Graded Lie Algebras. I

Contents

Graded Lie Algebras. I

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(Preliminary version)

1. Introduction. Recent applications of graded Lie algebras in physics have made it desirable to attempt to develop a comprehensive structure theory. This paper, hopefully the first of a series, is the beginning of such an attempt.

Right away I call attention to the fact that graded Lie algebras (finite-dimensional and of characteristic 0) appear to share many features of the characteristic p case of ordinary Lie algebra theory. I have therefore selected as the first target a study like the one Seligman pioneered in [3]. I assume that the algebra admits a representation whose attached form is nonsingular; in general the representation must be allowed to be "projective" as explained below. (The stronger assumption that the Killing form is nonsingular is of course the special case where the representation is the regular - also called adjoint - representation.) Very early in the work, however, I weaken the hypothesis to the following pair of assumptions: that there is a nonsingular associative form (not necessarily coming from any kind of representation), and that the even part of the algebra is semisimple plus abelian. (As far as I know at present, it is possible that no actual weakening of hypotheses has occurred.)

The plan of the paper is as follows. Various preliminary results are assembled in sections 2 to 7. At this point I launch an axiomatic study of the "root systems" that are emerging. The main difference between these root systems and those that arise in ordinary Lie algebra theory is the possible presence of isotropic vectors. The axiomatic treatment makes it possible to handle the geometric problem in a self-contained way, although admittedly the axioms are a little artificial. In sections 11 and 12 I return to graded Lie algebras to grapple with the possibility that root spaces may not be 1-dimensional. After this possibility has been severely limited I return to the root systems and complete the classification. The upshot is that there are two infinite families ("special linear" and "orthosymplectic") that might well be called <u>classical</u>, and in addition systems I label $\Gamma(A, B, C)$, Γ_2, and Γ_3. $\Gamma(A, B, C)$ is actually an infinite family, with parameters A, B, C free to roam, subject to $A + B + C = 0$. The ordinary Lie algebra G_2 makes an appearance inside Γ_2, but the other exceptional algebras F_4, E_6, E_7, and E_8 do not show up at all.

The arguments used in the classification of the root systems are elementary, tedious, and repetitive. I hope there will soon be a better way to do the job. But I venture to say that it was worth while pushing through the project: three times I received a surprise.

The next task of course is to answer the obvious questions concerning the existence and uniqueness of graded Lie algebras that go with these root systems. This study is in progress.

There exist simple graded Lie algebras which do not satisfy the assumptions in this paper. There are indications that they are related to Cartan's infinite pseudo-groups, just as is conjectured for characteristic p. Dr. Warren Nichols is at work on this.

Many further topics invite attention: representations, Whitehead and Levi theorems, cohomology, real forms, Ado's theorem, simplicity theorems of the Herstein type (Speers [4], [5] has made a start on this), graded Jordan algebras, grading by Z (instead on Z_2 as in this paper), etc.

I am greatly indebted to Peter Freund of Chicago's Physics Department for numerous stimulating conversations and for his patient attempts to teach me the role graded Lie algebras are playing in physics. In the joint announcement [1] the results contained in this paper are summarized.

2. Basic definitions. We present the bare minimum of background that is needed.

All grading in this paper is by Z_2, the integers mod 2. The first concept that is pertinent is that of a graded vector space V over a field k (all vector spaces are finite-dimensional throughout the paper). It appears to be technically convenient to take V as a set-theoretic union rather than a direct sum. So $V = V_0 \cup V_1$ where V_0 and V_1 are vector spaces over k, disjoint except for their common 0. An element in V_0 is even, one in V_1 is odd.

A graded algebra $A = A_0 \cup A_1$ is a graded vector space with a bilinear multiplication satisfying $A_i A_j \subset A_{i+j}$ (notice that this includes $A_1 A_1 \subset A_0$, since the subscripts are integers mod 2).

Linear transformations on a graded vector space V acquire a grading. Such a linear transformation is actually a pair of ordinary linear transformations. An odd linear transformation on $V = V_0 \cup V_1$ combines linear transformations from V_0 to V_1 and V_1 to V_0; an even one combines $V_0 \to V_0$ and $V_1 \to V_1$. The linear transformations on V form a graded (associative) algebra which is in fact simple. Here is an exercise for the reader: conversely any simple graded associative algebra over an algebraically closed field has this form. To extend the result to a field which is not algebraically closed, introduce graded vector spaces over a division ring.

Inaccurate!

We shall make use of the <u>trace</u> of a linear transformation. For an odd one we simply declare the trace to be 0. Let T be an even linear transformation, consisting of T_0: $V_0 \to V_0$ and T_1: $V_1 \to V_1$. We define

$$\mathrm{Tr}(T) = \mathrm{Tr}(T_0) - \mathrm{Tr}(T_1).$$

Let T and U be linear transformations on V. We have $\mathrm{Tr}(TU) = \mathrm{Tr}(UT)$ except that

(1) $$\mathrm{Tr}(TU) = -\,\mathrm{Tr}(UT)$$

when T and U are both odd.

The bracket [xy] of two elements in a graded algebra is defined as $xy - yx$ if at least one of the two elements is even and $xy + yx$ if both are odd. We have the following properties.

(a) $[xy] = -[yx]$ if at least one is even; $[xy] = [yx]$ if both are odd.

(b) The Jacobi identity

$$[[xy]z] + [[yz]x] + [[zx]y] = 0$$

holds if the number of even elements among x, y, z is 0, 2, or 3. If two are odd and one is even, take z to be even; then

$$[[xy]z] - [[yz]x] + [[zx]y] = 0.$$

We take (a) and (b) as the axioms for a <u>graded Lie algebra</u>.

Remarks. 1. In treating an abstract graded Lie algebra we write the operation merely as juxtaposition; the bracket is reserved for actual commutation in an associative algebra.

2. Additional axioms are needed for characteristics 2 and 3. We omit these since we shall shortly be assuming characteristic 0.

3. Observe that a graded Lie algebra is not a Lie algebra in the ordinary sense (though of course the even part is a Lie algebra). In the first place this is trivially true since our graded Lie algebras are not closed under addition. Naturally the missing sums can be supplied, but even then axioms (a) and (b) are twisted variants of the usual anticommutativity and Jacobi identity.

The notion of an ideal in a graded Lie algebra is the obvious one, as is that of simplicity. Our project is to study certain simple graded Lie algebras (over an algebraically closed field of characteristic 0).

3. <u>The form attached to a representation</u>. A representation $x \to S_x$ of a graded Lie algebra $J = J_0 \cup J_1$ is defined in the obvious way as a homomorphism of J into the graded Lie algebra of linear transformations on a graded vector space. The notion of a homomorphism includes the assumption that S preserves parity of elements. We associate with S the bilinear form

$$(x, y) = Tr(S_x S_y).$$

This form is symmetric on J_0 and skew on J_1. Also, J_0 and J_1 are orthogonal relative to the form. The form is associative (also called invariant):

(2) $$(xy, z) = (x, yz).$$

We give the verification of (2) in the most interesting case: x and y odd and z even. Then (2) becomes

(3) $$Tr\left[(S_x S_y + S_y S_x)S_z\right] = Tr\left[S_x(S_y S_z - S_z S_y)\right].$$

Since S_y and $S_x S_z$ are odd, (3) follows from (1).

The regular representation is defined by $x \rightarrow L_x$ where L_x is left multiplication by x on J, that is, $L_x y = xy$. The form arising from the regular representation is the <u>Killing form</u>.

It is too restrictive to assume that the Killing form is nonsingular. For instance, let L be the graded Lie algebra of linear transformations of trace 0 on $V_0 \cup V_1$. Let m and n be the dimensions of V_0 and V_1. If $m \neq n$, L is simple and has a nonsingular Killing form. But if m = n, L is not simple. The identity linear transformation lies in L and spans the one-dimensional center Z of L. It is the quotient algebra $L^* = L/Z$ that is simple for $n \geq 2$ (for n = 1, L is not simple). The Killing form vanishes identically on L and L*. But the very definition of L gives us a representation of L by smaller matrices; the induced form is non-zero on L and in the obvious way it induces a nonsingular form on L*.

This suggests introducing projective representations, as was done for characteristic p in [2]. A projective representation of a graded Lie algebra J is a homomorphism of J into an algebra L* = L/Z of the type just described. One easily sees that ordinary representations can be viewed as a special case of projective representations.

Now we can at last describe the class of graded Lie algebras to be studied in this paper: simple ones (over an algebraically closed field of characteristic 0) which admit a projective representation whose induced form is nonsingular.

4. Cartan subalgebras. We fix the notation $J = J_0 \cup J_1$ for a simple graded Lie algebra over an algebraically closed field k of characteristic 0. It is assumed that J possesses a projective representation whose induced form (,) is nonsingular. Recall that the form is symmetric on J_0, skew on J_1, and that J_0 and J_1 are orthogonal. When restricted to J_0 the form is still nonsingular. Since the notion of projective representation is vacuous for ordinary Lie algebras of characteristic 0, we might as well say that J_0 possesses an ordinary representation whose attached form is nonsingular. Hence [2, Th. 31 on p. 28] J_0 is the direct sum of its center and a semisimple algebra.

This is all the use we are going to make of the assumption that the form comes from a representation. So it is natural to start all over again with a weaker set of assumptions. We begin with a simple graded Lie algebra $J = J_0 \cup J_1$ over k. We assume $J_0 = P \oplus Q$ where P is semisiple and Q is abelian. We postulate the existence on J of a nonsingular bilinear form (,) which is symmetric on J_0, skew on J_1, makes J_0 and J_1 orthogonal, and satisfies the associativity condition (2).

Classical theory describes the form on J_0. It makes P and Q orthogonal, and on each simple summand of P it is a nonzero scalar multiple of the Killing form.

Let H be a Cartan subalgebra of J_0; H is the direct sum of Q and a Cartan subalgebra of P. One knows that H is abelian. When we decompose J_0 relative to H, the root spaces relative to the nonzero roots are one-dimensional and have numerous further properties. We shall typically write α, β , γ for these roots and K_α , K_β , K_γ for the corresponding root spaces.

We proceed to decompose J_1 relative to H. Temporarily we allow for the possibility of a root space in J_1 for the root 0 and write L_0 for it, but very shortly we shall prove that $L_0 = 0$. For the nonzero roots and root spaces in J_1 we typically write λ, μ , ν , ρ and L_λ , L_μ , L_ν , L_ρ. Many things work with virtually no change from the classical case. We have $K_\alpha L_\lambda \subset L_{\alpha+\lambda}$, $L_\lambda L_\mu \subset K_{\lambda+\mu}$, $(L_\lambda, L_\mu) = 0$ except for $\mu = -\lambda$, and the form is nonsingular between L_λ and $L_{-\lambda}$. In particular, L_λ and $L_{-\lambda}$ have the same dimension. Proofs are left to the reader. Of course the analogous statements about the K_α's are classical.

