GRADED LIE ALGEBRAS. II

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1. Introduction. In this sequel to [2] I settle affirmatively the existence and uniqueness of simple graded Lie algebras attached to the root systems that arose in [2].

By and large, definitions will not be repeated from [2].

2. Uniqueness. Let us recall the basic setup. $J = J_0 \cup J_1$ is a simple graded Lie algebra with an appropriate form. $J_0 = P \oplus Q$ with P semisimple and Q abelian. Any Cartan subalgebra of J_0 acts diagonally on J_1. The root spaces are 1-dimensional except possibly when J is 14-dimensional. One of the objectives of this paper is to settle that the exceptional case is unique (stated without proof in [2]). A root system is attached to J. Let J' be a second such algebra and suppose that J and J' have isomorphic root systems. Our problem is to prove that J and J' are isomorphic. The theory of ungraded Lie algebras shows that one can reconstruct from the root system the structure of J_0 and the representation of J_0 on J_1. Thus only the multiplication $J_1 J_1 \subset J_0$ is needed.

I give a unified argument for the cases where J_1 is irreducible under P. These cases are: OSp(m, n) with

$n > 1$, EOSp(m, n), $\Gamma(A, B, C)$, Γ_2 and Γ_3. (The case of no isotropic roots, treated in the last section of [2], fits in as EOSp(m, 0) and needs no further comment. Incidentally, this is the only case where J_0 is simple.) This leaves SL(m, n), OSp(m, 1), and the 14-dimensional exception; these I treat individually.

We assume then that we are in the case where J_1 is irreducible under P and proceed to build the desired isomorphism F of J onto J'. As was asserted above, we can begin with an isomorphism F of J_0 onto J_0' (we systematically put a prime on an object attached to J to get the corresponding object attached to J'). Moreover it is easy to see that F can be selected to be form-preserving.

Pick a system of simple roots in P, and let us write Σ for a set of root vectors, one for each simple root. Write Π for a similar set of root vectors for the negatives of the simple roots. Since J_1 is irreducible under P, the subset of J_1 annihilated by Π is a 1-dimensional subspace S. Likewise the annihilator in J_1 of Σ is a 1-dimensional subspace T. J_1 is spanned by S and the elements obtained by multiplying S repeatedly by members of Σ, and the same statement holds with S and Σ replaced bt T and Π. S and T are root spaces for roots which are negatives of each other. These facts follow from representation theory; they can also be checked by inspection in the root systems at hand.

Pick s and t different from 0 in S and T. We tentatively proceed to extend F to J_1 by defining F(s) to be any nonzero element in S'. There is then a unique extension of F to a module isomorphism of J_1 onto J_1', and in particular the extended F sends T onto T'. But F need not be multiplicative on J_1. For instance, $F(st) = cF(s)F(t)$ with c a nonzero scalar which need not be 1. By replacing F by $F/\sqrt{c}$ on J_1 we can make the map multiplicative at least on st. Let us do so and change notation so that the adjusted map is again F. We can then show that F is multiplicative on J_1 and thus furnishes the desired map.

We take $a, b \in J_1$ and set out to prove $F(ab) = F(a)F(b)$. We can take a and b in the form

$$a = L_p \ldots L_1 s, \quad b = M_q \ldots M_1 t,$$

where each L_i, resp. M_j, denotes left-multiplication by some member of Σ, resp. Π. We argue by induction on $p + q$. The induction starts successfully at $p + q = 0$, for we have prearranged $F(st) = F(s)F(t)$. So $p + q$ is positive; by symmetry we may assume that p is positive. Suppose that L_p is left-multiplication by x and write $a = xd$. We have

(1a) $$db.x - bx.d + xd.b = 0.$$

We study $xb = L_p M_q \ldots M_1 t$ by systematically pushing L_p to the right of the M's. For a given j there are two cases. If the root for M_j is not the negative of that for L_p, then L_p and M_j commute. In the contrary case, the commutator G of L_p and M_j is left-multiplication by an element of the Cartan subalgebra. The commutator of G and an M is a scalar multiple of that M, so we can systematically push G to the right till it hits t and sends it into a scalar multiple of itself. When L_p reaches t we have $L_p t = 0$. The upshot of all this is that we see by induction that

$$F(d.xb) = F(d)F(xb),$$

or equivalently,

(1b) $$F(bx.d) = F(bx)F(d).$$

This looks after the middle term of (1a). As regards the first term, we have $F(db) = F(d)F(b)$ by induction, $F(x.db) = F(x)F(db)$ since F is an isomorphism on J_0, and $F(xd) = F(x)F(d)$ since F preserves the action of J_0 on J_1. These combine to give

$$F(x.db) = F(x).F(d)F(b),$$

or equivalently,

(1c) $$F(db.x) = F(d)F(b).F(x).$$

By the Jacobi identity:

(1d) $$F(d)F(b).F(x) - F(b)F(x).F(d) + F(x)F(d).F(b) = 0.$$

In (1d) we may replace $F(b)F(x)$ by $F(bx)$ and $F(x)F(d)$ by $F(xd) = F(a)$, again since F preserves the action of J_0 on J_1. Now apply F to (1a) and use (1b), (1c), and the modified form of (1d). The result is the desired equation $F(ab) = F(a)F(b)$.

This concludes our discussion of uniqueness for all the cases where J_1 is irreducible under P. We turn to SL(m, n), the first of the remaining cases. Let J be a simple graded Lie algebra with SL(m, n) as its root system. We shall identify J with an appropriate "special linear algebra" in a quite straightforward way.

As in [2] we exhibit the algebra matrix style, using m + n by m + n matrices. The odd ones have the form

$$\begin{pmatrix} 0 & m \text{ by } n \\ n \text{ by } m & 0 \end{pmatrix},$$

and the even ones

$$\begin{pmatrix} m \text{ by } m & 0 \\ 0 & n \text{ by } n \end{pmatrix},$$

where the displayed m by m and n by n matrices have equal traces, so that the "graded trace" is 0. We assume $m \neq n$ at present; at the end of the discussion we shall indicate the changes needed when $m = n$. These matrices form a simple graded Lie algebra; call it J'.

It is convenient to change notation in J's root system. Replace $e_1, \ldots, e_m, f_1, \ldots, f_n$ by $g_1, \ldots, g_m, -g_{m+1}, \ldots, -g_{m+n}$. Then we can uniformly assert that the roots are given by all $g_i - g_j$ $(i \neq j, i. j = 1, \ldots, m + n)$. Pick any nonzero element in the root space for $g_1 - g_i$ $(i = 2, \ldots, m + n)$ and call it E_{1i}. Pick E_{i1} in the root space for $g_i - g_1$ satisfying $(E_{1i}, E_{i1}) = 1$. Define $E_{ij} = E_{i1}E_{1j}$ for for $i \neq j, i, j, = 2, \ldots, m + n$. We now have representatives for all the root spaces. Together with the elements $h_{g_1-g_i}$, $i = 2, \ldots, m + n$, which form a basis for the Cartan subalgebra of J, we have a basis of J. We now map J into J' by sending E_{ij} into e_{ij} (the usual matrix unit), $h_{g_1-g_i}$ into $e_{11} - e_{ii}$ for $2 \leq i \leq m$, and into $e_{11} + e_{ii}$ for $m + 1 \leq i \leq m + n$.

That this is an isomorphism is a fairly automatic verification, left to the reader.

This concludes the treatment of SL(m, n) for $m \neq n$. When $m = n$ we take $m > 2$, since $m = n = 2$ is the 14-dimensional case with 2-dimensional root spaces to be discussed below. Both J' and J undergo a 1-dimensional shrinkage. The identity matrix has trace 0 and lies in J'; it spans the 1-dimensional center Z of J'. We pass to J'/Z instead, and indulge in an abuse of notation by continuing to write e_{ij}, although strictly speaking we need a new symbol for the homomorphic image of e_{ij}. In J the new feature is that we have

$$g_1 + \dots + g_m - g_{m+1} - \dots - g_{2m} = 0.$$

Thus the Cartan subalgebra is $(2m - 2)$-dimensional and a basis for it is obtained by deleting (say) the last element $h_{g_1} - h_{g_{2m}}$ from the list used above. Subject to these changes, the isomorphism of J onto J' is defined as before.

