Amer. Math. Soc. Transl.
(2) Vol. 155, 1993

Subgroups of Chevalley Groups Containing a Maximal Torus

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This paper is a systematic survey of subgroups of those Chevalley groups over a field that contain a split maximal torus. It will be shown, for the first time, that subgroups of this class, in groups of all normal types over infinite fields, admit a standard description. Together with Seitz's result for the finite case, this gives an almost complete solution of the problem, thus completing 25 years of research by many authors. The main steps in the proof of this result will be outlined. The detailed calculations at each stage of the proof involve considerable technical difficulties: the full proofs for the exceptional groups occupy several hundred typewritten pages and will be published elsewhere. In addition, we will review results concerning subgroups that contain a not necessarily split maximal torus and state some unsolved problems.

This survey is a natural continuation of our previous survey [30], where this and similar problems for the classical Chevalley groups were discussed in detail. We therefore concentrate our attention on special groups and on the methods and aspects of the theory that were not considered in [30]. For the same reason, we have not tried to present a comprehensive bibliography on subgroups of classical groups over rings containing maximal tori. Additional references may be found in [42, 27, 30]. Complete proofs of all results formulated here for the classical groups (except, of course, SL_2, which was considered by O. King at the end of 1988), as well as an exhaustive bibliography, can be found in the author's thesis [24].

§1. Preliminaries

We assume the reader to be familiar with at least one of the following texts on the theory of Chevalley groups: [7, 43, 69, 73, 83, 107]. Here we

1991 *Mathematics Subject Classification*. Primary 22E40; Secondary 20D06.

recall only the basic notation and a number of general facts which will be used throughout.

1. Chevalley groups. Let Φ be a reduced irreducible root system, P a lattice that lies between the root lattice $Q(\Phi)$ and the weight lattice $P(\Phi)$, $G_P(\Phi)$ a Chevalley-Demazure group scheme of type Φ, P [86, 7], $T_P(\Phi)$ a split maximal torus in $G_P(\Phi)$. We usually omit the weight lattice from the notation, since it does not play any significant role in our context. Nevertheless, if we want to emphasize that we are considering a simply-connected group (which is usually convenient), we use the subscript sc; similarly, for adjoint groups, we use the subscript ad. Since the general case can be reduced by standard means to that of simple groups, we will assume Φ to be irreducible, but we will have to consider subsystems of Φ that are not irreducible. If R is a commutative ring with unit — we need only the case when R is a field K — we denote the corresponding Chevalley group and its split maximal torus simply by $G = G(\Phi, R)$ and $T = T(\Phi, R)$, respectively.

The construction of Chevalley groups is based on the fact that in every semisimple complex Lie algebra L of type Φ we can choose a Chevalley basis x_α, $\alpha \in \Phi$; h_i, $\alpha_i \in \Pi$, where Π denotes the set of simple roots in Φ in some ordering of Φ. A Chevalley basis is a normalization of a Weyl basis all of whose structure constants are integers. The integral lattice spanned by x_α, h_i is denoted by $L_{\mathbb{Z}}$ and is called a Chevalley order in L. If K is a field, then the Lie K-algebra $L_K = L_{\mathbb{Z}} \otimes K$ with Chevalley basis $x_\alpha = x_\alpha \otimes 1$, $e_i = e_i \otimes 1$ is also called a Chevalley algebra. The adjoint Chevalley group $G_{\mathrm{ad}}(\Phi, K)$ may be described most simply as the subgroup of the automorphism group of the Chevalley group generated by certain special automorphisms. This is how Chevalley groups are treated in [73] (see also [43, 83, 157]).

Now let π be a representation of a Lie algebra L in a finite-dimensional vector space V. We omit the symbol π in the notation for the action of L on V, writing simply xu for $\pi(x)u$, where $x \in L$, $u \in V$. A lattice $V_{\mathbb{Z}}$ in V is said to be admissible with respect to $L_{\mathbb{Z}}$ if it is stable with respect to all divided powers $x_\alpha^{(m)} = x_\alpha^m/m!$, that is, if $x_\alpha^{(m)} V_{\mathbb{Z}} \subseteq V_{\mathbb{Z}}$ for all $\alpha \in \Phi$ and $m \in \mathbb{N}$ (in other words, $V_{\mathbb{Z}}$ is stable under the action of the $\mathbb{Z}$-form $U(L)_{\mathbb{Z}}$ of the universal enveloping algebra $U(L)$ associated with the Chevalley basis, that is, the Kostant form). Such a lattice always exists [7, 69, 86]. Thus, if the weight lattice of π is P, then the group $G_P(\Phi, K)$ is defined from the start together with a representation in the vector space $V_K = V_{\mathbb{Z}} \otimes K$, which we denote by the same letter π.

2. Root elements. For every root $\alpha \in \Phi$ there exist root unipotent elements $x_\alpha(\xi)$, $\xi \in K$, and root semisimple elements $h_\alpha(\varepsilon)$, $\varepsilon \in K^*$. Recall that $h_\alpha(\varepsilon) = w_\alpha(\varepsilon) w_\alpha(1)^{-1}$, where $w_\alpha(\varepsilon) = x_\alpha(\varepsilon) x_{-\alpha}(-\varepsilon^{-1}) x_\alpha(\varepsilon)$ [69, 83]. If we want to emphasize that the root elements in question correspond to a fixed choice of a maximal torus T, we will call them elementary. In general, however, the term root unipotent (or semisimple) element, will be used for any element of G conjugate to an elementary unipotent (or semisimple) element, i.e., to a certain $x_\alpha(\xi)$ or $h_\alpha(\varepsilon)$, respectively. These elements are called long if α is a long root and short if it is short.

All calculations in Chevalley groups are based on certain relations, called the Steinberg relations, among the elements $x_\alpha(\xi)$, $w_\alpha(\xi)$, $h_\alpha(\xi)$ [69, 83]. The most important are additivity of the x_α, multiplicativity of the h_α (the latter does not hold in the "Steinberg group"!) and the Chevalley commutation formula:

$$[x_\alpha(\xi), x_\beta(\zeta)] = \prod x_{i\alpha+j\beta}(N_{\alpha\beta ij}\xi^i\zeta^j),$$

where $[x, y] = xyx^{-1}y^{-1}$ denotes the commutator of two elements x, y of G; the product on the right is taken over all $i, j \in \mathbb{N}$ such that $i\alpha + j\beta \in \Phi$ in some fixed order, and $N_{\alpha\beta ij}$ are certain constants, independent of ξ and ζ and equal to $\pm1, \pm2, \pm3$, called the structure constants of G ($N_{\alpha\beta 11} = N_{\alpha\beta}$ are simply the structure constants of L in the Chevalley basis: $[x_\alpha, x_\beta] = N_{\alpha\beta} x_{\alpha+\beta}$).

For a root $\alpha \in \Phi$, let X_α denote the corresponding elementary root unipotent subgroup, consisting of all $x_\alpha(\xi)$, $\xi \in K$. The root subgroups are the subgroups conjugate in G to elementary root subgroups. The subgroup $E = E(\Phi, R)$ of G is called an elementary Chevalley group. Elementary groups are, of course, especially important in the theory of Chevalley groups over rings, but the groups G and E need not coincide even in the case of a field (counterexample: the orthogonal group). Nevertheless, in the most important case, that of simply-connected groups, $G_{\mathrm{sc}}(\Phi, K) = E_{\mathrm{sc}}(\Phi, K)$. Generally speaking, the difference between G and E is not very essential in the case of a field, being related only to the fact that the torus T may not coincide with the group

$$H = H(\Phi, K) = \langle H_\alpha(\varepsilon), \alpha \in \Phi, \varepsilon \in K^* \rangle,$$

generated by all the elementary root semisimple elements. As we just noted,

$$H_{\mathrm{sc}}(\Phi, K) = T_{\mathrm{sc}}(\Phi, K) = \mathrm{Hom}(P(\Phi), K^*).$$

The center of G_{sc} consists of those elements of H_{sc} that are identically 1 on $Q(\Phi)$; hence it is isomorphic to $\mathrm{Hom}(P(\Phi)/Q(\Phi), K^*)$.

3. Bruhat decomposition. Now let $N = N(\Phi, K)$ be the subgroup of G generated by T and all $w_\alpha(1)$, $\alpha \in \Phi$. As is well known, N is the normalizer of T in G, except for a very few minor exceptions such as $|K| = 2, 3$ [69]. The quotient group N/T is canonically isomorphic to the Weyl group $W = W(\Phi)$ of the root system Φ; hence, for every element $w \in W$, we can consider a preimage n_w of w in N. Usually, speaking of subgroups containing T, we shall write simply w instead of n_w. Thus, for example, we shall write BwB instead of Bn_wB to denote double cosets modulo Borel subgroups (see below).

Fix an ordering on Φ and let Φ^+, Φ^-, and $\prod = \{\alpha_1, \dots, \alpha_l\}$ be the sets of positive, negative, and simple roots, respectively, with respect to this ordering. As usual, we set

$$U = U(\Phi, K) = \langle x_\alpha(\xi), \alpha \in \Phi^+, \xi \in K \rangle,$$

$$V = V(\Phi, K) = \langle x_\alpha(\xi), \alpha \in \Phi^-, \xi \in K \rangle,$$

where $\langle X \rangle$ denotes the subgroup of G generated by X. Then the subgroup $B = B(\Phi, K)$ — the product of T and U — is called the standard Borel subgroup of G corresponding to the given ordering in Φ, and U is called the unipotent radical of B. The subgroup $B^- = B^-(\Phi, K)$, which is equal to TV, is called the Borel subgroup opposite to B.

The Bruhat decomposition of G means that W is a system of representatives of the double cosets of G modulo B, i.e., any element $x \in G$ can be represented as $x = b_1 w b_2$ where $b_1, b_2 \in B$ and $w \in W$ is uniquely defined. The Bruhat decomposition can be made somewhat more precise. Indeed, given $w \in W$, we set $U_w^- = U \cap w^{-1} V w$. Then every element $x \in G$ can be uniquely represented as $x = uwvd$, where $u \in U$, $w \in W$, $v \in U_w^-$, $d \in T$. This decomposition, also called the canonical form, is a fundamental computational tool in the analysis of Chevalley groups over fields.

§2. Standard subgroups

In this section we consider certain subgroups of G that obviously contain T. The main result of the paper, formulated in the next section, asserts that, as a rule, there are no other subgroups containing T.

1. Standard subgroups. The groups U and V are special cases of a group $E(S)$ that we are now going to define for an arbitrary closed subset of roots S. Recall that a set of roots $S \subseteq \Phi$ is closed if, whenever $\alpha, \beta \in S$ and $\alpha + \beta \in \Phi$, then also $\alpha + \beta \in S$. Define $E(S)$ to be the subgroup generated by all X_α, $\alpha \in S$. Then U and V are the groups $E(\Phi^+)$ and $E(\Phi^-)$. The group $E(S)$ has a special meaning when S is a special set, that is, a closed subset such that if $\alpha \in S$, then $-\alpha \notin S$. In that case $E(S)$ is simply the product of all X_α, $\alpha \in S$, taken in any fixed order. Set $G(S) = T \cdot E(S)$.