At present we have no information about the dimension of L_λ or how H acts on it. Much will be proved in due course, but at this point we alert the reader to the fact that it is possible for L_λ to be two-dimensional.

The form remains nonsingular when restricted to H. So for each α or λ there is a unique h_α, $h_\lambda \in H$ inducing the linear function in question. It is often harmless and convenient to make an "abuse of language" and write α or λ when we really mean h_α or h_λ, and we shall freely do so.

There are two fairly immediate corollaries of the classical thory that we shall wish to quote. They appear in the first two lemmas.

Remark. To avoid a lot of repetition our lemmas will be stated in skeleton form. But the two major theorems of the paper will be stated in full.

Lemma 1. Assume that α and β are not orthogonal. Then the form is non-isotropic on the rational subspace spanned by h_α and h_β.

Proof. The elements h_α and h_β lie in P, the semisiple summand of J_0. Since they are non-orthogonal, they lie in the same simple component of P. The result now follows from classical theory.

Lemma 2. Suppose that $(\beta, \beta)/(\alpha, \alpha)$ is a negative rational number. Then for any γ, γ is orthogonal to at least one of α, β.

Proof. This time h_α and h_β must lie in different simple summands of P, for otherwise one knows that $(\beta, \beta)/(\alpha, \alpha)$ is a positive rational number. h_γ cannot lie in both of these summands, and so must be orthogonal to at least one of h_α, h_β.

The following fact is standard within J_0. Although the proof is identical in J_1 we give it for completeness.

Lemma 3. Let $s \in L_\lambda$ be characteristic under H and let t be an element of $L_{-\lambda}$, not necessarily characteristic. Then $st = (s, t)h_\lambda$.

Proof. We have $st \in H$. Since the form is nonsingular when restricted to H, it suffices to prove that $(h, st) = (s, t)\lambda(h)$ for any h H. Now

$$(h, st) = (hs, t) = (\lambda(h)s, t)$$

since s is characteristic.

Lemma 4. If $(\lambda, \mu) \neq 0$ then $L_\lambda L_\mu$ or $L_{-\lambda} L_\mu$ (or both) must be non-zero. The same is true if λ or μ or both are replaced by roots in J_0.

Proof. We give the proof only for λ and μ. Suppose that on the contrary $L_\lambda L_\mu = 0 = L_{-\lambda} L_\mu$. Pick s in L_λ characteristic under H and t in $L_{-\lambda}$

with $(s, t) = 1$. We have $st = h_\lambda$ by Lemma 3. For any $x \in L_\mu$ the Jacobi identity tells us that

$$st.x + tx.s + xs.t = 0.$$

Here the second and third terms vanish, but the first equals $h_\lambda x$ and does not vanish for nonzero x, since $(\lambda, \mu) \neq 0$ implies that h_λ is nonsingular on L_μ.

We shall use Lemma 4 so frequently that after a few initial references to Lemma 4 its use will be tacit.

We recall that $(\alpha, \alpha) \neq 0$ for any α. This is not necessarily true for λ. So we make a distinction, calling λ isotropic if $(\lambda, \lambda) = 0$ and non-isotropic if $(\lambda, \lambda) \neq 0$. This calls for a refinement of our fixed notation. From now on λ, μ, ν, ρ will always denote isotropic roots and σ will stand for a non-isotropic root (under the action of H on J_1). The notation for all these root spaces will continue to be L with the appropriate subscript: $L_\lambda, \ldots, L_\sigma$.

5. Odd non-isotropic roots, I.

Lemma 5. Assume that the nonzero element x in L_σ is characteristic under H. Then $x^2 \neq 0$.

Proof. Pick $y \in L_{-\sigma}$ with $(x, y) = 1$. Then (Lemma 3) $xy = h_\sigma$. By the Jacobi identity

$$x^2y + xy.x + yx.x = 0,$$

so that $x^2y = -2xy.x$ (note that $xy = yx$ since x and y are odd). We have $xy.x = h_\sigma x = (\sigma, \sigma)x$ since x is characteristic. Hence $xy.x \neq 0$ and $x^2 \neq 0$.

Thus the even element x^2 lies in the root space $K_{2\sigma}$ and in particular 2σ is a root in J_0. Since in J_0 no root can be twice another, and furthermore the roots in J_0 are non-isotropic, we conclude that the roots in J_0 and J_1 are distinct. So we are henceforth able to speak about even and odd roots unambiguously.

We emphasize the information that *every odd non-isotropic root is equal to half an even root*.

6. *Proof that* $L_0 = 0$. Recall that L_0 is the root subspace of J_1 (the odd part of J) corresponding to the zero linear function on H.

Lemma 6. $L_0 = 0$.

Proof. We must have $L_0K_\alpha = 0 = L_0L_\lambda$, for L_0K_α would lie in the non-existent L_α, and likewise $L_0L_\lambda \subset K_\lambda = 0$. Let E denote the set of all sums of products of elements in the K's and L's (since we have agreed never to add an even element and an odd element, the sums are to be confined within J_0 and J_1). An easy induction using the Jacobi identity shows that E is an ideal in J. By simplicity, $E = J$. Another use of the Jacobi identity shows that the annihilator of L_0 is a subalgebra. Since this annihilator contains the K's and L's, it equals J.

Hence $L_0 = 0$.

It can now reasonably be said that H is a Cartan subalgebra not just of J_0 but of all of J as well.

7. <u>An even root and an isotropic root</u>.

Lemma 7. Either $K_\alpha L_\lambda$ or $K_{-\alpha} L_\lambda$ is 0.

Proof. Assume that on the contrary both are nonzero. Then $\lambda+\alpha$ and $\lambda-\alpha$ are odd roots. There are two possibilities for $\lambda+\alpha$. If it is isotropic then

(4) $$2(\lambda,\alpha) + (\alpha,\alpha) = 0.$$

If $\lambda+\alpha$ is non-isotropic then $2(\lambda+\alpha)$ is an even root. Since the isotropic root λ lies in the rational subspace spanned by α and $2(\lambda+\alpha)$ we deduce from Lemma 1 that α and $\lambda+\alpha$ are orthogonal:

(5) $$(\lambda,\alpha) + (\alpha,\alpha) = 0.$$

Likewise we get from $\lambda-\alpha$ that (6) or (7) holds:

(6) $$-2(\lambda,\alpha) + (\alpha,\alpha) = 0,$$

(7) $$-(\lambda,\alpha) + (\alpha,\alpha) = 0.$$

Any of the four ways of combining (4) or (5) with (6) or (7) yields the contradiction $(\alpha,\alpha) = 0$.

Lemma 8. Assume that $(\alpha,\lambda) \neq 0$. Then:

(a) $2(\alpha,\lambda)/(\alpha,\alpha) = \pm 1$ or ± 2,

(b) If $2(\alpha,\lambda)/(\alpha,\alpha) = -1$ then $\lambda+\alpha$ is an isotropic root,

(c) If $2(\alpha,\lambda)/(\alpha,\alpha) = -2$ then $\lambda+\alpha$ is an odd non-isotropic root and $\lambda+2\alpha$ is an isotropic root.

(d) If $2(\alpha,\lambda)/(\alpha,\alpha) = -1$ or -2 there are no zero-divisors between K_α and L_λ, i. e. the result of multiplying a nonzero element of K_α by a nonzero element of L_λ is nonzero.

Proof. By Lemmas 4 and 7 exactly one of $K_\alpha L_\lambda$, $K_{-\alpha} L_\lambda$ is 0. We first study the consequences of $K_{-\alpha} L_\lambda = 0$. Let x be any nonzero element of L_λ. Then $K_{-\alpha} x = 0$ and the Jacobi identity shows that $K_\alpha x \neq 0$. Multiply x repeatedly by K_α as long as the product stays nonzero; say the last nonzero product occurs after r multiplications, so that it lies in $L_{\lambda + r\alpha}$. Then standard theory shows that $2(\alpha,\lambda)/(\alpha,\alpha) = -r$.

We have that $\lambda+\alpha$ is in any event a root and it is odd. If it is isotropic we find $r = 1$. If it is non-isotropic then $2(\lambda + \alpha)$ is an even root. Since λ lies in the rational subspace spanned by α and $2(\lambda + \alpha)$ we deduce from Lemma 1 that α and $\lambda+\alpha$ are orthogonal, whence $r = 2$.

If at the beginning we had assumed $K_\alpha L_\lambda = 0$ the discussion would have been entirely analogous and would have led to $2(\alpha,\lambda)/(\alpha,\alpha) = 1$ or 2. So the assumption in parts (b), (c), and (d) of the lemma that $2(\alpha,\lambda)/(\alpha,\alpha) = -1$ or -2 is inconsistent with $K_\alpha L_\lambda = 0$ and requires $K_{-\alpha} L_\lambda = 0$. Everything stated in the lemma has now been proved.

8. <u>Axioms for root systems</u>. It is a good idea in the classical theory of simple Lie algebras to pause at a suitable point and put down as axioms the crucial properties thus far established for root systems. Then one proceeds to the clean self-contained geometric problem thus posed.

We shall do the same here. We postulate a finite-dimensional vector space V over a field k of characteristic 0 (which need not be algebraically closed). V is equipped with a nonsingular symmetric form (,). In V a finite set Γ of nonzero vectors is given; we call the members of Γ "roots". Γ is a disjoint set-theoretic union of two subsets whose members we call "even" and "odd". We continue to write α, β, γ for even roots, λ, μ, ν, ρ for odd isotropic roots, σ for an odd non-isotropic root.

There are six axioms. In due course we shall add two more.

1. Along with any vector, Γ contains its negative. A root and its negative have the same parity.

2. The even roots in Γ constitute the system of roots of a semisimple (ungraded) Lie algebra. (As a consequence, all even roots are non-isotropic.)

3. For any two non-orthogonal odd roots the sum or the difference is an even root. In particular, for any σ, 2σ is an even root.

4. $2(\alpha, \lambda)/(\alpha, \alpha) = 0, \pm 1$, or ± 2.

5. If $2(\alpha, \lambda)/(\alpha, \alpha) = -1$ then $\lambda + \alpha$ is a root.

6. If $2(\alpha, \lambda)/(\alpha, \alpha) = -2$ then $\lambda + \alpha$ and $\lambda + 2\alpha$ are roots. $\lambda + \alpha$ is odd.

Note that in axioms 5 and 6 respectively, $\lambda + \alpha$ and $\lambda + 2\alpha$ are isotropic (and hence of course odd). It is also to be noted that axiom 2 makes Lemmas 1 and 2 available.

We can dispose at once of one question concerning parity.