We turn to the 14-dimensional exception. We take J' exactly as in the preceding paragraph, with $m = n = 2$. The algebra J is equipped with the following roots: $\pm\lambda, \pm\mu$ as ~~isotropic~~ odd roots, and $\pm\lambda\pm\mu$ (all four signs) as even roots. λ and μ are isotropic and we have $(\lambda, \mu) = 1$. The ~~even~~ odd root spaces $L_{\pm\lambda}$, $L_{\pm\mu}$ are 2-dimensional, while the ~~odd~~ even ones $K_{\pm\lambda\pm\mu}$ are 1-dimensional. Take E_{12} nonzero in $K_{\lambda+\mu}$ and $E_{21} \in K_{-\lambda-\mu}$ with $(E_{12}, E_{21}) = 1$. ~~Take E_{13} nonzero in L_λ. For E_{31} we first make a tentative choice~~

Multiplication induces a pairing from the 2-dimensional spaces L_λ, L_μ to the 1-dimensional space $K_{\lambda+\mu}$. We claim that this pairing is nondegenerate. Suppose on the contrary that there is a nonzero element r in L_λ annihilating L_μ. Multiplication by r sends the 2-dimensional space $L_{-\mu}$ into the 1-dimensional space $K_{\lambda-\mu}$. Hence there is a nonzero element s in $L_{-\mu}$ with rs = 0. Pick t in L_μ with st $\neq$ 0. Then the vanishing of

$$rs.t + st.r + tr.s$$

yields a contradiction.

We take E_{13} to be any nonzero element in L_λ. We next make a tentative selection of E_{31} in $L_{-\lambda}$ with $(E_{13}, E_{31}) = 1$. By what was proved in the preceding paragraph, the annihilator of E_{13} in of an element in L_- with $(E_{13}, E_{31}) = 1$. We have $L_- L \neq 0$ 2, Lemma 16. Thus multiplication between $L_>$ and L induces a nondegenerate pairing to the 1-dimensional space K_+. The annihilator of E_{13} in L_{μ} is [redacted] 1-dimensional, and we pick E_{14} to be any nonzero element in this 1-dimensional space. Likewise we find E_{41} in $L_{-\mu}$ annihilating E_{31}, unique up to a scalar. Now there is a difficulty. We wish to have $(E_{14}, E_{41}) \neq 0$ and may have failed. Let us retrace our steps and make a fresh choice $E_{31}' = E_{31} + z$ with $z \neq 0$, $(E_{13}, z) = 0$. This changes E_{41} to E_{41}', say. Since E_{31} and E_{31}' are linearly independent, so are their annihilators E_{41} and E_{41}' in $L_{-\mu}$. If (E_{14}, E_{41}') is also 0 then E_{14} annihilates all of $L_{-\mu}$, a contradiction of the nondegeneracy proved in thepreceding paragraph. So by our revised choice of E_{31} we have achieved (after a change of notation) $(E_{14}, E_{41}) \neq 0$. A further normalization (multiplication of E_{41} by a suitable scalar) allows us to assume $(E_{14}, E_{41}) = 1$.

For $i \neq j$, $i, j = 2, 3, 4$, we next define $E_{ij} = E_{i1}E_{1j}$. Before proceeding further we need to check that the 12 E's, along with the basis h_{λ}, h_{μ} for the Cartan subalgebra, form a basis of J. There are two typical points to settle.

(1) $E_{31}E_{14} = E_{34}$ is nonzero, and therefore spans $K_{\lambda-\mu}$. This is deduced at once from the Jacobi identity

$$E_{13}E_{31}\cdot E_{14} + E_{31}E_{14}\cdot E_{13} + E_{14}E_{13}\cdot E_{31} = 0,$$

for the last term vanishes while the first is nonzero, since $E_{13}E_{31} = h_\lambda$, $E_{14} \varepsilon L_\mu$, and $(\lambda, \mu) = 1$.

(2) E_{13} and E_{42} form a basis of L_λ. The alternative is that E_{42} is a scalar multiple of E_{13} and therefore annihilates E_{14}. Since E_{42} equals $E_{41}E_{12}$ by definition, this means the vanishing of the second term in

$$E_{14}E_{41}\cdot E_{12} - E_{41}E_{12}\cdot E_{14} + E_{12}E_{14}\cdot E_{41} = 0.$$

The third term vanishes since $E_{12} \varepsilon K_{\lambda+\mu}$, $E_{14} \varepsilon L_\mu$ and $2\lambda + \mu$ is not a root. The first term does not vanish since $E_{14}E_{41} = h_\mu$ and $h_\mu E_{12} = E_{12}$.