If S is an arbitrary closed set of roots, let S^r denote its reductive or symmetric part, i.e., the set of all $\alpha \in S$ for which $-\alpha \in S$, and S^u its unipotent or special part, i.e., the set of all $\alpha \in S$ for which $-\alpha \notin S$. We avoid the notation S^+ and S^-, which is often used in this context, since it is more convenient to reserve it for the sets $S^+ = S \cap \Phi^+$, $S^- = S \cap \Phi^-$. Clearly, S is the disjoint union of S^r and S^u. It is easy to see that $E(S) = E(S^r) \cdot E(S^u)$ and $G(S) = G(S^r) \cdot E(S^u)$. The subgroup $E(S^u)$ is called the unipotent radical of $G(S)$, and $G(S^r)$ is called the Levi subgroup or Levi component.

An element $w \in W$ is said to normalize S if $wS = S$. The set $X(S)$ of all $w \in W$ that normalize S is called the normalizer of S in the Weyl group. It is clear that $X(S)$ contains the Weyl subgroup $W(S) = W(S^r)$; moreover (see, e.g., [84]), if $S = S^r$ is a root system, then $X(S)$ is the normalizer of $W(S)$ in W. Two sets S_1 and S_2 are said to be conjugate if there exists $w \in W$ such that $wS_1 = S_2$. Let $N(S)$ denote the subgroup of G generated by $G(S)$ and the elements n_w for all $w \in X(S)$. It follows from the Tits theorem (see [70] or §7 of this paper) that $N(S)$ is almost always equal to the normalizer of $G(S)$ in G. This equality may fail to hold only when $|K| = 2, 3$, and even then not for all S.

The next definitions play a major role in what follows. A subgroup F of a Chevalley group $G = G(\Phi, K)$ is called standard if there exists a closed set of roots $S \subseteq \Phi$ such that $G(S) \leq F \leq N(S)$. It is clear that the standard subgroups contain T. We will say that G admits a standard description of subgroups containing a split maximal torus T if the converse holds, i.e., every subgroup of G containing T is standard. The goal of this paper is to obtain such standard descriptions for all "sufficiently big" fields.

2. Parabolic subgroups. Recall that a closed set of roots containing Φ^+ is called a standard parabolic subset; any set conjugate to a standard parabolic subset is called a parabolic subset. It is clear that the only maximal standard parabolic subsets are the sets P_r, $1 \leq r \leq l$, defined as the smallest closed subsets of Φ that contain Φ^+ and all roots $-\alpha_i$ where $\alpha_i \in \Pi \setminus \{\alpha_r\}$. Since two different sets P_r and P_s are never conjugate, there exist exactly l classes of maximal parabolic subsets, up to conjugacy. In general, the standard parabolic subsets correspond in one-to-one fashion with all the subsets of the fundamental root system Π. Indeed, given an arbitrary subset $J \subseteq \Pi$, consider the parabolic set defined as the closure of $\Phi^+ \bigcup \{-\alpha | \alpha \in J\}$. This construction gives different parabolic subsets P_I and P_J for different subsets I and J, and every standard parabolic subset is a P_J for some $J \subseteq \Pi$. Thus, the total number of standard parabolic subsets of a root system of rank l is 2^l.

The subgroups of a Chevalley group $G = G(\Phi, K)$ that contain the standard Borel subgroup $B = B(\Phi, K)$ are called standard parabolic subgroups; any subgroup conjugate to a standard parabolic subgroup is called a parabolic subgroup. The classical Tits theorem (see, e.g., [69, 83]) states that the mapping $P_J \to G(P_J)$ is a bijection of the set of standard parabolic subsets onto the set of standard parabolic subgroups. If P is a parabolic subgroup, let U_P and L_P denote its unipotent radical and Levi subgroup, respectively. Naturally, the maximal parabolic subgroups are of the greatest importance for our purposes. If $P = G(P_r)$, we will also write $U_P = U_r$, $L_P = L_r$.

Parabolic subgroups play a special role because every nonreductive standard subgroup is contained in a parabolic subgroup. Namely, the Borel-Tits theorem [9] asserts that if $S^u \neq \varnothing$ then $N(S)$ is contained in some proper parabolic subgroup. Recall that parabolic subgroups coincide with their normalizers; in particular, $W(S) = X(S)$ for any parabolic set of roots S.

3. Subsystems of roots. For reduction to groups of lower rank, and for the classification of maximal subgroups of Chevalley groups containing a split maximal torus, we have to determine the maximal standard subgroups. Thanks to the Borel-Tits theorem, this can be done by finding all maximal standard subgroups $N(\Delta)$, where Δ is a subsystem of roots in Φ or, what is the same, by finding the pairs $(\Delta, X(\Delta))$ that are maximal with respect to inclusion in the set of pairs $(S, X(S))$, where S is a closed subset of Φ.

A description of all subsystems of reduced systems of roots was obtained independently by Borel-de Siebenthal and Dynkin, who used the following strikingly simple construction. (Dynkin [39] in fact solved the much more general problem of describing all the semisimple subalgebras of semisimple Lie algebras.) Let $\overline{\Pi}$ be an extended system of simple roots, obtained from Π by adding the root $\alpha_0 = -\delta$, where δ is the maximal root of Φ in some specific ordering. Let Δ_r, $1 \le r \le l$, be the smallest closed set of roots containing all roots $\pm\alpha$, $\alpha \in \overline{\Pi} \setminus \{\alpha_r\}$. Then Δ_r is a subsystem of roots of "maximal rank", which may be reducible. It may also coincide with Φ; this is always the case if $\Phi = A_l$. We can now repeat the process, starting from any irreducible component of Δ_r, and so on. In this way we eventually obtain all subsystems of maximal rank [39, Table 10]. Any root subsystem of Φ may now be obtained as follows. Let Δ be any subsystem of maximal rank in Φ and Σ its set of simple roots. Take an arbitrary subset $J \subseteq \Sigma$ and consider the closure of $(-J) \cup J$. As a result, we obtain all closed root subsystems of Φ. For the classical root systems, the classification of subsystems is completely elementary [39, Table 9]. In special systems, the following numbers of proper root subsystems (i.e., different from both Φ and $\varnothing$), up to conjugacy are as follows: 4 in G_2, 22 in F_4, 19 in E_6, 45 in E_7 and, finally, 75 in E_8 [39, Table 11].

As a rule, two subsystems of the same type are conjugate to each other in Φ. The only exceptions are as follows. If all roots of Δ are of the same length, it may be imbedded in Φ in two (generally different) ways, that is, as long roots or short roots. In order to distinguish between these two imbeddings, we write $\tilde{\Delta} \subseteq \Phi$ for imbedding as short roots, reserving the notation $\Delta \subseteq \Phi$ for imbedding as long roots. Apart from this case, this occurs only in the systems B_l and D_l, owing to the difference between $2A_1$ and D_2, and also between A_3 and D_3, as well as in the systems D_l, E_7, and E_8 for certain subsystems $A_{k_1} + \cdots + A_{k_l}$ where all the k_i are odd.

4. Classification of maximal standard subgroups. The following theorem lists all the maximal standard subgroups of Chevalley groups. We use the following notation: $\Phi + \Delta$ denotes the orthogonal sum of root systems, $k\Delta$ the sum of k copies of Δ, $A_0 = B_0 = D_1$ the empty root system.

THEOREM 1. *Let Φ be a reduced irreducible root system, K a field. Then the Chevalley group $G = G(\Phi, K)$ contains the following conjugacy classes of maximal standard subgroups: l classes of maximal parabolic subgroups and the classes listed in the following table*:

A_l : $N\left(\frac{l+1}{k+1}A_k\right)$, $(k+1)|(l+1)$, $k \ne l$.

B_l : $N(B_k + D_{l-k})$, $0 \le k \le l-1$.

C_l : $N((l/k)C_k)$, $k|l$, $k \ne l$; $N(A_{l-1})$; $G(C_k + C_{l-k})$, $1 \le k \le l/2$.

D_l : $N((l/k)D_k)$, $k|l$, $k \ne l$; $N(A_{l-1})$; $N(D_k + D_{l-k})$, $1 \le k \le l/2$.

G_2 : $N(A_2)$, $G(A_1 + \tilde{A}_1)$.

F_4 : $G(B_4)$, $N(D_4)$, $G(A_1 + C_3)$, $N(A_2 + \tilde{A}_2)$.

E_6 : $G(A_5 + A_1)$, $N(3A_2)$, $N(D_4)$, N.

E_7 : $G(D_6 + A_1)$, $N(A_7)$, $N(A_5 + A_2)$, $N(D_4 + 3A_1)$, $N(7A_1)$, $N(E_6)$, N.

E_8 : $G(E_7 + A_1)$, $G(D_8)$, $N(E_6 + A_2)$, $N(A_8)$, $N(2D_4)$, $N(2A_4)$, $N(4A_2)$, $N(8A_1)$, N.

Each of the types listed in the table gives one conjugacy class in G, except $N(A_{l-1})$, which gives two conjugacy classes in $G(D_l, K)$ if l is even and is not maximal for odd l. If K is an infinite field, the subgroups $G(S)$, and of the standard subgroups they alone, are connected in the Zariski topology. Thus, Theorem 1 gives, in particular, a classification of the connected maximal standard subgroups: just choose those Δ for which $G(\Delta)$ occurs in the table. An explicit list of maximal standard subgroups is also of interest, since subgroups that are not contained in any proper subgroup conjugate to a standard subgroup play the same role in the theory of Chevalley groups as the primitive irreducible subgroups in the theory of linear groups.

A theorem of this type for complex semisimple Lie groups was obtained by Golubitsky and Rothschild [103, 104]. Since the main calculations here involve only root systems and their Weyl groups, the result should be the same for all fields. However, the tables in the papers just cited contain three errors [21]. Theorem 1 was established by the author in [19, 21].

§3. Formulation of main results

We are now going to present the first really general formulations of two basic theorems about subgroups of $G = G(\Phi, K)$ containing $T = T(\Phi, K)$ — the classification theorem and the conjugacy theorem. Previously, such results have been stated either for special classes of fields or for groups of nonsymplectic types. The classification theorem states that the descriptions of subgroups in G that contain T are almost always standard. The conjugacy theorem states that under the same restrictions T is a pronormal subgroup of G. The proofs are outlined in §§4–9.

The assumption char $K \neq 2$ for $\Phi = B_l, C_l, F_4$, char $K = 2, 3$ for $\Phi = G_2$, is a necessary condition for a standard description. Otherwise, even the Borel subgroup $B = B(\Phi, K)$ will have nonstandard subgroups containing T. In fact, however, the situation is easy to remedy for all groups, except those of type C_l, by admitting not only closed but also quasiclosed sets of roots [8]. For the symplectic group $\mathrm{Sp}(2l, K) = G_{sc}(C_l, K)$, in particular, $\mathrm{SL}(2, K) = \mathrm{Spin}(3, K) = G_{sc}(C_1, K)$, $\mathrm{Spin}(4, K) = G_{sc}(C_1, K)G_{sc}(C_1, K)$, and $\mathrm{Spin}(5, K) = G_{sc}(C_2, K)$, the situation is much more serious: If K is a nonperfect field of characteristic 2, there may be even infinitely many subgroups in G containing T. Taking this into consideration, we will assign the systems $A_1 = B_1 = C_1$ and $B_2 = C_2$ to the symplectic series. Thus, we assume $l \geq 2$ for A_l and $l \geq 3$ for B_l and D_l. Even so, it turns out that, apart from the forbidden characteristics, only extremely small fields may be exceptions as far as the standard description is concerned.

