Notes on Lie algebras and superalgebras

I. Unitary representations of superalgebras

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The following preliminary thoughts arose as the result of conversations with Bruno Zumino during August, 1982. During these conversations the phrase "unitary representation of a Lie superalgebra" popped up. After a while I wondered what this means. Here is my answer.

On page 230 of my joint note with Peter Freund (J. of Math. Phys. vol. 17) a Lie superalgebra labelled $SU(m|n)$ is displayed; it consists of all matrices of the form

$$\begin{pmatrix} a & b \\ ib^* & d \end{pmatrix}$$

where a and d are skew-Hermitian with equal traces and b* is the complex conjugate transpose of b. If a is m by m and d is n by n the (real) dimension is $(m + n)^2 - 1$. The algebra complexifies into the special linear superalgebra. I declare a unitary representation to be a homomorphism into this algebra.

I would like to redo this in a basis-free style, partly because that's always a good idea, and partly because in the infinite-dimensional case I prefer to avoid infinite matrices. I introduce the concept of a "super Hilbert space". This is an orthogonal direct sum $V = W \oplus X$, with W a Hilbert space and X like a Hilbert space but with a skew-Hermitian inner product. Rather than discuss this axiomatically I shall simply say that the inner product on X is obtained by multiplying a Hilbert space inner product by i.

V is made into a super vector space by declaring W even and X odd. A linear transformation T on V is skew if (Ta, b) = -(a, Tb), except that this is replaced by (Ta, b) = (a, Tb) if T and a are both odd. The skew linear transformations form a Lie superalgebra under super-commutation. If orthonormal bases are used in W and X we get the matrices displayed above.

The next notion is the tensor product of super Hilbert spaces. Let A and B be super Hilbert spaces. Take $a_1, a_2 \in A$, $b_1, b_2 \in B$. We have to decide on the value of the inner product $(a_1 \otimes b_1, a_2 \otimes b_2)$. The usual principle applies: we take $(a_1, a_2)(b_1, b_2)$ except when a_2, b_1 are both odd, in which case we change the sign.

It is a routine matter to check that the tensor poduct of two unitary representations is again unitary.

It seems reasonable to call a superalgebra <u>compact</u> if it has a faithful unitary representation.

As I see it, there are now two main problems: (1) Determine the compact superalgebras, especially the simple ones; (2) Classify the unitary representations of these algebras. (Incidentally, it is easy to see that any unitary representation is a direct sum of irreducibles.)

At present I have very little information. I did look at the unitary representations of the first interesting compact superalgebra: the 8-dimensional one. It has an irreducible unitary representations of each odd dimension;

this is seen by reducing the tensor powers of the basic 3-dimensional representation. (These representations -- when complexified -- are exactly the nontypical ones, in Kac's terminology.) I also checked that there is no irreducible unitary 4-dimensional representation; thus no member of the family of 4-dimensional irreducible representations (they are all typical) of the complexified algebra arises from a unitary representation.

II. Kac's K_n in characteristic p

The algebras K_n were studied in [1] and [2]. I became curious about them in characteristic p. Here are some facts without proof and some questions.

1. First I recall the definition. K_n is the Lie algebra defined by generators $h, e_1, \ldots, e_n, f_1, \ldots, f_n$ and relations $[he_i] = e_i$, $[hf_j] = -f_j$, $[e_i f_j] = \delta_{ij} h$. Take $n \geqq 2$. In characteristic 0, K_n is simple and has exponential growth.

2. In characteristic p, K_n is not simple. For instance, for p odd, $e_1(\mathrm{ad}\ e_2)^{p-1}$ generates a proper ideal; for $p = 2$, $e_1(\mathrm{ad}\ e_2)^3$ does. As usual we divide K_n by the maximal ideal disjoint from the -1, 0, 1 part (in the natural Z-grading of K_n) to get L_n, or $L_n(p)$ to emphasize p.

3. $L_2(2)$ is the algebra of Laurent polynomials over the simple 3-dimensional algebra.

4. $L_2(3)$ is finite-dimensional; it is the 7-dimensional algebra of 3 by 3 matrices of trace 0, modulo scalars.

5. A hasty inspection of $L_3(3)$ suggested that it is probably the algebra of Laurent polynomials over the 7-dimensional algebra just mentioned.

6. Except for the cases in items 3, 4, and 5 it may be that $L_n(p)$ has exponential growth.

7. For characteristic >3, is L_n defined by the relations $e_i(\mathrm{ad}\ e_j)^{p-1}$, $f_i(\mathrm{ad}\ f_j)^{p-1}$?

8. Now define K_n over Z instead of over a field. Let α be a product of $e_1, e_2, \ldots, e_m$ (in any association and order). Let β be a product of $f_1, f_2, \ldots, f_m$; the association and order may be different. Of course $\alpha\beta$ is an integral multiple of h. In every experiment this integer turned out to be a nonzero multiple of m.

9. The following special case is an easy exercise:

$$\left[[\cdots[[e_1 e_2]e_3]\cdots e_m],[\cdots[[f_1 f_2]f_3]\cdots f_m]\right] = m!h.$$

10. The statement in item 8 appears to be valid also when repetitions in the e's and corresponding repetitions in the f's are allowed.

1. Kac, Izv. 1968

2. Kac, Bull. AMS 1980

III. Superalgebras in characteristic p

(a) The Ramond-Neveu-Schwarz superalgebra. With ordinary Lie algebras of characteristic p still largely a mystery, it may seem premature to contemplate Lie superalgebras of characteristic p. But it is never too early to collect examples.

Here is the RNS superalgebra, formulated with reasonable generality. Let k be any field of characteristic $\neq 2$. Let Γ be any additive subgroup of k. We define a Lie superalgebra $L = H + M$. the even part H is the Albert-Zassenhaus algebra based on Γ; it has a basis u_α, α ranging over Γ, with $[u_\alpha u_\beta] = (\alpha-\beta)u_{\alpha+\beta}$. The odd part M also has a basis v_β indexed by Γ. We set $[u_\alpha v_\beta] = (\alpha/2 - \beta)v_{\alpha+\beta}$ and $[v_\alpha v_\beta] = cu_{\alpha+\beta}$, with c a fixed nonzero element of k. L is a simple Lie superalgebra, finite-dimensional if Γ is finite. Thus we get a family of simple Lie superalgebras of characteristic p.

When k is algebraically closed, it is known that the structure of H is determined by its dimension. The question promptly arises: is the same true for L?

(b) Can the even part of a simple superalgebra be solvable?

Again write $L = H + M$ for a simple superalgebra in characteristic p ($p \neq 2$). In characteristic 0 one knows that H cannot be solvable. In fact, the following is known and easy: M cannot have an H-invariant subspace of codimension 1. This rules out, for any characteristic, the possibility of H being abelian.

I have improved this to show that H cannot be nilpotent with a class less than the characteristic. I have also ruled out the case where H is the nonabelian 2-dimensional algebra. Pending the possibility of an idea that cuts deeper, I won't at present record the details.

P.S. Needless to say, this casual document is not intended for publication in anything like its present form. I am scattering a few copies to people who might be interested.

I. K.

Nov., 1982