Lemma 9. Suppose that $(\lambda, \mu) \neq 0$ and that $\lambda + \mu$ is a root. Then $\lambda + \mu$ is even.

Proof. If $\lambda + \mu$ is odd it is an odd non-isotropic root so that $\alpha = 2(\lambda + \mu)$ is even. But we find $2(\alpha, \lambda)/(\alpha, \alpha) = 1/2$, a contradiction.

We shall use Lemma 9 quite frequently and shall not quote it.

9. <u>Three isotropic roots</u>, I. In this section and the next it is to be understood that the axioms of §8 for root systems are in force.

We proceed to two lemmas concerning three distinct isotropic roots λ, μ, ν. We fix the notation $(\mu, \nu) = A$, $(\nu, \lambda) = B$, $(\lambda, \mu) = C$.

Lemma 10. Assume that $A = 0$, that B and C are nonzero, and that $\lambda + \mu$, $\lambda + \nu$ are roots. Then $B = -C$.

Proof. We have

$$\frac{2(\lambda + \mu, \nu)}{(\lambda + \mu, \lambda + \mu)} = \frac{B}{C}, \quad \frac{2(\lambda + \nu, \mu)}{(\lambda + \nu, \lambda + \nu)} = \frac{C}{B}.$$

Each of B/C, C/B is equal to ± 1 or ± 2. This forces each to be ± 1. It remains to exclude the possibility B = C. If B = C we have

$$(\lambda + \mu , \lambda + \nu) = 2B \neq 0.$$

The rational subspace spanned by $\lambda+\mu$ and $\lambda+\nu$ is non-isotropic (Lemma 1) and contains the isotropic vector $\mu - \nu$. Hence $\mu = \nu$, whereas we assumed μ and ν to be distinct.

Lemma 11. Assume that A, B, C are nonzero and that $\mu + \nu$, $\nu + \lambda$, $\lambda + \mu$ are roots. Then A + B + C = 0. Moreover, $\lambda + \mu + \nu$ is an isotropic root.

Proof. Suppose that A + B + C is not zero. Note that A + B + C is equal to the inner product of any two of $\mu + \nu$, $\nu + \lambda$, and $\lambda + \mu$. So these three vectors are pairwise non-orthogonal. The ordinary theory of Lie algebras tells us that the ratio between any two of

$$(\mu + \nu , \mu + \nu), (\nu + \lambda , \nu + \lambda), (\lambda + \mu , \lambda + \mu)$$

is 1, 2, 3, 1/2, or 1/3. These ratios are A/B, etc. Furthermore

$$\frac{2(\mu + \nu, \lambda)}{(\mu + \nu, \mu + \nu)} = \frac{B + C}{A}.$$

So B/A and C/A, two ratios from the list above, must add up to 1 or 2. This can only be done if both are 1/2 or both are 1. But if a fraction is 1/2 its reciprocal is 2. We see that A = B = C is forced.

We now have

$$(\lambda+\mu, \lambda+\mu) = (\lambda+\nu, \lambda+\nu) = 2A,$$

$$(\lambda+\mu, \lambda+\nu) = 3A.$$

This combination cannot occur in the roots of a semisimple Lie algebra. This contradiction arose from our initial assumption that A + B + C was not zero. We have proved A + B + C = 0.

To see that $\lambda+\mu+\nu$ is an isotropic root we need only note that

$$\frac{2(\mu+\nu,\lambda)}{(\mu+\nu,\mu+\nu)} = \frac{2(C+B)}{2A} = -1.$$

10. Sum and difference both roots. The lemmas in this section will be used in §12 in the study of the dimensions of root spaces.

Lemma 12. Assume that $(\lambda,\mu) \neq 0$ and that both $\lambda+\mu$ and $\lambda-\mu$ are roots. Then for any ν different from $\pm\lambda, \pm\mu$ one of the following is true: ν is orthogonal to both λ and μ or ν is not orthogonal to either λ or μ. In the latter case it is impossible for both $\lambda+\nu$ and $\lambda-\nu$ to be roots.

Proof. Assume that ν is not orthogonal to both λ and μ, say $(\lambda, \nu) \neq 0$. By changing the sign of ν if necessary we can suppose that $\lambda + \nu$ is a root. We proceed to rule out $(\mu, \nu) = 0$. If (μ, ν) does vanish, Lemma 10 is applicable to yield $(\lambda, \mu) = -(\lambda, \nu)$. Furthermore, Lemma 10 can be applied a second time to the triple $\lambda, -\mu, \nu$ to yield $(\lambda, -\mu) = -(\lambda, \nu)$, a contradiction.

Suppose now that $\lambda + \nu$ and $\lambda - \nu$ are both roots. It is harmless to assume that $\mu + \nu$ is a root. We now apply Lemma 11 to the triples λ, μ, ν and $-\lambda, \mu, \nu$. We get two equations

$$(\lambda, \mu) + (\lambda, \nu) + (\mu, \nu) = 0$$

$$-(\lambda, \mu) - (\lambda, \nu) + (\mu, \nu) = 0,$$

which combine to yield the contradiction $(\mu, \nu) = 0$.

A note is in order about the notation in the next lemma: by $\Gamma - \Delta$ we mean the set-theoretic complement of Δ within Γ, and $(\Delta, \Gamma - \Delta) = 0$ means that every member of Δ is orthogonal to every member of $\Gamma - \Delta$.

Lemma 13 is designed to avoid repeating an argument that will be used several times.

Lemma 13. Let Δ be a root subsystem of the root system Γ. Assume that Δ is spanned by its isotropic members. Assume further that every isotropic root in $\Gamma - \Delta$ is orthogonal to Δ. Then:

(a) Any even root in $\Gamma - \Delta$ which is not orthogonal to Δ has the form $\lambda + \mu$ or $(\lambda + \mu)/2$ with λ, μ non-orthogonal in Δ,

(b) If every even root in $\Gamma - \Delta$ is orthogonal to Δ, then $(\Delta, \Gamma - \Delta) = 0$.

Proof. (a) Take α in $\Gamma - \Delta$, $(\alpha, \Delta) \neq 0$. Then $(\alpha, \nu) \neq 0$ for some ν in Δ. It follows that one of $\nu \pm \alpha$, $\nu \pm 2\alpha$ is an isotropic root, and since it is not orthogonal to ν it must by hypothesis lie in Δ. This gives the desired result.

(b) It remains to prove $(\Delta, \sigma) = 0$ for $\sigma \in \Gamma - \Delta$. Assuming the contrary we have $(\lambda, \sigma) \neq 0$, $\lambda \in \Delta$. One of $\sigma \pm \lambda$ is an even root, and so we may suppose that $\alpha = \sigma + \lambda$ is an even root. α is not orthogonal to λ and so by hypothesis $\alpha \in \Delta$. If $2(\alpha, \lambda)/(\alpha, \alpha) = 1$ or 2 then $\sigma = \alpha - \lambda$ lies in Δ, a contradiction. Therefore $2(\alpha, \lambda)/(\alpha, \alpha) = -1$ or -2. But this also leads to trouble for we find

$$2(2\sigma, \lambda)/(2\sigma, 2\sigma) = -1/4 \text{ or } -1/3,$$

a contradiction since 2σ is an even root.

11. Odd non-isotropic roots, II. In this section and the next we return to our "concrete" simple graded Lie algebra J.

Lemma 14. L_σ is one-dimensional.

Proof. We continue the analysis of σ where Lemma 5 left off. Pick x as in that lemma. If

L_σ (and hence $L_{-\sigma}$) is more than one-dimensional, we can find a nonzero z in $L_{-\sigma}$ with $xz = 0$. From the Jacobi identity, as in Lemma 5, we get $x^2z = -xz.x = 0$. Now x^2, $h_{2\sigma}$, and an appropriate element in $K_{-2\sigma}$ form a simple 3-dimensional ordinary Lie algebra Y. We have $h_{2\sigma}x^2 = 4(\sigma,\sigma)x^2$, $h_{2\sigma}z = -2(\sigma,\sigma)z$. One knows (from the representation theory of Y) that this is inconsistent with $x^2z = 0$.

12. Two-dimensional root spaces.

Lemma 15. Assume $(\lambda,\mu) \neq 0$. Then L_λ and L_μ are at most 2-dimensional.

Proof. $L_\lambda L_\mu$ or $L_\lambda L_{-\mu}$ must be nonzero (Lemma 4). Let us say $L_\lambda L_\mu \neq 0$. We have a bilinear multiplication of L_λ and L_μ into the 1-dimensional space $K_{\lambda+\mu}$. Suppose, for instance, that the dimension of L_μ is 3 or more. Pick $x \in L_\lambda$, $y \in L_{-\lambda}$ with $xy = h_\lambda$ (Lemma 3). We can find $z \neq 0$ in L_μ annihilating x and y. In the equation

$$xy.z + yz.x + zx.y = 0$$

the first term is nonzero, the second and third zero.

Lemma 16. Assume that $(\lambda,\mu) \neq 0$ and that L_λ is 2-dimensional. Then L_μ is also 2-dimensional. Furthermore $L_\lambda L_\mu$ and $L_\lambda L_{-\mu}$ are both nonzero.

Again we quote Lemma 4 to assert that at least one of $L_\lambda L_\mu$, $L_\lambda L_{-\mu}$ is nonzero. Let us say

$L_\lambda L_{-\mu} \neq 0$. Write $\alpha = \mu - \lambda$. We have

$$\frac{2(\alpha,\lambda)}{(\alpha,\alpha)} = \frac{2(\mu-\lambda,\lambda)}{(\mu-\lambda,\mu-\lambda)} = -1.$$

By part (d) of Lemma 8, there are no zero-divisors between K_α and L_λ in the multiplication that lands in L_μ. Hence L_μ is at least 2-dimensional. By Lemma 15, it is exactly 2-dimensional. If we had initially assumed $L_\lambda L_\mu \neq 0$ we would have found $L_{-\mu}$ to be 2-dimensional. But L_μ and $L_{-\mu}$ have the same dimension.

We turn to the final statement of the lemma. Suppose that on the contrary $L_\lambda L_\mu = 0$. Multiplication of $L_{-\lambda}$ by L_μ lands in the 1-dimensional space $K_{\mu-\lambda}$. Hence we can find nonzero elements $x \in L_{-\lambda}$, $y \in L_\mu$ with $xy = 0$. Pick $z \in L_\lambda$ with $xz = h_\lambda$. In

$$xy.z + yz.x + zx.y = 0$$

the first two terms vanish but not the third.

We need a lemma assuring us that in suitable circumstances orthogonality of roots implies that the product of the corresponding root spaces vanishes.