THEOREM 2. *Let Φ be a reduced irreducible root system and let K be a field. Assume that $|K| \geq 13$ and, moreover,* char$K \neq 2$ *if $\Phi = B_l, C_l, F_4$,* char$K \neq 2, 3$ *if $\Phi = G_2$. Then the Chevalley group $G = G(\Phi, K)$ admits a standard description of its subgroups that contain the split maximal torus $T = T(\Phi, K)$. In other words, for every intermediate subgroup F, $T \leq F \leq G$ there exists a unique closed set of roots $S \subseteq \Phi$ such that $G(S) \leq F \leq N(S)$.*

The main steps toward a proof of this theorem were as follows (see §6 for the very important case of extended groups). The result was proved for algebraically closed fields in the classical work of Borel and Tits [8]. Formally speaking, their proof concerned only subgroups that were connected and closed in the Zariski topology, which were proved to coincide with $G(S)$ for appropriate closed (or quasiclosed, in the exceptional characteristics) sets of roots S. However, the famous theorem of Chevalley immediately implies that in the case of an algebraically closed field, any subgroup that contains T is automatically closed in the Zariski topology; and another classical paper of Tits [70] states that the normalizers $G(S)$ in G coincide with $N(S)$, not only for algebraically closed fields but for all fields K with $|K| > 5$.

The next important step was a 1979 paper by Seitz [146], in which he established a standard description for finite fields K, char $K \neq 2$, for all root systems, including the series $\Phi = A_l, D_l, E_l$ and $|K| \geq 13$. The proofs in [146] rely substantially on the theory of finite groups and they do not carry over to infinite fields.

In 1979, the author and Dybkova proved the following reduction theorem for symplectic groups: if char $K \neq 2$ and $|K| \geq 7$, the description of the subgroups of $\mathrm{Sp}(2l, K)$ containing a split maximal torus is standard if and only if it is standard for $\mathrm{SL}(2, K)$ [33, 38, 24]. This result made it possible, in particular, to prove Theorem 2 for symplectic groups under the additional hypothesis $|K^*| > |K^*/K^{*2}|$ [33, Theorem 5].

In 1983, the author proposed a method that enabled him to prove Theorem 2 for all nonsymplectic groups over an infinite field (announced in [17]). The key idea was to reduce the treatment of the nonsymplectic groups, not to ordinary Chevalley groups, but to extended Chevalley groups of lower rank, for which the theorem had already been proved (see §6). This proof, for $\mathrm{SL}(n, K)$, $n \geq 3$, over an infinite field, was published in [18, I]. Parts II–IV of [18] actually established a much stronger result, namely the truth of the theorem for $\mathrm{SL}(l+1, K) = G_{sc}(A_l, K)$, $l \geq 2$, for all K, $|K| \geq 7$. Later we showed that the theorem also holds for the spinor group $\mathrm{Spin}(2l, K) = G_{sc}(D_l, K)$, with the same bound $|K| \geq 7$ on the number of elements of the field (the proof will be published in a paper "On subgroups of the spinor group that contain a split maximal torus. I"). In [23] we lifted the restriction char $K \neq 2$ for groups of type E_l over a finite field.

Thus, it remained only to consider the case $\mathrm{SL}(2, K) = G_{sc}(A_l, K)$. Paradoxically, this case turned out to be difficult for infinite fields. In 1983 the author and Dybkova [33, II] verified that under the additional assumption $-1 \in K^{*2}$ the group $\mathrm{SN}(2, K)$ of monomial matrices is maximal in $\mathrm{SL}(2, K)$; in 1985 King proved this in full generality [118]. In 1986, analyzing King's proof, we noticed that an insignificant modification yields a proof for the group $\mathrm{SL}(2, K)$ and, simultaneously, thanks to the results of [33], for all symplectic groups as well, provided that $-1 \in K^{*2}$ [32]. Finally, in the fall of 1988, King successfully completed the treatment of $\mathrm{SL}(2, K)$ [119]; this was the last step in the proof of the classification theorem.

Theorem 2 is a substantial generalization of many previously known results, in particular, of the description of the parabolic subgroups of G. For example, Theorems 1 and 2 imply the following

COROLLARY. *Under the assumptions of Theorem 2, any maximal subgroup in G that contains T is either a maximal parabolic subgroup or conjugate to one of the groups listed in the table of Theorem 1.*

As particular cases, this corollary includes a great many previous results of Dye, Ki, King, Li Shangzhi and other authors, relating mostly to classical groups. We will not go into the details here, nor discuss geometrical versions of the corollary for classical groups; the reader is referred to [24, 27, 30] (in addition to the literature listed there, see also the recent work of Ton Dao-rong [167, 168]).

Our second main theorem, as we mentioned, is the conjugacy theorem. Recall that a subgroup T of a group G is said to be pronormal in G if any two subgroups conjugate to it are already conjugate in the subgroup that they generate. In other words, for every $x \in G$ there exists $y \in \langle T, xTx^{-1}\rangle$ such that $xTx^{-1} = yTy^{-1}$. The best known examples of pronormal subgroups are of course the Sylow subgroups in finite groups and the maximal tori in algebraic groups [6, 72]. Pronormality generalizes both normality and abnormality. Recall that a subgroup B of a group G is said to be abnormal if, for any $x \in G$, $x \in \langle B, xBx^{-1}\rangle$. In particular, the normalizer of a pronormal group is abnormal. It turns out [2] that pronormality is of paramount importance in describing the lattice of subgroups of an abstract group. The classical Tits theorem asserts that a Borel subgroup B of a Chevalley group G is abnormal. The following result is an analog — and essentially a substantial generalization — of this fact.

THEOREM 3. *Under the assumptions of Theorem 2, the split maximal torus $T = T(\Phi, K)$ is pronormal in the Chevalley group $G = G(\Phi, K)$.*

For the proof, see §7. The "prehistory" of the theorem, relating to extended groups, is described in §6. The following fairly useful assertion follows from Theorem 3.

COROLLARY. *Under the assumptions of Theorem 2, if two subgroups that contain T are conjugate to each other in G, they are conjugate by an element of N.*

In actual fact, pronormality of T is equivalent to a more precise assertion: if $F_2 = xF_1x^{-1}$ for two subgroups F_1, F_2 of G containing T, and for some $x \in G$, then $x = wy$ for a suitable $w \in N$ and $y \in F_1$.

§4. Long root semisimple elements

Underlying the proofs of Theorems 2 and 3 are calculations with long root semisimple elements. The starting point is the following well-known fact (see, e.g., [69]):

$$T_{sc}(\Phi, K) = \langle h_\alpha(\varepsilon),\ \alpha \in \Phi_l,\ \varepsilon \in K^* \rangle,$$

where Φ_l is the set of long roots of Φ. Moreover, the long root semisimple elements (LRSE) not only generate a split maximal torus of a simply-connected group, but generally speaking, they are in fact the simplest elements of the torus. Certain microweight elements (see §5) may indeed have simpler structure, but it is well known that groups of types E_8, F_4, G_2 have no microweight representations. The use of LRSE's here is therefore dictated by hard necessity. But even for groups of types A_l, D_l, E_6 the use of LRSE's, as a rule, produces more precise results than that of microweight elements. In this section we assemble several results due to the author and Semenov (see, in particular, [28, 29, 31,35, 66]) concerning the Bruhat decompositions of LRSE's; in §§7, 8 we shall show how this information is used to prove Theorems 2 and 3. In view of future applications, we are interested mainly not in the Bruhat decompositions of individual LRSE's, but in the Bruhat decompositions of "long root tori", i.e., of subgroups such as

$$Q_x = \{y(\varepsilon) = xh_\delta(\varepsilon)x^{-1},\ \varepsilon \in K^*\},$$

where $x \in G$ and δ is a maximal root.

1. Reduction to D_4. The following fact is crucial: any problem concerning Bruhat decompositions of long root tori in Chevalley groups of all types can be reduced to the analogous problem in the group $\mathrm{Spin}(8, K)$. Let Φ, Δ be root systems. We will say that Δ is obtained from Φ if Δ is a twisting (possibly, trivial) of a subsystem of Φ (possibly of Φ itself).

THEOREM 4. *Let Q_x, $x \in G$, be a long root torus in a Chevalley group $G = G(\Phi, K)$. Then there exist a subsystem $\Delta \subseteq \Phi$ obtained from D_4 and an element $y \in U(\Phi, K)$ such that $yQ_xy^{-1} \le G(\Delta, K)$.*

This is Theorem 1 of [35]. The idea of the reduction is contained (in a less rigorous form) in our papers [28, 29]. The main step in the proof consists in finding orbits of a Borel subgroup in certain special representations [28]; this was also done independently by A. G. Elashvili. Other applications of the theorem and another interpretation of the relevant orbit computations are discussed in [28, 29, 35]. As it happens, another problem which, at first glance, has nothing in common with our problem, also reduces to exactly the same computations: to find all possible configurations of triples of long root unipotent subgroups, two of which are opposite (see §9).

2. Number of degenerations. Using Theorem 4, in $\mathrm{Spin}(8, K)$, Semenov [35] obtained the following surprising result.

THEOREM 5. *A long root torus Q_x intersects at most four cosets in the Bruhat decomposition. Moreover, all elements $y(\varepsilon)$, except at most three, are in the same coset Bw_0B.*

In other words, there is at most one $\theta \neq 0$, 1 such that $y(\varepsilon) \in Bw_0B$ for $\varepsilon \neq 0$, 1, θ, θ^{-1}, but $y(\theta) \in BwB$ and $y(\theta^{-1}) \in Bw^{-1}B$ for some $w \neq w_0$. The elements $y(\varepsilon)$ in Bw_0B and the corresponding values of the parameter ε will be called typical; all others will be called degenerate. A comparison of the Bruhat decompositions of the elements $y(\varepsilon)$ and $y(\varepsilon^{-1}) = y(\varepsilon)^{-1}$ shows that w_0 is an involution. The author's own papers [28, 29] proved only that all but a finite set of the elements $y(\varepsilon)$ are in Bw_0B. Semenov previously proved Theorem 5 for the cases $\Phi = A_l$, C_l [66].

3. Factors from the Weyl group. The following corollary, relating to w_0, immediately follows from Theorem 4.

COROLLARY. *The element w_0 in the Bruhat decomposition of a long root torus is necessarily a product of reflections with respect to pairwise orthogonal roots. If r of these roots are long and s are short, then $r + 2s \le m$, where $m = 4$ for $\Phi = B_l$, $l \ge 3$, D_l, $l \ge 4$, E_l, F_4; $m = 3$ for $\Phi = G_2$; $m = 2$ for $\Phi = A_l$, $l \ge 3$, C_l, $l \ge 2$, and, finally, $m = 1$ for $\Phi = A_1$, A_2.*

More detailed information about what collections of orthogonal roots occur here may be found in [28]. It is also of considerable interest to find all values that w may take in the Bruhat decomposition of a LRSE.

For $\mathrm{SL}(n, K)$, and hence also for $\mathrm{Sp}(2l, K)$, all possible $w \in S_n = W(A_{n-1})$ were found in [31]. It turns out that, besides transpositions and products of two independent transpositions, there may occur only 3-cycles and certain (but not all!) 4-cycles. A more detailed analysis of this problem was presented in [66], including a table of all possible w_0, w, corresponding to a long root torus in $\mathrm{SL}(n, K)$. In particular, it was found that for a symplectic group $\mathrm{Sp}(2l, K)$ degeneration may occur only at the points $\varepsilon = \pm 1$; moreover, if w_0 is a product of reflections with respect to two orthogonal long roots, degeneration will invariably occur at $\varepsilon = -1$, that is, $y(-1)$ lies outside the typical coset Bw_0B. We will make essential use of this fact in §8. Very recently, Semenov extended the results of [31, 66] to all Chevalley groups, but his results are too cumbersome to be presented here.