Using this basis of J we map J to J' as before: E_{ij} to e_{ij}, h_λ to $e_{11} + e_{33}$, h_μ to $e_{22} + e_{44}$. The routine verification that we have an isomorphism is again left to the reader.

The final case of reducibility of J_1 which we must handle in a special way is OSp(m, 1). We follow the same pattern as the discussion of SL, first setting up a "concrete" algebra J' as a target. We exhibit J' as the set of all 2m + 2 by 2m + 2 matrices of the form

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} & B_1 \\ -A_{12}^* & A_{22} & \cdots & A_{2m} & B_2 \\ & & \cdots & & \\ -A_{1m}^* & -A_{2m}^* & \cdot & A_{mm} & B_m \\ -B_1^\dagger & -B_2^\dagger & & -B_m^\dagger & \begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix} \end{pmatrix}.$$

Here the A's and B's are 2 by 2, each A_{ii} has trace 0, and

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix}^\dagger = \begin{pmatrix} d & b \\ c & a \end{pmatrix}.$$

The roots in the algebra J are: $\pm 2e_i$ and $\pm e_i \pm f$ $(i = 1, \ldots, m)$. Pick any non-zero elements $s, s_2, \ldots, s_m, s'$ in the root spaces for $2e_1$, $e_1 - e_2, \ldots, e_1 - e_m, e_1 - f$. Pick $t, t_2, \ldots, t_m, t'$ in the root spaces for the negatives of these roots, making each inner product equal to 1. We claim that the following assignment determines uniquely an isomorphism of J onto J':

$$s \to e_{12}, \quad t \to e_{21}$$

$$s_i \to e_{1,2i-1} - e_{2i,2}, \quad t_i \to e_{2,2i} - e_{2i-1,1} \quad (i = 2, \ldots, m)$$

$$s' \to e_{1,2m+1} + e_{2m+2,2}, \quad t' \to e_{2,2m+2} + e_{2m+1,1}.$$

Details are omitted.

~~With this point established, it is easy to see that changing F by a scalar makes it multiplicative on ST; the main argument now applies.~~

3. Existence.

$\Gamma(A, B, C)$. We simply exhibit a multiplication table; this is not too onerous for a 17-dimensional algebra treated with a liberal dose of symmetry. But we shall briefly explain how the table was constructed. Pursuing the goal of symmetry, we use the notation a_i $(i = 1, \ldots, 4)$ for root vectors for λ, μ, ν, and $-\lambda-\mu-\nu$, x_i for the corresponding negative root vectors, and h_i for the corresponding elements of the Cartan subalgebra. The relation $h_1 + h_2 + h_3 + h_4 = 0$ is required. The products $a_i x_i$ are normalized to be h_i. In place of the scalars A, B, C the notation p_{ij} is used; here $i \neq j$, i and j range from 1 to 4, $p_{ij} = p_{ji}$, and $p_{ij} = p_{km}$ for i, j, k, m different. The restriction $p_{12} + p_{13} + p_{14} = 0$ is imposed, reflecting $A + B + C = 0$.

The relations below are now all inevitable, with one exception: the equations $x_i x_j = a_k a_m$ (and their consequences) need not hold. The elements $x_i x_j$ and $a_k a_m$ are related by a scalar, and it turns out that the scalar is invariant under permutations of i, j, k, m. A fresh selection of a_1 normalizes the scalar to be 1. It would be redundant to offer a proof since all this is a consequence of the uniqueness proved in the preceding section.

Here are the 17 basis elements: a_1, a_2, a_3, a_4, x_1, x_2, x_3, x_4, h_1, h_2, h_3, a_{12}, a_{13}, a_{14}, a_{23}, a_{24}, and a_{34}. The first 8 are odd and the remaining 9 are even. It is convenient to use both a_{ij} and a_{ji}, setting them equal. In the table, products are given in only one order, since parity determines the sign of the opposite product. All squares of basis elements are 0 and are omitted from the display. The subscripts are always distinct.

$$a_i a_j = a_{ij},\ a_i x_i = h_i,\ a_i x_j = 0,\ h_i a_i = 0,$$
$$h_i a_j = -p_{ij} a_j,\ a_i a_{ij} = 0,\ a_{ij} a_k = p_{ij} x_m,\ x_i x_j = a_{km},$$
$$h_i x_i = 0,\ h_i x_j = p_{ij} x_j,\ a_{ij} x_i = p_{ij} x_j,\ a_{ij} x_k = 0,$$
$$h_i h_j = 0,\ h_i a_{ij} = p_{ij} a_{ij},\ h_i a_{jk} = p_{jk} a_{jk},\ a_{ij} a_{ik} = 0,$$
$$a_{ij} a_{km} = -p_{ij}(h_i + h_j).$$

The task of verifying the Jacobi identity is left to the reader. It is helpful to begin by recognizing that the 9 even basis elements span the (ordinary) Lie algebra $A_1 \oplus A_1 \oplus A_1$, and that the 8 odd elements span a representation space for it. If symmetry is fully used, only a handful of easy verifications remain.