Lemma 17. (a) If $(\lambda,\mu) = 0$ for $\mu \neq -\lambda$ then $L_\lambda L_\mu = 0$. (b) If $(\lambda,\sigma) = 0$ then $L_\lambda L_\sigma = 0$. (c) If $(\lambda,\alpha) = 0$ then $L_\lambda K_\alpha = 0$.

Proof. (a) If $L_\lambda L_\mu \neq 0$ then $\lambda + \mu$ is an even root, hence non-isotropic. But the sum of two orthogonal isotropic vectors is isotropic.

(b) Assume $L_\lambda L_\sigma \neq 0$. Then $\lambda + \sigma$ is an even root. Moreover 2σ is an even root. We have $(\lambda + \sigma, 2\sigma) \neq 0$. But λ is in the rational subspace spanned by $\lambda + \sigma$ and 2σ, and this contradicts Lemma 1.

(c) From $L_\lambda K_\alpha' \neq 0$ we get, as in the proof of lemma 7, that either equation (4) or equation (5) holds. In conjunction with $(\lambda, \alpha) = 0$ this leads to the contradiction $(\alpha, \alpha) = 0$.

Lemma 18. For any λ there exists a μ with $(\lambda, \mu) \neq 0$.

Proof. Assume the contrary. Let Γ denote the system of all roots in J; let Δ denote the root subsystem consisting just of λ and $-\lambda$ (it is indeed a subsystem). The hypotheses of Lemma 13 are fulfilled. Evidently the conclusion reached in part (a) of Lemma 13 cannot be tolerated. So we pass to part (b) and conclude that λ is orthogonal to $\Gamma - \Delta$. Since λ is in addition orthogonal to itself, it is actually orthogonal to all roots. Lemma 17 now tells us that L_λ annihilates all root spaces other than $L_{-\lambda}$. Ditto, interchanging λ and $-\lambda$. Therefore $L_\lambda + L_\lambda L_{-\lambda} + L_{-\lambda}$ is an ideal and so, by simplicity, $= J$. Therefore $H = L_\lambda L_{-\lambda}$. Let x be chosen under H in L_λ. Then $x \cdot L_\lambda L_{-\lambda} = 0$ ∵ λ is isotropic; i.e. $Hx = 0$. But this says $\lambda \equiv 0$. ✗. ~~We now argue easily that $L_\lambda + L_\lambda L_{-\lambda}$ is an ideal in J. Simplicity gives a contradiction.~~

At this point we launch an investigation of the structure of J in the case where some L_λ is 2-dimensional. The discussion will proceed in several stages.

(1) Apply Lemma 18 to get μ with $(\lambda, \mu) \neq 0$.

(2) Apply Lemma 16 to learn that L_μ is also 2-dimensional and that $L_\lambda L_\mu$ and $L_\lambda L_{-\mu}$ are nonzero, making both $\lambda + \mu$ and $\lambda - \mu$ roots.

(3) Let ν be any isotropic root other than $\pm\lambda$, $\pm\mu$. We claim that ν is orthogonal to λ and μ. If, for example, $(\lambda, \nu) \neq 0$ we apply Lemma 16 again to get $\lambda \pm \nu$ both to be roots. Then Lemma 12 gives the desired contradiction.

(4) Let Γ be the full root system of J and Δ the subsystem consisting of $\pm\lambda$, $\pm\mu$, $\pm(\lambda + \mu)$, and $\pm(\lambda - \mu)$. What we have just seen in (3) shows that the hypotheses of Lemma 13 are fulfilled. The possibility of (a) in that lemma is ruled out at once. So we conclude that $(\Delta, \Gamma - \Delta) = 0$.

(5) Let W be the subspace of J spanned by $L_{\pm\lambda}$, $L_{\pm\mu}$, $K_{\pm(\lambda+\mu)}$, $K_{\pm(\lambda-\mu)}$, $L_\lambda L_{-\lambda}$, and $L_\mu L_{-\mu}$. We claim that W is a subalgebra of J. Closure of W under multiplication by H is apparent (and will be used again in (6) below). The rest of the closure of W under multiplication is either visible or covered by the following remarks. We have $h_\lambda \in L_\lambda L_{-\lambda}$ and $h_\mu \in L_\mu L_{-\mu}$ (Lemma 3). $K_{\lambda+\mu} K_{-\lambda-\mu}$ consists of scalar multiples of $h_{\lambda+\mu} = h_\lambda + h_\mu$. A similar remark covers $K_{\lambda-\mu} K_{\mu-\lambda}$. We have $K_{\lambda+\mu} K_{\lambda-\mu} = 0$, etc. Finnaly, all products like $L_\mu K_{\lambda+\mu}$ vanish, as follows for instance from Lemmas 7 and 8.

(6) We claim that W is actually an ideal in J. Since we know that W is a subalgebra closed under multiplication by H, it only remains to check closure of W under multiplication by a root space external to W, say R. It follows from item (4) above and Lemma 17 that R annihilates $L_{\pm\lambda}$ and $L_{\pm\mu}$. By the Jacobi identity, R also annihilates products such as $L_{\lambda} L_{-\lambda}$ and $L_{\lambda} L_{\mu}$. By Lemma 16, $L_{\lambda} L_{\mu} \neq 0$ and therefore spans the 1-dimensional space $K_{\lambda+\mu}$. Hence R annihilates $K_{\lambda+\mu}$ and the same is true for $K_{-(\lambda+\mu)}$ and $K_{\pm(\lambda-\mu)}$. This completes the verification that W is an ideal in L.

(7) By simplicity, W = J. We next examine $L_{\lambda} L_{-\lambda}$ more closely. Pick a basis a, b for L_{λ} with b characteristic under H, and likewise a basis c, d for $L_{-\lambda}$ with d characteristic. The four products ac, ad, bc, bd span $L_{\lambda} L_{-\lambda}$. By Lemma 3 the last three are scalar multiples of h_{λ}. We now provide an argument showing that the same is true for e = ac.

The Jacobi identity for the triple a, a, c, together with the fact that $a^2 = 0$, tells us that ea = 0. This shows that $\lambda(e) = 0$ and so e acts as a nilpotent linear transformation on L_{λ}. Thus e annihilated b as well as a and we have $eL_{\lambda} = 0$.

We proceed to argue that e acts as a scalar on L_{μ}. Pick $x \neq 0$ in $K_{\mu-\lambda}$. By part (d) of of Lemma 8, $L_\lambda x = L_\mu$. So it suffices to study the action of e on $L_\lambda x$. We have just seen that e annihilates L_λ ; ex is a scalar multiple of x since $K_{\mu-\lambda}$ is 1-dimensional; by the Jacobi identity e acts as a scalar on $L_\lambda x = L_\mu$.

All this applies equally well to $L_\mu L_{-\mu}$. The upshot is that we have at last proved that all of H acts diagonally on every root space. So (Lemma 3) $L_\lambda L_{-\lambda}$ is spanned by h_λ and $L_\mu L_{-\mu}$ is spanned by h_μ.

(8) At this point we interpolate the observation that we have acquired a new axiom for our abstract root systems. Since J is simple it is in particular equal to its square. We deduce that H is spanned by the h_α's and h_λ's. For the abstract root systems of §8, this means the following statement: V is spanned by Γ. (This is not really a significant addition if the subspace of V spanned by Γ is nonsingular, but up to this point the possibility existed that this subspace was singular.) This axiom will not be invoked until the final moments of the classification (§19).

(9) We have that J is 14-dimensional and that its root space structure is fully at hand. That such an algebra exists will be seen in §19. It is a fact that the multiplication can be pinned down uniquely (up to isomorphism), and so J must be isomorphic to this 14-dimensional algebra. This discussion will

fit better in the projected second paper of the series, and so is omitted here. But we venture to make the assertion a part of the following summarizing theorem.

Theorem 1. Let $J = J_0 \cup J_1$ be a simple graded Lie algebra over an algebraically closed field of characteristic 0. Assume that J_0 is the direct sum of a semisimple algebra and an abelian algebra. Assume further that J possesses a nonsingular associative form which is symmetric on J_0, skew on J_1, and makes J_0 and J_1 orthogonal. Decompose J relative to a Cartan subalgebra of J_0, and write α for a root on J_0, λ for an isotropic root on J_1, and δ for a non-isotropic root on J_1. Then the root space for α or δ is 1-dimensional. The root space for λ is at most 2-dimensional. The 2-dimensional case occurs only in a certain 14-dimensional algebra. The system of roots in J fulfils the axioms set forth in §8, augmented by the statement that V is spanned by Γ.

From now on we are entitled to assume that root spaces are 1-dimensional. We proceed to acquire a new piece of information.

Lemma 19. Assume that $(\lambda, \mu) \neq 0$ and that L_λ and L_μ are 1-dimensional. Then exactly one of $\lambda + \mu$, $\lambda - \mu$ is a root.

Proof. We can assume $L_\lambda L_\mu \neq 0$. We proceed to rule out the possibility that $\mu-\lambda$ is a root. If so, it is an even root (Lemma 9); call it α. We find $2(\alpha,\lambda)/(\alpha,\alpha) = -1$. Part (d) of Lemma 8 tells us that there are no zero-divisors in the multiplication of K_α and L_λ that lands in L_μ. Take nonzero elements a and x in K_α and L_λ. Since L_μ is 1-dimensional it is spanned by ax. The Jacobi identity for the triple x, x, a, in conjunction with the fact that $x^2 = 0$, yields the information x.ax = 0. Thus $L_\lambda L_\mu = 0$, a contradiction.

In our investigation of abstract root systems we are now in a position to adjoin still another axiom: $(\lambda,\mu) \neq 0$ implies that exactly one of $\lambda+\mu$, $\lambda-\mu$ is a root. This will be invoked for the first time in §19.

13. <u>Three isotropic roots</u>, II. The axioms of §8 are in force. In the notation of §9 there is one more case to explore.

Lemma 20. Assume that A, B, C are nonzero and that $\mu-\nu$, $\nu-\lambda$, $\lambda-\mu$ are roots. Then either A, B, C are all equal or two are equal and the third is equal to twice or three times this common value.

Proof. Note that

$$(\mu - \nu, \nu - \lambda) = A + B - C.$$

If two such expressions, say $A + B - C$ and $A + C - B$, are zero then $A = 0$, a contradiction. So at least two are nonzero. It follows that $\mu - \nu$, $\nu - \lambda$, $\lambda - \mu$ all lie in the same simple summand of the semisimple Lie algebra giving rise to the root system of even roots. Note that the sum of these three vectors is 0.

The situation then is that we have two non-orthogonal vectors in the root system of a simple Lie algebra with their sum again a root. They lie in a root system of type A_2, B_2, or G_2. The possibilities for the squares of thier lengths are:

(i) All equal in the case of A_2,

(ii) Two of them equal with the third twice as big in the case of B_2,

(iii) Two of them equal with the third three times as big in the case of G_2.