4. Factors from the Borel subgroup. It is also important to have information about the factors b_1 and b_2 from the Borel subgroup $B = B(\Phi, K)$ in the Bruhat decomposition of a LRSE: $y(\varepsilon) = b_1 w b_2$. The available results are too technical to state here in full detail. A rough impression may be gained from [20, 22, 24, 31], where explicit calculations of the factors from B are carried out for several microweight elements. Here we restrict ourselves to a simple proposition whose role will become clear in §7. Express the elements b_1 and b_2 as $b_1 = ud_1$, $b_2 = d_2 v$, where $u, v \in U = U(\Phi, K)$ and $d_1, d_2 \in T$, and factorize the elements u and v into elementary factors:

$$u = \prod x_\alpha(u_\alpha), \qquad v = \prod x_\alpha(v_\alpha)$$

extending over all positive roots in an arbitrary but fixed order.

THEOREM 6. *The elements u_α and v_α are rational functions of ε, whose numerators and denominators are at most quadratic functions of ε.*

In actual fact, if the ordering of Φ^+ is suitably chosen, most of these coefficients do not depend at all upon the choice of a typical ε and satisfy the condition $u_\alpha = -v_\alpha$.

§5. Weight elements

The proofs of Theorems 2 and 3 for the classical groups and for types E_6 and E_7 may be somewhat simplified if one exploits the fact that in these cases T generally contains elements of simpler structure than LRSE's. The existence of these elements is closely bound up with the possibility of constructing nontrivial diagonal extensions of groups of these types — "the extended Chevalley groups", which stand in the same relation to the ordinary Chevalley groups as do the groups $\mathrm{GL}(n, K)$ to $\mathrm{SL}(n, K)$. The extended Chevalley groups are generally much easier to study than the ordinary Chevalley groups, and thay may serve as models. At the same time, they often appear naturally as a stage in the theory of Chevalley groups proper, if only because the Levi components of parabolic subgroups in G are almost always not semisimple but reductive: a maximal torus of G induces nontrivial diagonal automorphisms of their semisimple parts. We now recall some

constructions and results due to Berman and Moody [81], and the author [20, 25, 28, 29].

1. Diagonal extensions of Chevalley groups. Assign a scalar $\chi(\alpha) \in K^*$ to each simple root $\alpha \in \Pi$. This correspondence extends linearly to a K-character, i.e., a homomorphism of the root lattice $Q(\Phi)$ into K^*:

$$\chi(m_1\alpha_1 + \dots + m_l\alpha_l) = \chi(\alpha_1)^{m_1} \cdots \chi(\alpha_l)^{m_l}.$$

It is well known that there exists a unique automorphism φ_χ of the Chevalley group $G = G(\Phi, K)$ such that $\varphi_\chi(x_\beta(\xi)) = x_\beta(\chi(\beta)\xi)$. Steinberg [69] calls such automorphisms of G diagonal. The main part of diagonal automorphisms is of course the inner automorphisms. Indeed,

$$h_\alpha(\varepsilon)x_\beta(\xi)h_\alpha(\varepsilon)^{-1} = x_\beta(\varepsilon^{\langle\beta,\alpha\rangle}\xi),$$

where $\langle\beta, \alpha\rangle = 2(\beta, \alpha)/(\alpha, \alpha) = (\beta, \alpha^\vee)$ is the Cartan number (here, as usual, $\alpha^\vee = 2\alpha/(\alpha, \alpha)$ is the dual root to α). Thus, conjugation by $h_\alpha(\varepsilon)$ is equivalent to the diagonal automorphism φ_χ corresponding to the K-character $\chi = \chi_{\alpha,\varepsilon}$ defined by

$$\chi_{\alpha,\varepsilon}(\beta) = \varepsilon^{\langle\beta,\alpha\rangle} = \varepsilon^{(\beta,\alpha^\vee)}.$$

In fact, conjugation by any element $h \in H$ is equivalent to some diagonal automorphism. These are precisely the diagonal automorphisms corresponding to the K-characters of the root lattice $Q(\Phi)$ that extend to K-characters of the weight lattice $P(\Phi)$ [83]. This means that the quotient group of the group of all diagonal automorphisms by the subgroup of inner automorphisms is isomorphic (in the simply-connected case) to a product of groups K^*/K^{*m_j}, where the m_j are all the elementary divisors of the finite abelian quotient group $P(\Phi)/Q(\Phi)$. Thus, for any field K such that $K^{*m_j} = K^*$, all diagonal automorphisms are inner automorphisms.

One would naturally like to construct an extension of the Chevalley group (for any field K) in which all diagonal automorphisms are inner automorphisms. This is easy to do for adjoint groups: for every K-character of $Q(\Phi)$ there is an automorphism $h(\chi) = h_{\mathrm{ad}}(\chi)$ of the Chevalley algebra L_K, defined by $h(\chi)h_i = h_i$, $h(\chi)x_\beta = \chi(\beta)x_\beta$. The automorphisms $h(\chi)$, where $\chi \in \mathrm{Hom}(Q(\Phi), K^*)$ constitute a subgroup $\overline{T}_{\mathrm{ad}} = \overline{T}_{\mathrm{ad}}(\Phi, K)$ of the automorphism group of the Chevalley group $G_{\mathrm{ad}}(\Phi, K)$ and, as one can easily see, conjugation by $h(\chi)$ defines a diagonal automorphism φ_χ of $G_{\mathrm{ad}}(\Phi, K)$:

$$h(\chi)x_\beta(\xi)h(\chi)^{-1} = x_\beta(\chi(\beta)\xi).$$

In particular, in this notation, the root semisimple element $h_\alpha(\varepsilon)$ is just $h(\chi_{\alpha,\varepsilon})$. All this can be found, for example, in [83, 157]; the latter also presents an identification of the groups $\overline{G}_{\mathrm{ad}} = \overline{T}_{\mathrm{ad}}G_{\mathrm{ad}}$ for the classical series.

It is much more difficult to define similar extensions for simply-connected groups (the dimension of the maximal torus must be increased by 1 or 2); this has been done only by Berman and Moody [81]. We shall not reproduce the details of their construction here (see also [20]); instead, in the next subsection we present a straightforward definition of weight elements — not as rigorous as might be based on [81], but quite sufficient for our purposes.

2. Weight elements. Let us return once more to the element $h_\alpha(\varepsilon)$. It corresponds to the character $\chi_{\alpha,\varepsilon}$ which is defined on $Q(\Phi)$ because $(\beta, \alpha^\vee) = \langle\beta, \alpha\rangle \in \mathbb{Z}$. But by definition, $(\beta, \omega) \in \mathbb{Z}$ for any $\beta \in \Phi$ and any $\omega \in P(\Phi^\vee)$, where $\Phi^\vee$ is the root system dual to Φ, i.e., $\Phi^\vee = \{\alpha^\vee \mid \alpha \in \Phi\}$. Thus, for any $\omega \in P(\Phi^\vee)$ and any $\varepsilon \in K^*$, there exists a K-character $\chi_{\omega,\varepsilon} \in \mathrm{Hom}(Q(\Phi), K^*)$; it is defined by $\chi_{\omega,\varepsilon}(\beta) = \varepsilon^{(\beta,\omega)}$. This character linearly depends on ω, i.e.,

$$\chi_{m_1\omega_1+m_2\omega_2,\varepsilon} = \chi^{m_1}_{\omega_1,\varepsilon} \cdot \chi^{m_2}_{\omega_2,\varepsilon}.$$

In addition, it is clear that $\chi_{m\omega,\varepsilon} = \chi_{\omega,\varepsilon^m}$. We can now consider an element $h_\omega(\varepsilon)$, which may be called a weight element, by analogy with the root semisimple element $h_\alpha(\varepsilon)$ (or as we should denote it now, $h_{\alpha^\vee}(\varepsilon)$), such that conjugation by $h_\omega(\varepsilon)$ defines a diagonal automorphism of $G = G(\Phi, K)$ corresponding to $\chi_{\omega,\varepsilon}$, i.e.,

$$h_\omega(\varepsilon)x_\beta(\xi)h_\omega(\varepsilon)^{-1} = x_\beta(\varepsilon^{(\beta,\omega)}\xi).$$

In the adjoint case, $h_\omega(\varepsilon)$ is uniquely defined by this condition, because then $h_\omega(\varepsilon) = h_{\mathrm{ad}}(\chi_{\omega,\varepsilon})$. However, in the simply-connected case $h_\omega(\varepsilon)$ is defined, generally speaking, only up to a factor in the center of the extended Chevalley group $\overline{G} = \overline{G}(\Phi, K)$; but this is not very important since the elements $h_\omega(\varepsilon)$ enter our calculations only in expressions like $[x, h_\omega(\varepsilon)]$. This essentially means that, in the context of extended groups, instead of subgroups containing $\overline{T}$ we could deal with subgroups invariant under diagonal automorphisms; nevertheless, the notation $h_\omega(\varepsilon)$ is rather convenient, especially since the construction of [8] can be used to refine the above definition so that the elements $h_\omega(\varepsilon)$ are uniquely defined in the simply-connected case, too, while all the necessary identifications for classical groups remain valid (see below).

In what follows we call any element $xh_\omega(\varepsilon)x^{-1}$, $x \in \overline{G}$, $\varepsilon \in K^*$, a weight element in the extended Chevalley group $\overline{G} = \overline{G}(\Phi, K)$ of type $\omega \in P(\Phi^\vee)$. Not all weight elements are equally important. The most common case in applications is when ω is $\overline{\omega}_i(\Phi)$, the fundamental weight of Φ. The following theorem, established in [25], defines the action of the elements $h_\omega(\varepsilon)$ in representations. As in §1, let π be a representation of $G = G(\Phi, K)$ in the K-vector space V_K defined as the tensor product of K and a certain admissible lattice in an irreducible representation V_L of the Lie algebra $L_\mathbb{C}$ (in other words, V_K is the Weyl module). Let $\Lambda(\pi)$ denote the set of weights of π.

THEOREM 7. *If $V_\mathbb{C}$ is irreducible, then $h_\omega(\varepsilon)$ acts diagonally on each weight subspace V^λ, $\lambda \in \Lambda(\pi)$ as multiplication by a certain scalar $c_\lambda \neq 0$, in such a way that for any two weights $\lambda, \mu \in \Lambda(\pi)$:*

$$c_\lambda c_\mu^{-1} = \varepsilon^{(\lambda,\omega)-(\mu,\omega)}.$$

We notice that the number in the exponent on the right is an integer, because the difference between two weights of an irreducible module belongs to $Q(\Phi)$. It is not always true that $c_\lambda = \varepsilon^{(\lambda,\omega)}$, since we have defined $h_\omega(\varepsilon)$ only up to a central factor and (λ, ω) need not be an integer. Considering an example, let us see what the weight elements look like in the usual representations (i.e., representations with highest weight $\overline{\omega}_1$) of the classical groups. For $\omega = \omega_k(A_l)$, we have

$$h_\omega(\varepsilon) = \mathrm{diag}(\varepsilon, \dots, \varepsilon, 1, \dots, 1),$$

where the number of ε's is k. For $\Phi = B_l$, C_l, D_l and $\omega = \overline{\omega}_k$, with the exception of $\overline{\omega}_{l-1}(D_l)$, $\overline{\omega}_l(D_l)$, $\overline{\omega}_l(C_l)$, we have

$$h_\omega(\varepsilon) = \mathrm{diag}(\varepsilon, \dots, \varepsilon, 1, \varepsilon^{-1}, \dots, \varepsilon^{-1}),$$

where the number of ε's and the number of ε^{-1}'s are both k. Finally, for the weight $\omega = \overline{\omega}_l$ in C_l, D_l, the element $h_\omega(\varepsilon)$ is the same as for the weight $\overline{\omega}_l(A_{2l-1})$. In general, the action of the weight elements may be quite surprising. Thus, a weight element of type $\overline{\omega}_1(E_6)$ in the representation of the highest weight $\overline{\omega}_1$ has one eigenvalue ε, ten eigenvalues ε^{-1}, and sixteen eigenvalues 1.