Γ_2. Take J_0 to be $G_2 \oplus A_1$. Take the 14-dimensional J_1 to be $C \otimes V$ where C denotes the 7-dimensional space of elements of trace 0 in a Cayley matrix algebra and V is a 2-dimensional space carrying a nonsingular alternate product (,). One has G_2 acting on C in the standard way and A_1 on V in the natural way.* To facilitate use of the material on pages 142-3 of [1] I will in this discussion place linear transformations on the right.

* As lin. transfs. skew relative to the form on V.

It remains to define the multiplication on J_1. We first define certain maps $\emptyset$ and ψ.

$\emptyset$: $C \times C \to G_2$. This is the map which appears on [1, p. 143]: $\emptyset(c, d) = [L_c L_d] + [L_c R_d] + [R_c R_d]$, L and R denoting left and right multiplication. It turns out that $\emptyset(c, d)$ defines a derivation of the Cayley matrix algebra and so lies in G_2. Note that $\emptyset$ is alternate.

ψ: $V \times V \to A_1$. For v, w in V define $\psi(v, w)$ to be the linear transformation on V given by

$$x\psi(v, w) = (x, v)w + (x, w)v.$$

We need to know that ψ is alternate relative to the form, i. e. we need

$$(x\psi, y) = -(x, y\psi). \tag{2}$$

Equation (2) is correct since it reduces to

$$(x, v)(w, y) + (x, w)(v, y) = -(y, v)(x, w) - (y, w)(x, v).$$

So $\psi \in A_1$. Note also that ψ is symmetric as a function of v and w: $\psi(v, w) = \psi(w, v)$.

The product from $J_1 \times J_1$ to J_0 is now defined by

$$(c \otimes v)(d \otimes w) = (v, w)\emptyset(c, d) + 4tr(cd)\psi(v, w),$$

where tr denotes the trace on the Cayley matrix algebra, normalized so that tr(1) = 1. Since (v, w) and $\emptyset(c, d)$ are both alternate bilinear functions, while tr(cd) and $\psi(v, w)$ are both symmetric, this multiplication is commutative.

The Jacobi identity must now be verified. Although this is a task that can be mechanized, some detail is offered. There are two major cases.

I. Two elements of J_1 and one of J_0. Take the members of J_1 as $c \otimes v$ and $d \otimes w$, and the member of J_0 as $D + T$, with T an alternate linear transformation on V and D a derivation of C (more accurately, a derivation of the Cayley matrix algebra of which C is the subset of elements of trace 0). The Jacobi identity reads

(3) $$(c \otimes v)(d \otimes w).(D + T) - (d \otimes w)(D + T).(c \otimes v) + (D + T)(c \otimes v).(d \otimes w) = 0.$$

In working with (3), it is to be observed that D or T to the left of an element gives a result which is the negative of what is obtained when it is placed on the right. Thus $(d \otimes w)(D + T) = dD \otimes w + d \otimes wT$ and $(D + T)(c \otimes v) = - cD \otimes v - c \otimes vT$. Note also that the first term of (3) is really a commutator. We check the four constituents of (3) separately.

(i) The A_1-component arising from T. After suppressing a factor $4\mathrm{tr}(cd)$ we find

(4) $$[\psi(v, w), T] - \psi(wT, v) - \psi(vT, w)$$

as the expression we have to prove equal to 0. Apply (4) to $x \in V$. The result is

$$\{(x, v)w + (x, w)v\} T - \{(xT, v)w + (xT, w)v\}$$
$$- \{(x, wT)v + (x, v)wT\} - \{(x, vT)w + (x, w)vT\}$$

which does indeed vanish (use the fact that T is alternate).