This proves the lemma.

14. The root system Γ_2. Let us investigate the last case that occurs in Lemma 20. Suppose A is the number which is the triple of B and C. Let us normalize the form so that $B = C = 1$, $A = 3$. We have

$$\frac{2(\mu - \lambda, \nu)}{(\mu - \lambda, \mu - \lambda)} = \frac{2(A - B)}{-2C} = -2.$$

So $\nu + \mu - \lambda$ and $\nu + 2\mu - 2\lambda$ are odd roots, non-isotropic and isotropic respectively. $2(\nu + \mu - \lambda)$ is an even root. By symmetry, $\mu + 2\nu - 2\lambda$ is an isotropic root. Next

$$\frac{2(\nu-\lambda, \nu+2\mu-2\lambda)}{(\nu-\lambda, \nu-\lambda)} = \frac{2(2A-3B-2C)}{-2B} = -1,$$

so that $2\nu + 2\mu - 3\lambda$ is an isotropic root. Finally

$$\frac{2(\lambda-\mu, \mu-\nu)}{(\lambda-\mu, \lambda-\mu)} = \frac{2(C-B+A)}{-2C} = 3,$$

so that $\mu - \nu + 2(\lambda - \mu) = 2\lambda - \mu - \nu$ and $\mu - \nu + 3(\lambda - \mu) = 3\lambda - 2\mu - \nu$ are even roots, as is $3\lambda - 2\nu - \mu$ by symmetry.

This mass of roots can be put in a prettier form. Invent vectors p, q, r with $p + q + r = 0$, $(p, p) = (q, q) = (r, r) = -2$, $(q, r) = (r, p) = (p, q) = 1$. Throw in a vector f perpendicular to p, q, r with $(f, f) = 2$. Pair off as follows.

λ	$f - p$
μ	$f + q$
ν	$f + r$
$\mu + \nu - \lambda$	f
$2(\mu + \nu - \lambda)$	$2f$
$\nu + 2\mu - 2\lambda$	$f - r$
$\mu + 2\nu - 2\lambda$	$f - q$
$2\mu + 2\nu - 3\lambda$	$f + p$
$\mu + \nu - 2\lambda$	p
$\mu - \nu$	$q - r$
$\lambda - \nu$	q
$\lambda - \mu$	r
$2\mu + \nu - 3\lambda$	$p - r$
$2\nu + \mu - 3\lambda$	$p - q$

In the new version we have the even roots p, q, r, $q - r$, $r - p$, $p - q$, $2f$; the isotropic roots $f \pm p$, $f \pm q$, $f \pm r$; and the odd non-isotropic root f (and the negatives of all these roots). It is toleably easy to see that the system closes off nicely and satisfies all the axioms. The system is 31-dimensional and its even part is the root system of $G_2 \oplus A_1$. Let us call this root system Γ_2.

15. Four isotropic roots, I. Let us return to the setup of Lemma 11. The isotropic roots $\pm\lambda$, $\pm\mu$, $\pm\nu$, $\pm(\lambda+\mu+\nu)$, together with the even roots $\pm(\mu+\nu)$, $\pm(\nu+\lambda)$, $\pm(\lambda+\mu)$ close splendidly to form a root system fulfilling the axioms. We shall call it $\Gamma(A, B, C)$. We proceed to study the possibility of enlarging $\Gamma(A, B, C)$.

Lemma 21. Let   be a root system which does not contain a copy of Γ_2. Assume that Γ contains a subsystem Δ of the form $\Gamma(A, B, C)$ with no two of A, B, C equal. Suppose that ρ lies in $\Gamma - \Delta$. Write $(\rho, \lambda) = D$, $(\rho, \mu) = E$, $(\rho, \nu) = F$, and assume that D, E, and F are not all 0. Then:

(a) D, E, F are all nonzero,

(b) It cannot be the case that $\rho+\lambda$, $\rho+\mu$, and $\rho+\nu$ are all roots,

(c) Suppose that $\rho+\lambda$, $\rho+\mu$, and $\rho-\nu$ are roots. Then $C = -3A$ and furthermore either $B = 2A$ or $A = 2B$. If $B = 2A$ we have either

$$D = 2A,\ E = A,\ F = -2A$$

or

$$D = A,\ E = 2A,\ F = -A.$$

Proof. (a) Suppose that on the contrary $F = 0$. We can assume $D \neq 0$, and by changing the sign of ρ if necessary we can arrange that $\rho+\lambda$ is a root. We apply Lemma 10 to the triple λ, ν, ρ to get $B = -D$.

If E = 0 we similarly get C = -D from the triple λ, μ, ρ. So E $\neq$ 0. We now make a case distinction.

I. $\rho + \mu$ is a root. Lemma 11 is applicable to the triple λ, μ, ρ to yield C + D + E = 0. From Lemma 10 applied to the triple ρ, μ, ν we get A = -E. So C + D + E = 0 becomes C * B * A = 0. In conjunction with A + B + C = 0 we get C = 0, a contradiction.

II. $\rho - \mu$ is a root. Lemma 20 is applicable to the triple $-\lambda, \mu, \rho$. The pertinent triple of inner products is -C, -D, E. So two of these three are equal and the third is equal or double (but not triple, since we have excluded the presence of Γ_2). Now we know that -D and B are equal so the triple can be rewritten -C, B, E. Equality of -C and B is ruled out since it would force A = 0. So we must be in the case where one of -C, B, E is double the others and it must be -C or B that is the double. In any event B = -2C or C = -2B. These two statements are symmetric, combining with A + B + C = 0 to make A equal to B or C. This finishes part (a) of Lemma 21.

(b) Apply Lemma 11 to the three following triples.

μ, ν, ρ: $\qquad A + E + F = 0,$

ν, λ, ρ: $\qquad B + D + F = 0,$

λ, μ, ρ: $\qquad C + D + E = 0.$

Adding, we get

$$(A + B + C) + 2(D + E + F) = 0$$

and we deduce $D + E + F = 0$. Hence $D = A$, $E = B$, $F = C$. Now we find

$$(\lambda + \mu, \lambda + \mu) = 2C,$$

$$(\nu + \rho, \nu + \rho) = 2C,$$

$$(\lambda + \mu, \nu + \rho) = B + A + D + E = -2C.$$

This forces $\lambda + \mu = -(\nu + \rho)$ and yields $\rho = -(\lambda + \mu + \nu)$, a contradiction since ρ was assumed not to be in Δ.

(c) As in part (b), the triple λ, μ, ρ yields

(8) $\qquad C + D + E = 0.$

Lemma 20 is applicable to $-\lambda, \nu, \rho$, with inner products

(9) $\qquad -B, -D, F$

and $-\mu, \nu, \rho$, with inner products

(10) $\qquad -E, F, -A.$

Now for a raft of cases.

(I) Equality holds in both (9) and (10). Then $-B = F = -A$, a contradiction.

(II) Equality holds in just one of (9) and (10). We shall assume that it is (9) where equality holds; the opposite assumption gives identical results except that the roles of A and B are interchanged. With $F = -B$ in hand from (9) we cannot tolerate $F = -A$. Therefore F or $-A$ is the one that is the double in (10).

(II_1) $-A$ is the double. We have

$$-B = -D = F, \tag{11}$$

$$-E = F = -A/2. \tag{12}$$

Equations (8), (11), and (12) enable us to express everything in terms of B. We find $E = B$, $A = 2B$, $C = -D - E = -2B$. But this, in conjunction with $A + B + C = 0$, forces $B = 0$.

(II_2) F is the double. We have

$$-E = -A = F/2. \tag{13}$$

We use (8), (11), and (13) to express everything in terms of A:

$$B = 2A, \ C = -3A, \ D = 2A, \ E = A, \ F = -2A.$$

(III) Equality holds in neither (9) nor (10). On the face of it we might make nine case distinctions. But symmetry cuts this down to the six we display in tabular form.

Element which is double in (9)	Element which is double in (10)	Result
F	F	A = B
-B	-A	A = B
-D	-E	A = B
F	-E	We have B = D, F = -2B, F = -A, E = 2A. In terms of B we get A = 2B, D = B, E = 4B. Then (8) yields C = -5B and A + B + C = 0 is violated
F	-A	We have B = D, F = -2B, -E = F, A = 2E. In terms of B we get A = 4B, D = B, E = 2B. From (8), C = -3B. Again A + B + C = 0 is violated.
B	-E	We have -D = F, B = 2D, -A = F, E = 2A and deduce B = 2A, C = -3A, D = A, E = 2A, and F = -A.

With this the proof of Lemma 21 is complete.

16. <u>The root system</u> Γ_3. Here is a root system which we label Γ_3. Its system of even roots is a direct sum of B_3 and A_1, as follows. We represent B_3 by the vectors

$$e_1, e_2, e_3, e_2 \quad e_3, e_3 \quad e_1, e_1 \quad e_2$$

and their negatives, where the e_i's are orthogonal and satisfy $(e_i, e_i) = 2$. The A_1 part is of course orthogonal to the e_i's and is spanned by a vector f with $(f, f) = -6$. The 16 odd vectors are

$$(\pm e_1 \pm e_2 \pm e_3 \pm f)/2,$$

all combinations of signs being used. We leave it to the reader to check that all the axioms are indeed fulfilled.

Lemma 22. Let Γ be a root system which does not contain a copy of Γ_2. Suppose that contains a copy Δ of $\Gamma(1, 2, -3)$ and also a root ρ not in Δ and not orthogonal to Δ. Then Γ contains a copy of Γ_3.

Proof. We apply Lemma 21. By changing the sign of ρ if necessary we can arrange to be in the situation covered by part (c) of that lemma, and we now show how to build up Γ_3 in each of the two cases.

We pair λ with $(-e_1-e_2 - e_3 + f)/2$, μ with $(e_1 + e_2 + e_3 + f)/2$ and ν with $(-e_1 - e_2 + e_3 - f)/2$. In the case $D = 2$, $E = 1$, $F = -2$ we pair ρ with $(-e_1 + e_2 - e_3 - f)/2$. In the case $D = 1$, $E = 2$, $F = -1$ we pair ρ with $(e_1 - e_2 + e_3 - f)/2$. It is a routine matter to verify that all inner products thus far arecorrect and that Γ_3 gets generated.

17. Injectivity lemmas.

Lemma 23. Let Γ be a root system containing Γ_2 as a subsystem. Then $(\Gamma_2, \Gamma - \Gamma_2) = 0$.