3. Bruhat decomposition of microweight elements. The weight elements of type ω have an especially simple structure when ω is a microweight. Recall that in that case the set $\Sigma_\omega = \{\alpha \in \Phi^+ \mid (\omega, \alpha) \neq 0\}$ is abelian, i.e., the sum of any two of its elements is not a root. All the fundamental weights for A_l; $\overline{\omega}_l$ for B_l; $\overline{\omega}_1$ for C_l; $\overline{\omega}_1, \overline{\omega}_{l-1}, \overline{\omega}_l$ for D_l; $\overline{\omega}_1, \overline{\omega}_6$ for E_6 and $\overline{\omega}_7$ for E_7 are microweights [10]. Microweight elements are particularly important because, as a rule, the torus $\overline{T}_{\mathrm{ad}}$ is generated by the microweight elements of the given type ω that it contains: it is sufficient that ω generate the weight lattice $P(\Phi^\vee)$ over the root lattice $Q(\Phi^\vee)$. Hence, in the general case, too, $\overline{T}$ is generated (under this assumption) by weight elements of type ω together with the center of the group $\overline{G}$. Thus the microweight elements play the same role in the theory of extended Chevalley groups that the LRSE's play in the theory of ordinary Chevalley groups. From the viewpoint of the Bruhat decomposition, the microweight elements are much simpler in structure than the LRSE's, as will be seen from the following theorem [22, 20, 25, 29]. Let $\overline{B} = \overline{B}(\Phi, K)$ denote a Borel subgroup of the extended Chevalley group $\overline{G}$, and let m denote the width of the partially ordered set Σ_ω. As usual, ω_α is reflection with respect to the root α.

THEOREM 8. *Let ω be a microweight; if $\Phi = B_l, C_l$, we also assume that* char$K \neq 2$. *Then every weight element of type ω belongs to one of the cosets*

$$\overline{B}w\overline{B} = \overline{B}w_{\gamma_1}\cdots w_{\gamma_{r+s}}\overline{B},$$

where $\gamma_1, \dots, \gamma_{r+s}$ are pairwise strictly orthogonal roots, and if $\gamma_1, \dots, \gamma_r$ are long roots and $\gamma_{r+1}, \dots, \gamma_{r+s}$ short roots, then $r + 2s \leq m$. Moreover, for fixed $x \in G$, the element ω in the Bruhat decomposition of $xh_\omega(\varepsilon)x^{-1}$ does not depend upon the choice of $\varepsilon \neq 0, 1$.

In the above-mentioned papers one can also find quite detailed information about the factors in the Borel subgroup. We list here all the microweights together with the appropriate values of m:

A_l,	$\omega = \varpi_k$,	$m = \min(k, l+1-k)$;
B_l,	$\omega = \varpi_1$,	$m = 2$;
C_l,	$\omega = \varpi_l$,	$m = l$;
D_l,	$\omega = \varpi_1$,	$m = 2$;
	$\omega = \varpi_{l-1}, \varpi_l$,	$m = [l/2]$;
E_6,	$\omega = \varpi_1, \varpi_6$,	$m = 2$;
E_7,	$\omega = \varpi_7$,	$m = 3$.

It follows from this table that the "smallest" semisimple element in a group of type E_6 is approximately of the same complexity as in the classical groups, while the complexity of the "smallest" semisimple element in a group of type E_7 is higher. As we have already said, the LRSE's are the simplest semisimple elements in groups of the type E_8, F_4, G_2.

§6. Extended Chevalley groups

Here we formulate analogs of the classification and conjugacy theorems for extended Chevalley groups. These theorems are actually simpler than Theorems 2 and 3 and were proved earlier. Throughout this section we will assume that $\Phi \neq E_8, F_4, G_2$. Of course, going over from G and T to $\overline{G}$ and $\overline{T}$, one has to modify the notion "standard", that is, replace the groups $G(S)$ and $N(S)$ in the definition by the groups $\overline{G}(S)$ and $\overline{N}(S)$ obtained from them by multiplication by $\overline{T}$.

THEOREM 9. *Let $|K| \geq 7$; then also if $\Phi = B_l, C_l$,* char$K \neq 2$. *Then the extended Chevalley group $\overline{G} = \overline{G}(\Phi, K)$ admits a standard description of subgroups containing a split maximal torus $\overline{T} = \overline{T}(\Phi, K)$. In other words, for every intermediate subgroup F, $\overline{T} \leq F \leq \overline{G}$, there exists a unique closed set of roots $S \subseteq \Phi$ such that*

$$\overline{G}(S) \leq F \leq \overline{N}(S).$$

This theorem was known before Theorem 2 and is used in the proof of the latter. For $GL(n, K)$, which is an extended Chevalley group of type A_{n-1}, the theorem was proved in 1976 by Borevich [1]. In 1978–1981 Borevich and the author proved a similar result for the general linear group over an arbitrary, not necessarily commutative semilocal ring Λ, (see, in particular, [3, 14] and a detailed survey of results of this kind in [42, 30]). This naturally requires redefining the notion "standard", to take the structure of the ideals of the underlying ring into account. The modification involved the notions of a net of ideals and net subgroups, first defined for the linear case in 1976 by Borevich and myself (see references in [1, 3, 30, 41, 42]), and, in the context of Chevalley groups, by Suzuki [164, 165] and the author and E. B. Plotkin [11, 34, 16]. The proofs relating to subgroups of $GL(n, \Lambda)$ were based on matrix technique, but in 1979 we found an invariant proof [13, 22].

Similar results were obtained in 1979, 1980 for other split classical groups: the general symplectic group $GSp(2l, R)$ and the general orthogonal group $GO(2l, R)$, i.e., extended Chevalley groups of types C_l and D_l, respectively [33, 169]. In these papers R is any commutative semilocal ring such that $2 \in R^*$ and all the residue fields of R contain at least 7 elements. A similar result for the odd-dimensional general orthogonal group $GO(2l+1, R)$, which is an extended Chevalley group of type B_l, was also obtained at that time, but published only in [27]. When writing [169], we did not realize that the results are valid not only for extended but also for ordinary orthogonal groups $SO(n, R)$. This fact, noted only in [27], may be explained by the observation that the groups $SO(n, R)$ are already extended by means of certain, though not necessarily all, diagonal automorphisms. The correct analogs of the groups $SL(n, R)$ and $Sp(2l, R)$ are of course the spinor groups $Spin(n, R)$.

In 1981 we proved Theorem 9 for extended Chevalley groups of types E_6, E_7 too. The result was announced in [14], and parts of the proof were presented in [20, 25, 26]. The treatment of all these cases, except the case of symplectic groups, is based on the analysis of the microweight elements contained in F (in symplectic groups the structure of LRSE's is much simpler than that of microweight elements, so the latter are used in [33] only at the last stage).

An analog of the conjugacy theorem for extended groups is valid under somewhat weaker assumptions on the base field.

THEOREM 10. *Under the assumptions of Theorem 9, the split maximal torus $\overline{T} = \overline{T}(\Phi, K)$ is pronormal in the extended Chevalley group $\overline{G} = \overline{G}(\Phi, K)$. In particular, if two subgroups $\overline{T} \leq F_1, F_2 \leq \overline{G}$ are conjugate in $\overline{G}$, then they are already conjugate by an element of $\overline{N}$.*

This theorem was first proved for the general linear group over a local matrix ring [12, 170]. (Recall that a ring Λ is called a local matrix ring if its quotient ring over the Jacobson radical is the complete matrix ring over a division ring.) Koĭbaev then showed that the conjugacy theorem is valid for the complete linear group even when there are no standard descriptions, namely, for the fields F_4 and F_5. Moreover, he found a way to modify the formulation of the conjugacy theorem so that it also included the case of the field F_3. Of course, the group $\overline{T} = D(n, F_3)$ of diagonal matrices is not pronormal in $\overline{G} = \mathrm{GL}(n, F_3)$. In fact, if $n = 2$ the groups $\overline{T}$ and $x\overline{T}x^{-1}$, where

$$x = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

are contained in the group $\overline{N} = N(2, F_3)$ of monomial matrices, but are certainly not conjugate in it. However, it was shown in [48] that this counterexample is, essentially, the only one. Theorem 10 was established for the symplectic and orthogonal cases in [33, 27]; in [26] one can find a general proof based on calculating the Bruhat decomposition for microweight elements (Theorem 8).

§7. Conjugacy and reduction theorems

We will now outline the proofs of Theorems 3 and 10, and create the basis for the induction on $|\Phi|$ used to prove Theorems 2 and 9.

1. Extraction of unipotent elements. The following theorem gives a description of unipotent subgroups normalized by a maximal split torus.

THEOREM 11. *Let* $|K| \geq 5$. *Then for any element* $u = \prod x_\alpha(u_\alpha) \in U$, *where the product is taken over all positive roots in arbitrary order,* $x_\alpha(u_\alpha) \in \langle H, uHu^{-1}\rangle$ *for any* $\alpha > 0$.

This result was actually proved in [145, 87] for finite ground fields, in a stronger form. It is clear that the proof for an infinite field can only be simpler (see, e.g., [79]). In fact, far more general results than Theorem 11, pertaining to arbitrary commutative rings, were proved in [164, 11].

COROLLARY. *Under the assumptions of Theorem* 11, T *is an abnormal subgroup of* B.

If the characteristic of K satisfies the same conditions as in Theorem 2, it follows immediately from Theorem 11 that $\langle T, uTu^{-1}\rangle = \langle T, u^{-1}Tu\rangle = \langle T, u\rangle$ is just $G(S)$, where S is the closure of the set of positive roots α such that $u_\alpha \neq 0$ in a certain fixed order. The following result is basic for the study of subgroups contained in a proper parabolic subgroup.

THEOREM 12. *Let* $|K| \geq 7$, *and let* P *be a standard parabolic subgroup of a Chevalley group* G. *Represent* $x \in P$ *as* $x = yu$, *where* $y \in L_P$ *and* $u \in U_P$. *Then* $u \in \langle T, x^{-1}Tx\rangle$.

2. Conjugacy theorems. The Tits theorem proved in [70] can be stated as follows in the present context.

TITS THEOREM. *Let* $|K| \geq 4$. *If there exist* $S \subseteq \Phi$ *and* $x \in G$ *such that* $xTx^{-1} \leq G(S)$, *then* $x = yw$ *for some* $w \in N$ *and* $y \in G(S)$.

Tits actually proved his theorem for any groups with "normal root data". The conjugacy theorem clearly follows immediately from the classification theorem and the following result, which we will refer to as the conjugacy theorem for standard subgroups. Being a generalization of the Tits theorem, it is one of the key steps in the proof of the classification theorem itself.

THEOREM 13. *Let* $|K| \geq 7$. *If there exist* $S \subseteq \Phi$ *and* $x \in G$ *such that* $xTx^{-1} \leq N(S)$, *then* $x = yw$ *for some* $w \in N$ *and* $y \in G(S)$.