(ii) The G_2-component arising from T. This is simply

$$- (wT, v)\phi(d, c) - (vT, w)\phi(c, d)$$

and vanishes because ϕ is alternate, the form on V is alternate, and T is alternate relative to the form.

(iii) The A_1-component arising from D. After suppression of a factor $-4\psi(v, w)$ this is $tr(dD.c) + tr(cD.d)$, whose vanishing is a known property of derivations of a Cayley algebra.

(iv) The G_2-component arising from D. We need

(5) $(v, w)[\phi(c, d), D] - (w, v)\phi(dD, c) - (v, w)\phi(cD, d) = 0.$

The fact that D is a derivation implies

(6) $$[L_c D] = L_{cD}, \quad [R_c D] = R_{cD}$$

for any $c \in C$. By two applications of the first half of (6) we get

(7) $$L_c L_d D = L_c(L_{dD} + DL_d) = L_c L_{dD} + (L_{cD} + DL_c)L_d.$$

In (7) interchange c and d and subtract the two equations. The result is

(8) $$[[L_c L_d]D] = [L_c L_{dD}] + [L_{cD} L_d].$$

By two similar computations we get

(9) $$[[R_c R_d]D] = [R_c R_{dD}] + [R_{cD} R_d],$$

(10) $$[[L_c R_d]D] = [L_c R_{dD}] + [L_{cD} R_d].$$

Add (8), (9), and (10):

(11) $$[\phi(c, d), D] = \phi(c, dD) + \phi(cD, d).$$

In view of the fact that (,) and ϕ are both alternate, the desired equation (5) is a consequence of (11).

II. Three elements of J_1.

(i) Suppose that the V-components of the three elements are linearly dependent in pairs. Since the definition of ψ shows that $v\psi(v, v) = 0$, it follows readily that each of the triple products appearing in the Jacobi identity vanishes.

(ii) We suppose the contrary. We may then take the three elements to be $c \otimes v$, $d \otimes v$, and $e \otimes w$, with $(v, w) = 1$. We take the requisite Jacobi identity in the form

$$(12)\quad (e \otimes w).(c \otimes v)(d \otimes v) + (c \otimes v).(d \otimes v)(e \otimes w) + (d \otimes v).(e \otimes w)(c \otimes v) = 0.$$

Expand (12) by the definitions, and use $w\psi(v, v) = -2v$, $v\psi(v, w) = v$. We reach

$$(13)\quad -8\mathrm{tr}(cd)(e \otimes v) + (c \otimes v)\phi(d, e) + 4\mathrm{tr}(de)(c \otimes v) + (d \otimes v)\phi(c, e) + 4\mathrm{tr}(ce)(d \otimes v).$$

Since the second factor of the tensor is always v, (13) is the tensor product of v with

$$(14)\quad -8\mathrm{tr}(cd)e + c\phi(d, e) + 4\mathrm{tr}(de)c + d\phi(c, e) + 4\mathrm{tr}(ce)d.$$

Expand $c\phi(de)$ by the definition of ϕ:

$$(15)\quad c\phi(d, e) = e.dc - d.ec + dc.e - d.ce + cd.e - ce.d.$$

To get $d\phi(c, e)$ we interchange c and d in (15). The upshot is that we need the following identity for Cayley numbers of trace 0:

$$(16)\quad 2(cd + dc)e + e(cd + dc) - d(ec + ce) - c(de + ed) - ce.d - de.c = 8\mathrm{tr}(cd)e - 4\mathrm{tr}(de)c - 4\mathrm{tr}(ce)d.$$

So our discussion concludes with a verification of (16). The fundamental ingredient is the fact that the square of any Cayley number of trace 0 is a scalar. Linearizing this we get that $cd + dc$ is a scalar for $c, d \in C$. This scalar is equal to $2\mathrm{tr}(cd)$. So we have

$$(17)\quad e(cd+dc) = 2\mathrm{tr}(cd)e,$$

$$(18)\quad 2(cd + dc)e + e(cd + dc) = 6\mathrm{tr}(cd)e,$$

$$(19)\quad d(ec + ce) = 2\mathrm{tr}(ce)d,$$

$$(20)\quad c(de + ed) = 2\mathrm{tr}(de)c.$$

A fundamental property of alternative rings is that the associator ec.d - e.cd is an alternating function of its arguments. Hence

(21) $(ec.d - e.cd) + (ed.c - e.dc) = 0.$

By combining (17)-(21) we get (16).