Proof. In this proof we depart from the procedure envisaged in Lemma 13 and tackle first an even $\alpha \in \Gamma - \Gamma_2$. The vectors

$$\pm p, \pm q \pm r, \pm(q - r), \pm(r - p), \pm(p - q)$$

constitute a copy of the root system of G_2. We know that it is impossible to adjoin to G_2 a non-orthogonal vector; hence α is orthogonal to p, q, and r. We are done with α if it is also orthogonal to f and so we assume $(\alpha, f) \neq 0$. Then α is not orthogonal to any isotropic root, for instance p + f. We have

$$\frac{2(\alpha, f)}{(\alpha,\alpha)} = \frac{2(\alpha, p+f)}{(\alpha,\alpha)} = \pm 1 \text{ or } \pm 2.$$

But ± 2 is ruled out since the consequence $2(\alpha, 2f)/(\alpha,\alpha) = \pm 4$ is an impossible relation between the even roots α and 2f. We normalize the sign of α so that $2(\alpha, f)/(\alpha,\alpha) = -1$. The relation $2(\alpha, 2f)/(\alpha,\alpha) = -2$ between α and 2f shows that $(2f, 2f)/(\alpha,\alpha) = 2$. Since $(f, f) = 2$, we get $(\alpha,\alpha) = 4$, and then $(\alpha, f) = -2$. In particular, $(\alpha + f, f) = 0$. Since $\alpha + f$ is orthogonal to f, p, q, r it cannot lie in the subspace they span, and neither can α.

We have that $\alpha + p + f$ is an isotropic root. Its inner product with q + f is (p, q) = 1. So either the sum or the difference of $\alpha + p + f$ and q + f is an even root. But neither the sum nor the difference is orthogonal to p. So then even root in question must lie in Γ_2. This forces α to lie in the subspace spanned by Γ_2, which we have seen to be impossible. We have proved α to be orthogonal to Γ_2.

We tackle an isotropic λ in $\Gamma - \Gamma_2$. It suffices to prove λ orthogonal to all the isotropic roots in Γ_2, for they span Γ_2. Suppose that λ is not orthogonal to p + f. Normalize λ so that $\lambda + p + f$ is a root, say β. We claim that β cannot lie in Γ_2, for if we check $\beta - p - f$ for isotropy in every case we find that the successful cases put λ into Γ_2. So $\beta \notin \Gamma_2$. By what we have already proved, $(\beta, \Gamma_2) = 0$. In particular, $(\lambda + p + f, p + f) = 0$, yielding the contradiction $(\lambda, p + f) = 0$.

Part (b) of Lemma 13 completes the proof.

In the remaining "injectivity" proofs the plan set forth in Lemma 13 will be used. At this point we make a remark that simplifies the use of Lemma 13. Suppose that the subsystem Δ* of that lemma has the following property (which will become an axiom in §19): $(\lambda, \mu) \neq 0$ implies that exactly one of $\lambda \pm \mu$ is a root. Then the possibility $\alpha = \lambda + \mu$ in (a) of Lemma 13 is clearly ruled out; only $\alpha = (\lambda + \mu)/2$ need be contemplated.

Lemma 24. Let Γ be a root system containing a subsystem Δ of the form $\Gamma(A, B, C)$ with no two of A, B, C equal and none equal to twice another.

*this is a slip. We need the axiom on all of Γ. this comes up in Lemmas 24, 25.

Assume further that Γ does not contain a copy of Γ_2. Then $(\Delta, \Gamma - \Delta) = 0$.

Proof. Let ρ be isotropic in $\Delta - \Gamma$. We prove $(\rho, \Delta) = 0$. Assume on the contrary that (ρ, λ), (ρ, μ), and (ρ, ν) are not all 0. Lemma 21 can be quoted, first telling us that these three numbers are all nonzero. By symmetry, and by changing the sign of ρ if necessary, we can arrange that either

$$\rho + \lambda, \quad \rho + \mu, \quad \rho + \nu$$

are all roots, or

$$\rho + \lambda, \quad \rho + \mu, \quad \rho - \nu$$

are all roots. Lemma 21 then yields a contradiction.

In view of the remark just preceding this lemma, Lemma 13 finishes the proof as soon as we rule out the possible existence in $\Gamma - \Delta$ of an even root equal to half the sum of two non-orthogonal isotropic roots in Δ. In view of the symmetry that holds among the isotropic roots of $\Delta = \Gamma(A, B, C)$ there is really just one case to examine, say $\alpha = (\mu - \lambda)/2$. We cannot have $(\alpha, \nu) = 0$ for this implies A = B. So $(\alpha, \nu) \neq 0$ and $\nu \pm \alpha$ or $\nu \pm 2\alpha$ is an isotropic root. Since this isotropic root is not orthogonal to ν it lies in Δ by what we proved in the preceding paragraph. But identifying it with an isotropic root in Δ

gives a linear dependence between λ, μ, and ν, and this is unacceptable for they are linearly independent (their matrix of inner products

$$\begin{pmatrix} 0 & C & B \\ C & 0 & A \\ B & A & 0 \end{pmatrix}$$

has nonzero determinant).

Lemma 25. Let Γ be a root system which does not contain a copy of Γ_2. Assume that Γ contains a subsystem Δ isomorphic to Γ_3. Then $(\Delta, \Gamma - \Delta) = 0$.

Proof. We take Δ in the form exhibited above for Γ_3. Our first task is to prove $(\rho, \Delta) = 0$ for $\rho \in \Gamma - \Delta$. Now e_i is an even root. Hence $2(e_i, \rho)/(e_i, e_i) = 0, \pm 1$, or ± 2. Since $(e_i, e_i) = 2$ this means $(e_i, \rho) = 0, \pm 1$, or ± 2. Next, $e_i + e_j$ $(i \neq j)$ is an even root. Hence

$$2((e_i + e_j), \rho)/(e_i + e_j, e_i + e_j)$$

is also $0, \pm 1$, or ± 2, whence

$$[(e_i, \rho) + (e_j, \rho)]/2 = 0, \pm 1, \text{ or } \pm 2,$$

and in particular is integral. Therefore (e_1, ρ), (e_2, ρ), and (e_3, ρ) all have the same parity. f is also an even root, so that $2(f, \rho)/(f, f) = 0, \pm 1$, or ± 2. Since $(f, f) = -6$ this means $(f, \rho) = 0, \pm 3$, or ± 6.

Now take the three isotropic roots

$$\lambda = (-e_1 - e_2 - e_3 + f)/2,$$

$$\mu = (e_1 + e_2 + e_3 + f)/2,$$

$$\nu = (-e_1 - e_2 + e_3 - f)/2.$$

They satisfy $(\mu, \nu) = 1$, $(\nu, \lambda) = 2$, $(\lambda, \mu) = -3$ and the sum of any two of them lies in Δ. So Lemma 21 is applicable. One possibility is that ρ is orthogonal to λ, μ, ν. Otherwise either final conclusion of Lemma 21 tells us that (ρ, λ), (ρ, μ), and (ρ, ν) are all ± 1 or ± 2. In any event these numbers are integral. It follows that (f, ρ) has the same parity as the (e_i, ρ)'s. There are now two cases.

1. The (e_i, ρ)'s and (f, ρ) are odd. Symmetry in Γ_3 (change the signs of e_1, e_2, e_3, f as necessary) allows us to assume that each $(e_i, \rho) = 1$ and $(f, \rho) = 3$. But then $(\mu, \rho) = 3$.

2. The (e_i, ρ)'s and (f, ρ) are even. It follows that (ρ, λ), (ρ, μ), and (ρ, ν) all have the same parity. But either final conclusion of Lemma 21 is now violated.

So we must retreat to the conclusion that ρ is orthogonal to λ, μ, ν. Since (for instance) μ could have been any isotropic root of Γ_3 by symmetry, we deduce that ρ is orthogonal to every isotropic root in Γ_3 and hence to all of $\Gamma_3 = \Delta$.

As in the proof of Lemma 24, the remaining task is to rule out the possible presence in $\Gamma - \Delta$ of an even root α equal to half the sum of two non-orthogonal isotropic roots in Δ. The typical possibilities are $\alpha = (e_1 + e_2 + e_3)/2$ and $\alpha = -(e_1 + e_2 + f)/2$. In the first case

$$\frac{2(\alpha, (e_1 - e_2 + e_3 + f)/2)}{(\alpha, \alpha)} = \frac{1}{6},$$

and in the second

$$\frac{2(\alpha, (-e_1 - e_2 + e_3 + f)/2)}{(\alpha, \alpha)} = \frac{1}{6}.$$

18. <u>Four isotropic roots</u>, II.

Lemma 26. Let Γ be a root system not containing a copy of $\Gamma(1, 2, -3)$. Then the following is impossible in Γ: $(\lambda, \mu_1) = 1$, $(\lambda, \mu_2) = \pm 2$, $(\lambda, \mu_3) = \pm 2$, $\lambda + \mu_i$ is a root $(i = 1, 2, 3)$, $\mu_3 \neq \pm\mu_2$.

Proof. We begin by noting that Lemma 10 rules out $(\mu_1, \mu_2) = 0$ and $(\mu_1, \mu_3) = 0$. We make a survey of the possibilities for μ_1, μ_2. If $\mu_1 + \mu_2$ is a root, then Lemma 11 applies to the triple λ, μ_1, μ_2. The choice $(\lambda, \mu_2) = 2$ leads to the forbidden presence of $\Gamma(1, 2, -3)$. So $(\lambda, \mu_2) = -2$ and $(\mu_1, \mu_2) = 1$. If $\mu_1 - \mu_2$

is a root, Lemma 20 applies to the triple $-\lambda$, μ_1, μ_2. We have $(-\lambda, \mu_1) = -1$, $(-\lambda, \mu_2) = \pm 2$ thus far, and so $(\lambda, \mu_2) = 2$ and $(\mu_1, \mu_2) = -1$ are forced. Note that in any event $(\mu_1, \mu_2) = \pm 1$. All this applies equally well with μ_2 replaced by μ_3. There now follows the usual array of cases and subcases.

(I) $(\mu_2, \mu_3) = 0$. By Lemma 10, $(\lambda, \mu_2) = -(\lambda, \mu_3)$. Symmetry between μ_2 and μ_3 allows us to write $(\lambda, \mu_2) = 2$, $(\lambda, \mu_3) = -2$. The remarks in the preceding paragraph yield: $(\mu_1, \mu_2) = -1$, $(\mu_1, \mu_3) = 1$, $\mu_1 - \mu_2$ and $\mu_1 + \mu_3$ roots. But now the triple μ_1, $-\mu_2$, μ_3 fails to fulfil Lemma 10.

(II) $(\mu_2, \mu_3) \neq 0$, $\mu_2 + \mu_3$ a root. We apply Lemma 11 to λ, μ_2, μ_3. The conclusion is that (λ, μ_2) and (λ, μ_3) must have the same sign, and $(\mu_2, \mu_3) = \pm 4$. But in conjunction with $(\mu_1, \mu_2) = \pm 1$, $(\mu_1, \mu_3) = \pm 1$, we have a conflict with Lemma 11 or Lemma 20, whichever is applicable.