Theorem 13, in turn, follows immediately from the Tits theorem and the following result, which we shall call the connectedness theorem. For an infinite ground field, this theorem follows immediately from the fact that $G(S)$ is a connected component of $N(S)$ in the Zariski topology.

THEOREM 14. *Let* $|K| \geq 7$. *For any* $x \in G$, *if* $xTx^{-1} \leq N(S)$, *then* $xTx^{-1} \leq G(S)$.

An analogous result for extended Chevalley groups was proved in [26]. However, the results of [26], applied to ordinary Chevalley groups, give a less sharp bound on $|K|$ than the bound in Theorem 14. Only Theorems 5 and 6 allow us to lower the bound to $|K| \geq 7$.

The connectedness theorem is proved by induction on $|\Phi|$. Theorem 12 at once reduces the case in which $N(S)$ is contained in a proper parabolic subgroup (as we know from the Borel-Tits theorem, this is always the case if $S^u \neq \varnothing$) to groups of lower rank. In what follows, therefore, we may assume that $S = \Delta$ is a subsystem of roots such that $N(\Delta)$ is not contained in a proper parabolic subgroup. We first examine the Bruhat decompositions of elements of $N(\Delta)$.

LEMMA. *Let* $|K| \geq 4$ *and* $\Delta \subseteq \Phi$. *Then every element of* $N(\Delta)$ *can be expressed uniquely as* $x = uwvd$, *where* $u \in U(\Delta)$, $w \in X(\Delta)$, $v \in U^-_{w_0}(\Delta)$. *Here* $w_0 \in W(\Delta)$ *is defined by* w *in the following way*: $w_1 = ww_0^{-1}$ *preserves the basis of* Λ *contained in* Φ^+.

It is now clear that a proof of the connectedness theorem can proceed as follows: we have to consider the Bruhat decomposition of the LRSE $y(\varepsilon) = xh_\delta(\varepsilon)x^{-1}$, calculated in §4, and check that, if it is of the type described in the lemma, then necessarily $w \in W(\Delta)$. A relatively simple inductive argument reduces everything to the case in which $N(\Delta)$ is maximal with respect to the standard subgroups and not connected, i.e., $N(\Delta) \neq G(\Delta)$. All such groups are listed in Theorem 1.

Indeed, let $y(\varepsilon) \in N(\Delta)$. By Theorem 5 we can assume that for all ε except $\varepsilon = 0, 1$, and at most two other values of ε, the element $y(\varepsilon)$ lies in a coset BwB, where $w = w_{\beta_1} \cdots w_{\beta_m}$ and $\beta = \beta_1, \ldots, \beta_m$ are pairwise strictly orthogonal roots, $m \le 4$, and w does not depend on ε. Calculations of factors in B described in §4 imply that the coefficients $u(\varepsilon)_\beta$ and $v(\varepsilon)_\beta$ in the Bruhat decomposition $y(\varepsilon) = u(\varepsilon)wd(\varepsilon)v(\varepsilon)$ are nonzero rational functions of ε (their denominators may vanish at the excluded values of ε). By Theorem 6, the numerators and denominators of these functions are of degree at most two. Hence, excluding at most two more values (this gives at most 6 forbidden values, so ε exists if K contains at least 7 elements), we may assume, for example, that $u(\varepsilon) \ne 0$. But then by the lemma, $\beta \in \Delta$. As this argument may be applied to all β_i, it follows that $w \in W(\Delta)$, i.e., $y(\varepsilon) \in G(\Delta)$ for all typical ε. It remains to notice that, since $|K| \ge 7$, the root torus is generated by its typical elements, and therefore $y(\varepsilon) \in G(\Delta)$ for all ε, proving the connectedness theorem.

3. Reduction theorem. The following result reduces the description of all subgroups containing T to those subgroups that are not contained (up to conjugacy) in a proper standard subgroup.

THEOREM 15. *Assume that $|K| \ge 7$ and that the classification theorem for all proper root subsystems in Φ is valid. If F is a subgroup of the Chevalley group $G = G(\Phi, K)$ containing $T = T(\Phi, K)$ and contained in a subgroup conjugate to a proper standard subgroup, then F is itself standard.*

Therefore, a similar assertion holds for extended Chevalley groups. Since T in fact induces nontrivial diagonal automorphisms on most of the standard subgroups, it is usually sufficient to assume that, in the statement of Theorem 15, the standard description holds in the extended Chevalley group corresponding to a proper subsystem Δ of Φ. In particular, there is no nonsymplectic system for which SL_2 admits standard descriptions.

The proof of Theorem 15 may be sketched as follows. It follows easily from the results of the two preceding subsections that for $|K| \ge 7$ any subgroup that contains T and is conjugate to a standard subgroup, is itself standard. Thus, we may assume from the start that $F \le N(S)$, where $S \subseteq \Phi$ (if desired, we may even assume that $N(S)$ is a maximal standard subgroup). Now, if $N(S) = P$ is a parabolic subgroup then, representing $x \in F$ as $x = yu$, where $y \in L_P$ and $u \in U_P$, we conclude by Theorem 12 that $u \in F$. But, by assumption, y together with the maximal torus of the group $[L_P, L_P]$ generate a standard subgroup of the latter. Therefore, y and T generate a standard subgroup of G. Hence, the subgroup generated by T and an arbitrary element $x \in F$ is standard (as the composite of standard subgroups $\langle T, u\rangle$ and $\langle T, y\rangle$). But then, of course, F itself is standard. If F is contained in a reductive standard subgroup, the proof is even simpler. Indeed, let $T \le F \le N(\Delta)$, where Δ is a root subsystem in Φ. As we have just pointed out, the assumptions of the theorem imply that the subgroup $F \cap G(\Delta)$ is a standard subgroup. But the conjugacy theorem for standard subgroups states, in particular, that the normalizer of a standard subgroup is standard.

It is convenient to use Theorem 15 in the following form.

COROLLARY. *Under the assumptions of the theorem, let $x \in G$ be an element of a proper standard subgroup. Then if $x \notin N$, $\langle T, x\rangle$ contains an elementary root unipotent subgroup X_α.*

In fact, we shall see in the following section that an analogous assertion is true without the assumption that x lies in a proper standard subgroup.

§8. Extraction of a root unipotent

The following results are the crucial — and the most difficult — steps in the proofs of Theorems 2 and 8. Throughout, we will continue to assume that the characteristic of the ground field satisfies the same conditions as in Theorem 2.

THEOREM 16. *Let K be an infinite field. Then any subgroup of the Chevalley group $G = G(\Phi, K)$ that contains the split maximal torus $T = T(\Phi, K)$ but is not contained in its normalizer $N = N(\Phi, K)$ contains an elementary root subgroup X_α, $\alpha \in \Phi$.*

THEOREM 17. *Let $\Phi = A_l, B_l, C_l, D_l, E_6, E_7$, and $|K| \ge 7$. Then any subgroup $\overline{G} = \overline{G}(\Phi, K)$ containing the split maximal torus $\overline{T} = \overline{T}(\Phi, K)$, but not contained in its normalizer $\overline{N} = \overline{N}(\Phi, K)$ contains an elementary root subgroup X_α, $\alpha \in \Phi$.*

In this section we shall only illustrate some very general ideas underlying the proof. The point is that all proofs known to me are technical in nature and need voluminous calculations, with each case examined separately. For the classical groups the proofs, formulated in Lie and matrix languages, are scattered over a large number of papers (see references in Subsections 3, 6 and the bibiliography in [30]). All the details are collected in my thesis [24]. The proofs for the exceptional groups have not yet been published in full because of their length.

1. General strategy. By the corollary to Theorem 15, it suffices to show that Φ contains an element $g \notin N$ that belongs to a proper standard subgroup. We will do this as follows. Since $F \not\le N$, it follows from the connectedness theorem that we may choose a one-parameter weight subgroup of F,

$$Q = \{y(\varepsilon) : xh_\omega(\varepsilon)x^{-1}, \ \varepsilon \in K^*\},$$

not contained in N. For ordinary Chevalley groups, we shall as a rule define $h_\omega(\varepsilon)$ as a LRSE $h_\alpha(\varepsilon)$; for extended Chevalley groups it will be a microweight element corresponding to the weight $\overline{\omega}_1$ (for the symplectic case, see subsection 4). Nevertheless, there are other alternatives; for example, in the proofs of Theorems 16 and 17 for groups of type B_l, it proves most convenient to use the elements $h_{\overline{\omega}_l}(\varepsilon)$ [24], which are not microweight elements.

We want to show that the required element g may already be found in the subgroup $\langle T, Q\rangle$ generated by T and Q (of course, we may assume that Q itself is not contained in a proper standard subgroup). As a rule, however, we will not be able to combine elements from T and Q at once so as to obtain an element with the required properties. A remarkable exception is the general linear group, for which g is always available in QTQ; this is yet another confirmation of the Verma principle (the famous Harish-Chandra principle claims that all reductive groups have the same structure, Verma stated that the structure of the general linear group is even more uniform than that of all others). In the general case one has to proceed by induction: starting with Q, one determines another one-parameter subgroup Q' of F, also not in a proper standard subgroup, which is in some sense simpler than Q itself.

Recall that almost all elements of Q lie in the same coset Bw_0B of the Bruhat decomposition, and that, moreover, w_0 is an involution. Our criteria for the complexity of the group Q are as follows, in order of decreasing priority: the number m of root reflections in the element $w_0 = w_{\beta_1}, \dots, w_{\beta_m}$ (i.e., the number of eigenvalues of w_0 other than 1, denoted by $\bar{l}(w_0)$ in [84]); the length $l(w_0)$ of w_0; the height h of the highest of the roots $\beta_1, \dots, \beta_m$; the number of nonzero coefficients in factors from the Borel subgroup; the validity of certain equations or inequalities for these coefficients. In each case one can show that $\langle T, Q\rangle$ contains a one-parameter subgroup $Q' \not\leq N$ of weight elements of simpler structure compared with Q, i.e., with smaller h, and so on. One then begins to analyze the group $\langle T, Q'\rangle$, eventually locating the required g in one of these subgroups. We shall now show approximately how this is done in a few simple cases (for the classical groups see also [1, 3, 15, 18, 22, 24, 27, 32, 33, 38]).

2. Group SL_2. The only groups of semisimple rank 1 among the groups of normal types are SL_2 and GL_2. We will reproduce a quite fantastic proof by King for the group SL_2 [119]; the much easier case of GL_2 was known long before and will be considered in the next subsection, in a far more general context.

