. \end{cases}$$

6 On positive energy representations of superconformal algebras

Let $\mathfrak{g}$ be a Lie superalgebra and let L_0 be an even element of $\mathfrak{g}$ such that adL_0 is diagonalizable with finite-dimensional eigenspaces and real eigenvalues. We have the triangular decomposition:

$$\mathfrak{g} = \mathfrak{n}_- + \mathfrak{h} + \mathfrak{n}_+,$$

where $\mathfrak{h}$ is the 0-eigenspace of L_0 and $\mathfrak{n}_-$ (resp. $\mathfrak{n}_+$) is the sum of all eigenspaces with negative (resp. positive) eigenvalues.

A representation π of $\mathfrak{g}$ on a vector space V is called a *positive energy representation* if $\pi(L_0)$ is diagonalizable with real eigenvalues, the minimal eigenvalue exists, and the eigenspaces are finite-dimensional. Note that each eigenspace of $\pi(L_0)$ is invariant with respect to $\mathfrak{h}$; we denote the representation of $\mathfrak{h}$ on the eigenspace V^0 with minimal eigenvalue by π_0.

LEMMA 6.1
The map $\pi \mapsto \pi_0$ is a bijection between irreducible positive energy representations of $\mathfrak{g}$ and finite-dimensional irreducible representations of $\mathfrak{h}$.

Proof.
An irreducible positive energy representation π if $\mathfrak{g}$ on V is a quotient by the unique maximal subrepresentation of the representation of $\mathfrak{g}$ by left multiplication on the space

$$\tilde{V} = U(\mathfrak{g}) \otimes_{U(\mathfrak{h}+\mathfrak{n}_-)} V^0 \ (= U(\mathfrak{n}_+)V^0).$$

Here $U(\mathfrak{p})$ denotes the universal enveloping algebra of the Lie algebra $\mathfrak{p}$ and the representation π_0 of $\mathfrak{h}$ on V is extended trivially to $\mathfrak{n}_-$. □

Recall that the power series

$$ch_\pi := trq^{\pi(L_0)} = \sum_{\lambda \in Spec\ \pi(L_0)} (\dim V_\lambda) q^\lambda$$

is called the *character* of π. Let $q = e^{2\pi i \tau}, Im\ \tau > 0$. The representation π is called *modular invariant* if $q^a ch_\pi$ is a modular function in τ on the upper half-plane, for some $a \in \mathbb{R}$ [12].

Let ω be an antilinear anti-involution of $\mathfrak{g}$ (i.e. ω is antilinear, $\omega^2 = 1$ and $\omega([x,y]) = [\omega(y), \omega(x)]$), and assume that $\omega(L_0) = L_0$. Given a representation π of $\mathfrak{g}$ on V, a Hermitian form on V is called *contravariant* if, with respect to this form,

$$(\pi(x))^* = \pi(\omega(x)),$$

where * denotes the adjoint operator. Note that, with respect to a contravariant form, the eigenspaces of $\pi(L_0)$ are orthogonal. For an irreducible positive energy representation, this form is determined by its restriction to V^0. The representation π is called *unitary* if the contravariant Hermitian form is positive definite. Then, of course, the representation π_0 of $\mathfrak{h}$ on V^0 is unitary, but the converse is not true.

Note that if σ is an antilinear involution of $\mathfrak{g}$, we obtain the associated antilinear anti-involution ω as follows:

$$\omega \mid_{\mathfrak{g}_{\bar{0}}} = -\sigma; \omega \mid_{\mathfrak{g}_{\bar{1}}} = i\sigma.$$

EXAMPLE 6.1
Let σ be an antilinear involution of the superalgebra $\Lambda_t(N)$ defined by $\sigma(t) = t^{-1}, \sigma(\theta_j) = it^{\epsilon_j - 1}\theta_j$. Then $\sigma(\omega_\epsilon) = -t^2\omega_\epsilon$, hence σ induces an antilinear involution of $K(N;\epsilon)$,

Let now $\hat{\mathfrak{g}}$ be a central extension of a superconformal algebra $\mathfrak{g}$ and let Vir be its subalgebra corresponding to $\overline{\text{Vir}} \subset \mathfrak{g}$. Let ω be an antilinear anti-involution of $\mathfrak{g}$ such that $\omega(L_n) = L_{-n}$. Note that if the restriction of the 2-cocycle to $\overline{\text{Vir}}$ is trivial (in particular if $\hat{\mathfrak{g}} = \mathfrak{g}$), then $\mathfrak{g}$ has no non-trivial unitary positive irreducible energy representations (since $\overline{\text{Vir}}$ has no such representations). One can check that the only involutions of $K(N;\epsilon)$ that allow unitary representation (for $N \leq 4$) are those associated to σ of Example 6.1. There has been a great deal of work done recently on classification of these unitary representations. On the other hand, the classification of the much more universal class of modular invariant representations is known only for the Virasoro, Neveu-Schwarz and Ramond algebras [12]. At present it is even not clear which of the superconformal algebras admit non-trivial irreducible modular invariant representations.

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