(III) $(\mu_2, \mu_3) \neq 0$, $\mu_2 - \mu_3$ a root. Lemma 20 is applicable to $-\lambda$, μ_2, μ_3. The conclusion is that (λ, μ_2) and (λ, μ_3) are equal, and that (μ_2, μ_3) is ± 2, ± 4, or ± 6. The possibility ± 4 is ruled out as in the preceding paragraph, and essentially the same argument applies to ± 6. So $(\mu_2, \mu_3) = \pm 2$.

(IIIa) $(\lambda, \mu_2) = (\lambda, \mu_3) = 2$. Then $(\mu_2, \mu_3) = -2$, $(\mu_1, \mu_2) = (\mu_1, \mu_3) = -1$, $\mu_1 - \mu_2$ and $\mu_1 - \mu_3$ are roots. Since

$$\frac{2(\mu_1 - \mu_2, \mu_3)}{(\mu_1 - \mu_2, \mu_1 - \mu_2)} = 1,$$

$\mu_1 - \mu_2 - \mu_3$ is a root. The triple λ, μ_1, $\mu_1 - \mu_2 - \mu_3$ has the forbidden array of inner products: 1, 2, -3.

(IIIb) $(\lambda, \mu_2) = (\lambda, \mu_3) = -2$. Then $(\mu_2, \mu_3) = 2$, $(\mu_1, \mu_2) = (\mu_1, \mu_3) = 1$, $\mu_1 + \mu_2$ and $\mu_1 + \mu_3$ are roots. Lemma 20 applied to the triple $-\mu_1$, μ_2, μ_3 gives the unacceptable inner products -1, -1, 2.

19. <u>The general case</u>. In this section we shall at last complete the classification of root systems. As stated earlier, we add at this point two axioms to join the six in §8. From now on we assume: Γ spans V, and $(\lambda, \mu) \neq 0$ implies that only one of $\lambda + \mu$, $\lambda - \mu$ is a root.

There is a natural notion of a direct sum of root systems: take V to be an orthogonal direct sum of subspaces V_i, and Γ a set-theoretic union of subsystems Γ_i, each Γ_i spanning V_i. It is an easy exercise that the expression of a root system as a direct sum of indecomposable ones is absolutely unique. So it suffices for us to study indecomposable systems.

The numerous statements $(\Delta, \Gamma - \Delta) = 0$ that have occurred throughout the paper can now be rephrased as assertions that Δ is a direct summand of Γ, and the "injectivity" of 17 is now seen to have the usual meaning of being a universal direct summand. (But note that the injectivity in Lemmas 24 and 25 is proved only in the absence of copies of Γ_2.)

The systems $\Gamma(A, B, C)$, Γ_2, and Γ_3 are indecomposable; we accept them as entries on the final list of answers, and put them aside. Now if we assemble all the strands of our previous work (especially Lemmas 23, 24, and 25) we see that the following may be assumed: (a) in the situation of Lemma 11, two of A, B, C must be equal (so that the triple is proportional to 1, 1, -2), (b) in the situation of Lemma 20, the possibility of one inner product being three times the others is excluded. Furthermore, Lemma 26 is available.

Subject to all this, let us take an indecomposable root system Γ and analyze it. The method will be to fix λ and then collect all μ_i with $(\lambda, \mu_i) \neq 0$. Right away we make a normalization: we pick λ so that the number of μ's is as large as possible.

The second of our new axioms tells us that exactly one of $\lambda + \mu_i$, $\lambda - \mu_i$ is a root. A choice of sign for μ_i is available, and we take advantage of it by arranging that $\lambda - \mu_i$ is a root for all i.

Given μ_i and μ_j it might be the case that $(\mu_i, \mu_j) = 0$; if so, Lemma 10 tells us that $(\lambda, \mu_i) = -(\lambda, \mu_j)$. If $(\mu_i, \mu_j) \neq 0$ there are various possibilities. In view of the remarks above all of them lead to $(\lambda, \mu_i) = \pm(\lambda, \mu_j)$, $(\lambda, \mu_i) = \pm 2(\lambda, \mu_j)$, or $(\lambda, \mu_j) = \pm 2(\lambda, \mu_i)$.

Now comes a major case distinction. It may be that we always have $(\lambda, \mu_i) = \pm(\lambda, \mu_j)$. We put this case aside for the moment. Suppose that at least once we have $(\lambda, \mu_i) = \pm 2(\lambda, \mu_j)$. Lemma 26 tells us that the element μ_i for which this occurs is unique. Let us give this preferred element the symbol λ^*. Furthermore let us normalize the form by multiplying by an appropriate scalar so as to arrange $(\lambda, \lambda^*) = -2$. Then for every other μ_i we have $(\lambda, \mu_i) = \pm 1$. We change notation: write μ's for the elements with $(\lambda, \mu_i) = -1$ and ν's for the elements with $(\lambda, \nu_j) = 1$.

We can introduce similar notation in the case where there is no λ^* (i. e. (λ, μ_i) always equals $\pm(\lambda, \mu_j)$) but note that the choice of which elements are μ's and which ν's is then arbitrary. At any rate we again have (after normalizing the form) $(\lambda, \mu_i) = -1$, $(\lambda, \nu_j) = 1$, but there is no λ^*.

In both cases inspection of the triple λ, μ_i, ν_j, in the light of Lemmas 11 and 20 leads to the conclusion $(\mu_i, \nu_j) = 0$. From this point on we distinguish the cases.

Case I (no λ^*). We begin by pinning down the possible behavior of a pair of μ's, say μ_i and μ_r. Lemma 10 shows that (μ_i, μ_r) is nonzero. Lemmas 11 and 20 then leave three possibilities:

(a) $\mu_i - \mu_r$ is a root and $(\mu_i, \mu_r) = -1$,

(b) $\mu_i - \mu_r$ is a root and $(\mu_i, \mu_r) = -2$,

(c) $\mu_i + \mu_r$ is a root and $(\mu_i, \mu_r) = -2$.

But the absence of an element like λ^* enables us to rule out (b) and (c). In (c), with $\alpha = \mu_i + \mu_r$, we have $2(\alpha, \lambda)/(\alpha, \alpha) = 1$, so $\mu_i + \mu_r - \lambda$ is an isotropic root. Its inner product with λ is -2. In (b) the argument is longer. With $\beta = \lambda - \mu_i$ and $\gamma = \mu_i - \mu_r$ we have $2(\beta, \gamma)/(\beta, \beta) = -2$. Hence $\delta = 2\beta + \gamma = 2\lambda - \mu_i - \mu_r$ is an even root. Next $2(\delta, \lambda)/(\delta, \delta) = 1$. Hence $\lambda - \delta$ is an isotropic root, and its inner product with λ is -2.

Thus only the possibility (a) survives. The situation with the ν's is virtually identical: we have $\nu_j - \nu_s$ a root and $(\nu_j, \nu_s) = -1$.

Now we exhibit the model which will be our target. We call it a <u>special linear root system</u> and write it SL(m, n). Here m and n are any positive integers except that the cases m = n = 1 or 2 degenerate and are discarded. Take an inner product space W as the orthogonal direct sum $X \oplus Y$, where X has an orthonormal basis $e_1, \ldots, e_m$ and Y has a "negative orthonormal" basis $f_1, \ldots, f_n$ (this means that the f's are orthogonal and each $(f_j, f_j) = -1$). In SL(m, n) all odd roots are isotropic; they are the 2mn vectors $\pm(e_i + f_j)$. The even roots consist of all $e_i - e_r$ and all $f_j - f_s$ $(i \neq r, j \neq s)$. (Note, for instance, that if n = 1 no even roots are are differences of f's; they are all differences of e's.) The subspace V spanned by SL(m, n) is (m + n - 1)-dimensional and is the orthogonal complement of the vector $\sum e_i + \sum f_j$. For $m \neq n$, V is nonsingular. For m = n, the vector $\sum e_i + \sum f_j$ is isotropic, lies in V, and spans the 1-dimensional annihilator of V. We therefore redefine SL(m, n) by dividing by this 1-dimensional subspace, thus depressing the dimension to m + n - 2. We shall continue to write e_i, f_j, etc. although strictly speaking we need new symbols for their homomorphic images.

We pause to note the two cases of collapse. When m = n = 1, the only roots present are $\pm(e_1 + f_1)$ and they map to 0. When m = n = 2, the identification of $e_1 + e_2 + f_1 + f_2$ with 0 coalesces the isotropic roots $e_1 + f_1$ and $-(e_2 + f_2)$, and also $e_1 + f_2$ and $-(e_2 + f_1)$. If we write $\lambda = e_1 + f_1$ and

$\mu = e_1 + f_2$ then both $\lambda - \mu = f_1 - f_2$ and $\lambda + \mu$, equal to $2e_1 + f_1 + f_2$ and congruent to $e_1 - e_2$, are roots, violating our most recent axiom. We shall return to these degenerate cases once more below, when the algebras that accompany them make an appearance.

Now we proceed to identify the indecomposable root system Γ under study with the appropriate SL(m, n). Pair λ with $e_1 + f_1$, the μ's with $e_i + f_1$ ($i = 2, \ldots, m$) and the ν's with $e_1 + f_j$ ($j = 2, \ldots, n$). All properties are preserved and we see that Γ contains a copy (say Δ) of SL(m, n). It remains to argue that there is nothing else in Γ.

The pattern of the argument is the familiar one that uses Lemma 13. Take ρ in $\Gamma - \Delta$. If ρ is not orthogonal to Δ, it fails to be orthogonal to some λ' in Δ. The symmetry in SL(m, n) shows that, within SL(m, n), λ and λ' admit the same number of non-orthogonal isotropic roots. But then replacing λ by λ' gives us a larger total number of non-orthogonal isotropic roots; this contradicts the maximality we assumed for λ. Hence $(\rho, \Delta) = 0$.

To finish the job by Lemma 13 we have to exclude the possible presence in $\Gamma - \Delta$ of an even root α equal to half the sum of two non-orthogonal isotropic roots in Δ. There is an exceptional case here, given by $m = 1$, $n = 2$ or (the same by symmetry) $m = 2$, $n = 1$. We shall return to it at the end of the argument. A sufficiently typical choice for α is $\alpha = (2e_1 + f_1 + f_2)/2$. We have available as an extra isotropic root either $e_2 + f_1$ or $e_1 + f_3$. Both give a contradiction. In the case of $e_1 + f_3$ we have $2(\alpha, e_1 + f_3)/(\alpha, \alpha) = 4$. In the case of $e_2 + f_1$ we have $2(\alpha, e_2 + f_1)/(\alpha, \alpha) = -2$ so that $e_2 + f_1 + 2\alpha = 2e_1 + e_2 + 2f_1 + f_2$ is an isotropic root. But it does not lie in Δ and it is not orthogonal to Δ since it is not orthogonal to $e_2 + f_1$.