Let $\Phi = A_1$ and $T \leq F \leq G = SL(2, K)$, but $T \not\leq N$, B, B^-. In the case of a finite field, all the subgroups of G are known. We may therefore assume that K is infinite. Replacing F by a subgroup conjugate to it by a diagonal matrix in $GL(2, K)$, we may assume that F contains an element

$$f = \begin{pmatrix} 1 & 1 \\ \alpha & 1+\alpha \end{pmatrix}.$$

Replacing F, if necessary, by $\langle T, fTf^{-1}\rangle$, we may also assume that α does not belong to a fixed finite subset in K^*. Denote the root element $h_\delta(\varepsilon)$ by

$$h(\varepsilon) = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}.$$

If now $\operatorname{char} K \neq 2, 3, 5$, King chooses $\alpha \neq 0, 1, \pm 2, \pm 3, -4, 5, -6$ and forms the following product:

$$\begin{aligned} g = {} & f^{-1}h\left(\frac{2}{\alpha}\right)f^{-1}h\left(\frac{\alpha+4}{\alpha+2}\right)fh\left(\frac{3}{4}\right)f^{-1}h\left(\frac{3}{\alpha-3}\right)f^{-1} \\ & \times h\left(\frac{\alpha+6}{\alpha+3}\right)fh\left(\frac{5}{6}\right)f^{-1}h\left(\frac{\alpha-2}{\alpha-5}\right)fh(3)f^{-1}h\left(\frac{\alpha+3}{\alpha+2}\right)f \\ & \times h\left(\frac{2}{3}\right)f^{-1}h\left(\frac{\alpha-1}{\alpha-2}\right)fh(\alpha-3)fh\left(\frac{4}{3}\right)f^{-1}h\left(\frac{\alpha+2}{\alpha+4}\right)fh\left(\frac{\alpha}{2}\right)f. \end{aligned}$$

Calculations show that $g_{21} = 0$, and g_{12} vanishes only for a finite number of values of α. Therefore, a certain group conjugate to F by a diagonal matrix contains an elementary transvection, and hence $F = G$. If $\operatorname{char} K = 3, 5$, King constructs other, shorter formulas expressing a transvection in terms of f and $h(\varepsilon)$.

Of course, our search for an element g in groups of rank greater than 1 does not require such ingenuity as King's formula, but it is not reasonable to expect it to be substantially simpler.

3. Reflections with respect to a single root. Let us return to the notation of subsection 1. Let w be a microweight. We will outline a proof of Theorem 17 for the simplest case, when F contains a one-parameter subgroup Q of weight elements of type W, such that if $\varepsilon \neq 0, 1$ the element $\mu(\varepsilon)$ belongs to the class $\overline{B}w_\alpha\overline{B}$. In particular, this will prove Theorem 17 and also — by McLaughlin's theorem (see [141] and §9 of this paper) — Theorem 9 for the simplest case of the general linear group. In fact, our proof exactly models the original proof [1, 3, 15, 22] for this case.

Namely, we claim that elements $\varepsilon, \eta, \theta \in K \setminus \{0, 1\}$ can be chosen in such a way that

$$z = y(\varepsilon)h_\omega(\theta)y(\eta^{-1})$$

is contained in a proper parabolic subgroup, but not in N. We may of course assume that the support of the root α is Π (otherwise $\overline{B}w_\alpha\overline{B}$ is contained in a standard parabolic subgroup). As we know from §5 (see also [20, 22, 24, 25, 29] for the details), the Bruhat decomposition of $y(\varepsilon)$ is $y(\varepsilon) = u(\varepsilon)n(\varepsilon)v(\varepsilon)$, where

$$n(\varepsilon) = w_\alpha(\pm c(\varepsilon-1)^{-1})h_{w\omega}(\varepsilon) \in N,$$

and $u(\varepsilon)$, $v(\varepsilon) \in U$. In fact, $v(\varepsilon) = x_\alpha(v(\varepsilon)_\alpha)v_1v_2$ belongs to the Heisenberg group $Y_\alpha = U^-_{w_\alpha}$, where $v(\varepsilon)_\alpha$ is a rational function of ε, with denominator and numerator of degree at most 1; v_1 and v_2 do not depend on ε and are products of root elements $x_\gamma(v_\gamma)$ over all γ_1 and γ_2, respectively, where γ_1 and γ_2 are positive roots such that $\alpha = \gamma_1 + \gamma_2$, and $\gamma_1 \in \Sigma_\omega$ (see the definition of this set in §6).

Now choose ε and η, differing from 0, 1, such that $v(\varepsilon)_\alpha, v(\eta)_\alpha \neq 0$, $v(\varepsilon)_\alpha \neq v(\eta)_\alpha$, and then take $\theta = v(\varepsilon)_\alpha v(\eta)_\alpha^{-1}$. It is clear that $\theta \neq 0, 1$

and, by the very choice of θ,

$$x_\alpha(v(\varepsilon)_\alpha)h_\omega(\theta)x_\alpha(v(\eta)_\alpha)^{-1} = h_\omega(\theta).$$

Obviously, v_2 commutes with $h_\omega(\theta)$ and

$$[v_1, h_\omega(\theta)] = \prod x_\gamma(v_\gamma(1-\theta)),$$

where the product is taken over all $\gamma \in \Sigma_\omega$ such that $\alpha - \gamma \in \Phi^+$. Let $\omega = \overline{\omega}_i$; then the simple root α_i occurs in the factorization of each of these roots γ with the coefficient 1. Therefore, the support of each of the roots $w_\alpha\gamma = \gamma - \alpha$ is contained in $\Pi \setminus \{\alpha_i\}$. This means that

$$z_0 = n(\varepsilon)v(\varepsilon)h_\omega(\theta)v(\eta)^{-1}n(\eta)^{-1}$$

is an element of the subgroup $\overline{G}(\pm(\Pi\setminus\{\alpha_i\}), K)$, so that z lies in a proper standard parabolic subgroup $\overline{G}_i = \overline{G}(P_i)$.

Careful calculation of the factors from the Borel subgroup will now show that, by excluding at most six values of ε, we can choose η with the required properties, in such a way that z does not belong to $\overline{N}$; this concludes the proof. Note that the above argument goes through almost completely for LRSE's in the symplectic group, except that the numerator and denominator of $v(\varepsilon)_\alpha$ then depend on ε not as linear but as quadratic functions, so we have to impose two restrictions on ε instead of only one; hence the stronger requirements on the order of the field K.

4. Symplectic group. The case of Sp_{2l} was the last to be considered, because it finally reduces to the group SL_2 and not to GL_2, like the others. However, the actual reduction to a lower rank is much easier for Sp_{2l}, $l \geq 2$, than for all other groups, except GL_n. We will now translate the proof of [33] into our language (see also [38, 24]), with due attention to the simplifications suggested in [66].

We have to show that there exists a long root torus Q in F lying in the coset $Bw_\gamma B$; having done this, we will be able to refer to subsection 3. Let $Q = \{y(\varepsilon), \varepsilon \in K^*\}$ be a long root torus in F that is not contained in N. According to §4, all the elements with $\varepsilon \neq 0, \pm 1$ lie in the same coset $Bw_\alpha w_\beta B$, where α and β are two orthogonal long roots, but the element $y = y(-1)$ cannot be in this coset. A computation shows that $y(-1)$ belongs either to B or to $Bw_\gamma B$, where γ is a short root. Without loss of generality, we may assume that $\gamma = \varepsilon + \varepsilon_i$, where $2 \leq i \leq l$ (here we are using the ordinary notation for root systems, see [10]), for otherwise Q is contained in a proper standard parabolic subgroup G_1 or G_l.

Now it is easy to see that the root torus $\widetilde{Q} = \{yh_\delta(\varepsilon)y^{-1}, \ \varepsilon \in K^*\}$ is contained either in a proper parabolic subgroup G_1 or in $Bw_\gamma B$, depending on whether the coefficient $v(-1)_\gamma$ vanishes or not (this is because $w_\gamma X_\alpha w_\gamma^{-1} \leq G_1$ for all roots $\alpha \in \Sigma$, $\alpha \neq \gamma$). Thus, referring to §7 and subsections 2, 3, we can complete the proofs of Theorems 16 and 17 for the symplectic case, provided only that $\widetilde{Q}$ can be chosen nondiagonal. If $\widetilde{Q}$ is diagonal, it follows from what we know about the conjugacy classes of semisimple elements and their centralizers (see, e.g., [67, 85]) that y is of the form $y = wz$, where $w \in W$ and $z \in G(C_{l-1}, K)$ for an appropriate imbedding $C_{l-1} \leq C_l$. Since $\langle T, y^{-1}Ty\rangle = \langle T, z^{-1}Tz\rangle$, we may again refer to Theorem 15. Thus, the required $\widetilde{Q}$ exists, provided the matrix y itself does not belong to N.

Now, if indeed $y \in N$, this actually imposes certain conditions on the coefficients of the factors in the Bruhat decomposition of the element x such that $y = xh_\delta(-1)x^{-1}$. We must therefore change x. Indeed, replace x by $y(\varepsilon)$ and, naturally, y by $z(\varepsilon) = y(\varepsilon)h_\delta(-1)y(\varepsilon^{-1})$. If $z(\varepsilon) \in N$ for all ε, then (since $|K| \geq 7$) in fact $z(\varepsilon) \in T$, and once again our information about the centralizers of semisimple elements implies that $y(\varepsilon)$ belongs to the subset $W \cdot G(C_{l-1} + A_1, K)$. Since, by assumption, Q itself is not contained in N, we may apply the inductive hypothesis, to conclude that $X_\alpha \leq F$ for some $\alpha \in \Phi$.

The proofs of Theorems 16 and 17 in other cases are based on repeated applications of the above and similar techniques, but even for SL_n, Spin_{2l}, and SO_{2l+1} — not to speak of the exceptional groups — they are far more complicated. Complete arguments at every stage of the proofs involve huge amounts of calculation.

§9. Subgroups generated by root subgroups

In this section we shall complete our outline of the proofs of Theorems 2 and 9 and discuss what must still be done to make this part of the proof completely natural.

1. Completing the proof of the classification theorems. We continue to assume that the characteristic of the ground field satisfies the usual conditions. The following result is the last step in the proof of the classification theorems.

THEOREM 18. *Let* $|K| \geq 7$. *Then there are no proper subgroups in* G *that contain* TX_α *for some* $\alpha \in \Phi$ *but are not contained in proper standard subgroups.*

It is clear that the classification theorems follow immediately from Theorem 18 modulo the previous results. Indeed, let F, $T \leq F \leq G$, be an intermediate subgroup. If F lies in a proper standard subgroup, then by Theorem 15 it is standard itself. Suppose, therefore, that F is not contained in any proper subgroup. Then, by Theorems 16 (or 17, for extended groups) and 18, F coincides with G.

Theorem 18 is proved as follows. Let S be the maximal set of roots such that $G(S) \leq F$. By assumption, $\alpha \in S$, so S is nonempty. If $S \neq \Phi$, we choose a unipotent root subgroup X in F, not lying in $G(S)$ (such a subgroup exists, since F is not contained in $N(S)$). Commuting X with elements of $G(S)$, one can construct an elementary root subgroup X_β in F not lying in $G(S)$, and this contradicts the definition of S. The proof of this fact imitates proofs of Theorems 16 and 17, but it is substantially simpler, since the computations are done with unipotent root subgroups instead of one-parameter subgroups of semisimple elements. The former have a much simpler structure, which is the same for all series; for example, a long root subgroup X always lies in one of the cosets $Bw_\alpha B$; to be precise, it is of the form $uX_\alpha u^{-1}$ for a suitable $u \in U(\Phi, K)$.

In reality, however, Theorem 18 imposes a needlessly stringent condition on F, demanding that it contains not just X_α but also T. It would be far more natural to list all the "irreducible" subgroups — in some sense of the word — that contain the root subgroups or the subgroups generated by the root subgroups, and then to show that the only group in the list that contains T is G itself. Suitable concepts of "irreducibility" might be the property of not being contained up to conjugacy in a proper standard subgroup, which is an analog of primitivity and irreducibility; or in a proper subgroup of the form $G(S)$ — an analog of ordinary irreducibility; or in a proper parabolic subgroup (in the linear case this is equivalent to the preceding property but it is generally weaker).