We return to the omitted case $m = 1$, $n = 2$. The only isotropic roots present are $\pm\lambda$ and $\pm\nu$. Recall that $\lambda - \nu$ is a root and $(\lambda, \nu) = 1$. We can regard this equally well as falling under Case II where this an element λ^*; renormalize the form by multiplying all inner products by -2 and take ν to be λ^*. With this the discussion of Case I is finished.

We mention briefly that there exist "classical" simple graded Lie algebras that have these root systems. Take the set of all matrices of the form

$$\begin{array}{cc} m & n \end{array}$$
$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{array}{c} m \\ n \end{array}$$

with Tr(A) = Tr(D). More exactly, we should say the set-theoretic union of the even ones of the form

$$\begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}$$

and the odd ones, of the form

$$\begin{pmatrix} 0 & B \\ C & 0 \end{pmatrix}$$

If m = n the identity matrix is central and the 1-dimensional subspace it spans should be divided out. Take as Cartan subalgebra all diagonal matrices. With e_i, f_j written for the usual diagonal matrix units in the entries A, D, respectively, we do indeed find the root system SL(m, n).

For m = n = 1 the 2-dimensional algebra obtained is not simple (it is nilpoent). For m = n = 2 we have a 14-dimensional algebra. It has 2-dimensional root spaces and is the algebra referred to in Theorem 1 and the discussion immediately preceding that theorem. For instance, a 2-dimensional root space is spanned by

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Case II (there is a root λ^*). Let us inspect the triple λ, λ^*, μ_i. Recall that we have $(\lambda, \lambda^*) = -2$, $(\lambda, \mu_i) = -1$, $\lambda - \lambda^*$ and $\lambda - \mu_i$ are roots. Lemma 10 shows that $(\lambda^*, \mu_i) = 0$ is ruled out. If $\lambda^* + \mu_i$ is a root then Lemma 11 is applicable to the triple $-\lambda$, λ^*, μ_i and gives us the forbidden inner products 1, 2, -3. Therefore $\lambda^* - \mu_i$ is a root, and we deduce $(\lambda^*, \mu_i) = -1$ from Lemma 20. The similar argument for ν_j (where we have $(\lambda, \nu_j) = 1$) is left to the reader; the conclusion this time is that $\lambda^* + \nu_j$ is a root and $(\lambda^*, \nu_j) = -1$.

Write $\alpha = \lambda - \lambda^*$. Then $2(\alpha, \nu_j)/(\alpha, \alpha) = 1$ so that $\alpha - \nu_j = \lambda - \lambda^* - \nu_j$ is an isotropic root. Its inner product with λ is 1 and the difference between it and λ is a root. Thus it is one of the ν's. So: the ν's come in pairs adding up to $\alpha = \lambda - \lambda^*$.

The setup is not symmetric here and the corresponding argument for the μ's is more involved. Let $\beta = \lambda - \mu_i$. We have $2(\alpha, \beta)/(\beta, \beta) = 2$ so that $\gamma = \alpha - 2\beta$ is an even root. Next $2(\gamma, \mu_i)/(\gamma, \gamma) = 1$ so that $\mu_i - \gamma$ is an isotropic root. In fact, $\mu_i - \gamma$ works out to be $\lambda + \lambda^* - \mu_i$. Its inner product with λ is -1 and its difference with λ is a root. It is one of the μ's. So: the μ's come in pais adding up to $\lambda + \lambda^*$. Notice that $\lambda + \lambda^*$ is not a root.

A change of notation is indicated at this point. Let $\{\mu_i\}$ stand for half the old μ's, where we arbitrarily pick one of each pair adding up to $\lambda + \lambda^*$. Treat the ν's similarly, picking one of each pair adding up to $\lambda - \lambda^*$. Note that the inner product $(\mu_i, \lambda + \lambda^* - \mu_i)$ between μ_i and its mate is -2, and for ν_j the corresponding inner product is $(\nu_j, \lambda - \lambda^* - \nu_j) = 2$.

We investigate (in this revised notation) the relation between μ_i and μ_r for $i \neq r$. The triple λ, μ_i, μ_r a priori admits all three cases listed at the beginning of the discussion of Cse I. But now we promptly rule out (b) and (c) by Lemma 26, for μ_i already has an element - its mate - whose inner product with it is -2. The same remark applies to the ν's.

We are ready for the next model. We call it an orthosymplectic root system and use the notation OSp(m, n). The vector space $W = X \oplus Y$ and the e's and f's are exactly the same as in SL(m, n). This time the isotropic roots are twice as numerous, consisting of all $\pm e_i \pm f_j$ (with all four choices of sign). The even roots on the e's consist of all $\pm 2e_i$ and all $\pm e_i \pm e_r$ (this is the root system of the Lie algebra C_m of all 2m by 2m skew-symplectic matrices). On the f's the even roots are all $\pm f_j \pm f_s$ (the root system of D_n - all 2n by 2n skew-symmetric matrices).

As before we pair λ with $e_1 + f_1$, μ_i with $e_i + f_1$, ν_j with $e_1 + f_j$. Now in addition we pair λ^* with $-e_1 + f_1$. The correspondence is perfect and we find inside Γ a copy Δ of OSp(m, n).

Let $\rho \varepsilon \Gamma - \Delta$. The proof that $(\rho, \Delta) = 0$ is identical with the corresponding proof in Case I.

Take $\alpha \varepsilon \Gamma - \Delta$. There are three typical choices for α. The first is the same as in Case I: $\alpha = (2e_1 + f_1 + f_2)/2$. This time we do not need extra room to demolish it; we just observe $2(\alpha, f_1 + f_2)/(\alpha, \alpha) = 4$. The second choice is $\alpha = (\lambda - \lambda^*)/2 = e_1$, which is ruled out since it is half an even root already present. The third is $\alpha = (\lambda + \lambda^*)/2 = f_1$. For the first time in the numerous arguments of this type, something different happens. This adjunction is legal!

We present our last model, calling it EOSp(m, n), the "E" suggesting "enlarged". The enlargement consists of throwing in $\pm f_1, \ldots, \pm f_n$ as even roots and $\pm e_1, \ldots, \pm e_m$ as odd (of course non-isotropic) roots. The augmented system of even roots built out of the f's is the root system of B_n, the Lie algebra of 2n + 1 by 2n + 1 skew-symmetric matrices. The even roots attached to the e's constitute C_m unchanged, but half of each "long" root has been adjoined as an odd root. Note that

OSp(m, n) and EOSp(m, n) have the same isotropic roots.

It is quite routine to see that once f_1 has been adjoined to OSp(m, n) we move all the way to EOSp(m, n). So we have the final context for an application of Lemma 13: Γ has a subsystem Δ isomorphic to EOSp(m, n). The handling of an isotropic root in $\Gamma - \Delta$ works exactly as before. Since there are no new isotropic roots, as compared with OSp(m, n), there are no new possibilities for an even root in $\Gamma - \Delta$. We have completed the proof of Theorem 2.

Theorem 2. Let Γ be a root system satisfying the axioms of § 8 and in addition the two axioms added at the beginning of this section. Assume that Γ is indecomposable and that it contains isotropic roots. The up to homothety (multiplication of the form by a nonzero scalar) Γ is isomorphic to one of the following: $\Gamma(A, B, C)$, Γ_2, Γ_3, SL(m, n), OSp(m, n), EOSp(m, n).

To conclude this section, the classical algebras that go with the orthosymplectic root systems (and give rise to the name) will be briefly described. A basis-free description will be given, rather than one by matrices. Take a vector space direct sum $M = S \oplus U$, regarded as a graded vector space with S odd and U even. Impose on S and U nonsingular bilinear forms, skew for S and symmetric for U. Say

S is 2m-dimensional and U is q-dimensional. We take the set of all skew linear transformations on M. The skewness condition on a linear transformation T is

$$(Tx, y) = -(x, Ty) \tag{14}$$

as usual, with the appropriate graded modification: if x and T are both odd, then the minus sign in (14) is dropped. When q is even, q = 2n, we get the root system OSp(m, n). For q = 2n + 1 we get EOSp(m, n).

20. <u>No isotropic roots</u>. The abstract root systems we have defined do not furnish a suitable framework for studying graded simple Lie algebras in the case where there are no isotropic roots. Rather than invent a modifies abstraction, we shall, in this final section, give a direct discussion of the algebras.

So let J be a simple graded Lie algebra with our standard properties: the even part is semisiple plus abelian and there exists a nonsingular associative form. Assume that all roots are non-isotropic (relative to a given Cartan subalgebra).

Let α and β be non-orthogonal even roots. Suppose that $\sigma = \alpha/2$ is an odd root. Then $(\sigma, \beta) \neq 0$. Then $\sigma + \beta$ or $\sigma - \beta$ is a root, necessarily odd and non-isotropic. We claim that it is not possible for both to be roots, for if they are then $\alpha + 2\beta$ and $\alpha - 2\beta$ are even roots, a forbidden array of even roots. So exactly one of $\sigma + \beta$, $\sigma - \beta$ is a root.

Let us say $K_\beta L_\sigma \neq 0$, $K_{-\beta} L_\sigma = 0$. It is impossible to continue further with multiplication by K_β: if $K_\beta L_{\sigma+\beta} \neq 0$ then $\sigma + 2\beta$ is an odd root and its double $\alpha + 4\beta$ is an even root, a contradiction. So the string ends at $\sigma+\beta$. This shows that $2(\beta, \sigma)/(\beta, \beta) = -1$, whence $2(\beta, \alpha)/(\beta, \beta) = -2$, and $(\alpha, \alpha) = 2(\beta, \beta)$ follows. We thus find that α and β generate a root system of type C_2 (or equivalently B_2) with α a long vector, β short.

We next notice that the even subalgebra J_0 is actually simple. The absence of isotropic roots implies that there is no abelian summand in J_0. So J_0 is semisimple. An orthogonal decomposition of the root system of J_0 carries with it a decomposition for all the roots (since each odd root is half an even root) and this in turn is readily seen to make J an algebra direct sum. Therefore J_0 must be simple. The only simple Lie algebras with a root system containing a copy of C_2 are B_n, C_n, and F_4.

Let us return to the root α above. We saw that any even root not orthogonal to α must be short. This promptly rules out B_n and F_4, leaving only C_n. A routine argument moreover shows that half of <u>every</u> long vector in C_n must occur as an odd root.

In sum, we have identified the root system of J as follows: the even roots form a copy of the roots of C_n, and the odd roots are obtained by taking half of every long vector in C_n. This of course does not yet identify the algebra, a project left for a later paper.

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