We shall now briefly discuss the current situation. More detailed information and references can be found in reviews by Zalesskiĭ [41, 42], Kantor [116], Kondrat′ev [54], and the author [29]. Of the more recent publications, not mentioned there, Brown and Humphries [82, 108–110], Li Shangzhi and Zha Jianguo [127–130, 134, 135] should be mentioned. We are preparing a survey "Generation in Chevalley groups", in which the available results will be presented in detail, an exhaustive bibliography will be compiled, and unsolved problems will be formulated. For the moment, therefore, we restrict ourselves to the essentials.

2. Classical groups. In the classical cases, a full range of results is available. To illustrate, we cite the following theorem, first proved in [141]. Recall that $H \sim F$ means that the subgroups H and F are conjugate in G.

McLaughlin's Theorem. *Let L be an irreducible subgroup of* $\mathrm{GL}(n, K)$, $|K| \geq 3$, *generated by root subgroups. Then either* $L = \mathrm{SL}(n, K)$ *or* $n = 2l$ *is even and* $L \sim \mathrm{Sp}(2l, K)$.

Of course, if F is a primitive irreducible subgroup of G containing X_α then, by Clifford's theorem (see, e.g., [69]), the normal subgroup L of F generated by all the subgroups of F conjugate to X_α is also irreducible, so that either $\mathrm{SL}(n, K) \leq F \leq \mathrm{GL}(n, K)$ or $n = 2l$ and $\mathrm{Sp}(2l, K) \leq F \leq \mathrm{GSp}(2l, K)$. However, it is easy to verify that $\mathrm{GSp}(2l, K)$ does not contain, up to conjugacy, the group $D(2l, K)$ of all diagonal matrices.

McLaughlin in fact proved a similar result for the field F_2, where some additional subgroups appear. Subsequently, Wagner, Zalesskiĭ and Serezhkin, Pollaczek, in a long series of papers, described all the finite irreducible subgroups generated by transvections (see references in [41, 42, 46]). Mclaughlin's theorem has been generalized and specialized in other directions. In particular, Brown and Humphries have "effectivized" it, that is, constructed an algorithm that makes it possible to establish, given a collection of root subgroups, which subgroup they generate.

Subgroups of orthogonal groups generated by root subgroups in the case of a finite base field have been studied by Stark (Saltzberg) in [158–160]. However, her result involved an inaccuracy, which was corrected by Kantor in the more general context of subgroups generated by root elements [114]. An attempt to generalize these results in part to the case of an infinite field was made by Andreassian [74]. The proofs of her results, however, were never published; and, the author has been informed by Kantor that the proof presented in Andreassian's thesis contains an error. The first proof of the following result was published by Li Shangzhi [128]. Independently, but later, this result was obtained by the author [24, 29].

Theorem 19. *Let L be an irreducible subgroup of $G = \mathrm{SO}(n, K)$,* $\mathrm{char} K \neq 2$, $|K| \geq 5$, *generated by long root subgroups. Then one of the following possibilities holds*:

(1) $L = \Omega(n, K)$,
(2) $n = 4m$, $L \sim \mathrm{SU}(m, E)$, $[E : K] = 2$,
(3) $n = b$, $L \sim \mathrm{Spin}(7, K)$,
(4) $n = 7$, $L \sim \mathrm{G}_2(K)$.

In [128] — and elsewhere — Li Shangzhi has actually done more: he determined all the maximal subgroups of the classical groups that contain root subgroups. The following results, proved by Li Shangzhi in [128, 130] (see also [159]), complete the description of subgroups of the classical groups that are generated by root subgroups.

Theorem 20. *Let* $\mathrm{char} K \neq 2$. *Then there are no irreducible subgroups in* $G = \mathrm{SO}(m, K)$ *that are generated by short root subgroups but do not contain long root subgroups.*

Theorem 21. *Let* $\mathrm{char} K \neq 2$ *and let L be an irreducible subgroup of* $G = \mathrm{Sp}(2l, K)$ *generated by short root subgroups and containing no long root subgroups. Then* $l = 2m$ *and* $L \sim \mathrm{Sp}(2m, E)$ *for some extension E,* $[E : K] = 2$.

Thus, for the case of the classical groups everything is clear, although there are of course quite a lot of unsolved problems such as the "effectivization" of Theorems 19 and 21.

3. Exceptional groups. Here the situation is totally different. All we have is a series of papers by Cooperstein [89–91], in which he describes subgroups of the finite exceptional groups that are generated by long root subgroups but not contained in a proper parabolic subgroup. Here, for instance, one has such imbeddings as $G(F_4, K)$, $G(C_4, K) \le G(E_6, K)$, but we shall not give the exact formulation of the result [54]. Later Cooperstein also described the "irreducible" subgroups of finite Chevalley groups generated by long root elements [92]. For an algebraically closed field, the maximal connected subgroups containing a long root subgroup were recently described by Seitz [156].

For the time being, the generalization of these results to arbitrary fields is an open problem. There are some partial results by the present author: the existence of two opposite root subgroups in any subgroup generated by long root subgroups but not contained in a proper parabolic subgroup; a description of orbits of a Chevalley group, acting by conjugacies on the sets of pairs and triples of long root subgroups; and so on (see [24, 29] and the references given there). Cooperstein and the author are presently working on this problem and hope to complete the solution in the near future.

As for subgroups generated by short root subgroups, the situation is even less clear. The problem has not been solved even for a finite base field. It is not even known exactly what subgroups can be generated by two short root subgroups of a Chevalley group of type F_4.

§10. Overgroups of nonsplit maximal tori

Thus, for almost any ground field we have a satisfactory description of subgroups of Chevalley groups that contain a split maximal torus. I do not believe there is any hope of obtaining an equally explicit description for overgroups of an arbitrary maximal torus over general fields. For some important classes of fields, such as local or global fields, there are nevertheless significant results, and for the finite case the results are almost definitive. A brief survey of these results follows.

1. Steinberg's theory. A remarkable paper of Seitz [150] gives an almost exhaustive description of subgroups of finite groups of Lie type that contain an arbitrary maximal torus. This description relies essentially on Steinberg's theory [162, 67, 85, 86]. It is convenient, therefore, to alter our notation slightly as compared to the rest of the paper. Thus, $\overline{G} = G(\Phi, \overline{K})$ will now denote not an extended Chevalley group but a Chevalley group of type Φ over the algebraic closure $\overline{K}$ of a finite field $K = F_q$ considered as an algebraic group. Similarly, $\overline{T} = T(\Phi, \overline{K})$, $\overline{B} = B(\Phi, \overline{K})$, etc.

Recall that a standard Frobenius endomorphism $\sigma = \sigma_q$ of $\mathrm{GL}(n, \overline{K})$ is defined by raising the matrix coefficients of σ to the qth power : $(x_{ij}) \to (x_{ij}^q)$. A standard Frobenius endomorphism of a Chevalley group G is the mapping $\sigma : \overline{G} \to \overline{G}$ induced by σ_q under a certain imbedding $i : \overline{G} \to \mathrm{GL}(n, \overline{K})$. Finally, a Frobenius endomorphism is a mapping $\sigma : \overline{G} \to \overline{G}$, some power of which is a standard Frobenius endomorphism. Steinberg showed that in this case the group $\overline{G}^\sigma$ of fixed points of σ is finite. Conversely, any finite group G of Lie type is such that $O^{p'}(\overline{G}^\sigma) \le G \le \overline{G}^\sigma$ for some σ, where p is the characteristic of K and $O^{p'}(G)$ denotes, as usual in the theory of finite groups, the subgroup of G generated by all p-elements. The major technical tool of the theory is the Lang-Steinberg theorem, according to which the mapping $g \mapsto g^{-1}\sigma(g)$ is surjective on $\overline{G}$.

By definition, a maximal torus T of G is of the form $G \cap \overline{T}$, where $\overline{T}$ is some σ-invariant algebraic maximal torus, that is, $\sigma(\overline{T}) = \overline{T}$. It is well known [67, 85, 86, 95, 162] that the conjugacy classes of maximal tori in G are in one-to-one correspondence with the σ-conjugacy classes of the Weyl group $W = W(\Phi)$. Recall that σ acts naturally on W and that two elements $x, y \in W$ are said to be σ-conjugate if there exists $w \in W$ such that $w^{-1}x\sigma(w) = y$. Since an untwisted Frobenius endomorphism acts trivially on W, it follows that, in Chevalley groups of normal types, maximal tori correspond to ordinary conjugacy classes of the Weyl group [44, 84]. Namely, if $\overline{T}$ is a σ-invariant torus contained in a σ-invariant Borel subgroup $\overline{B}$, then to each $w \in W$ there corresponds a torus T_w constructed in the following way: take $g \in \overline{G}$ such that $g^{-1}\sigma(g) = w$ and $T_w = G \cap g^{-1}\overline{T}g$. The order of the torus T_w, its index in the normalizer, etc., can easily be expressed in terms of w [44, 84, 85].

Maximal tori, especially minisotropic ones (that is, those not contained in proper parabolic subgroups), play a crucial role in the Deligne-Lusztig theory of complex representations of finite Chevalley groups [95, 139] (see also [85, 86]). Given a character of a torus T, one uses l-adic cohomology theory to construct a virtual character $R_{T,\theta}$ of a Chevalley group G, known as the Deligne-Lusztig character. If θ is in general position, then $R_{T,\theta}$ is, up to sign, a real character of G and, moreover, an irreducible one. In this procedure minisotropic tori lead to cuspidal characters. This explains the intense interest in maximal tori in Chevalley groups; they are also of extreme importance in describing conjugacy classes of elements. Besides the papers already cited, we also mention Veldcamp [171, 172].

2. Seitz's theory. We retain the notation of the previous subsection. Though Seitz [150] studies all finite groups of Lie type, we restrict ourselves to the normal types, i.e., the case in which σ is an untwisted Frobenius endomorphism.

Let $\overline{X}_\alpha$, $\alpha \in \Phi$, be a collection of elementary root subgroups of the group $\overline{G}$ with respect to $\overline{T}$. Since $\overline{T}$ is σ-invariant, these subgroups are permuted by the action of σ. Let $\Delta = \{\overline{X}_1, \dots, \overline{X}_m\}$ be a σ-orbit of these root subgroups. Set $\overline{X} = \langle \overline{X}_1, \dots, \overline{X}_m \rangle$. Then the groups $X = O^{p'}(\overline{X}^\sigma)$ will be called T-root subgroups of G [148, 150]. Every such group is either unipotent or a group of Lie type over a certain extension of the base field. If

T is split, all its root subgroups are unipotent; this is the case we considered previously. On the other hand, if T is minisotropic, i.e., its split rank is the least possible, then all the root subgroups are semisimple. In the general case one has subgroups of both types. Moreover, in the case of a nonsplit torus there may be nontrivial inclusions among the root subgroups.

Now let S be a set of the T-root subgroups that is closed in the sense that if X, Y, Z are three T-root subgroups such that $X, Y \in S$ and $Z \le \langle X, Y\rangle$, then $Z \in S$. Let $G(T, S)$ be the subgroup generated by T and all the T-root subgroups $X \in S$. Let $N(T, S)$ be the normalizer of $G(T, S)$ in G. Then the main result of Seitz's paper can be stated as follows.

SEITZ'S THEOREM. *Let K be a finite field,* $\mathrm{char} K \ne 2, 3$, $|K| \ge 13$, *and T a maximal torus in a Chevalley group $G = G(\Phi, K)$ over K. Then, for any subgroup F of G that contains T, there exists a unique closed set S of T-root subgroups such that*

$$G(T, S) \le F \le N(T, S).$$

Seitz actually established more detailed and precise results in [150]. Among other important results, he showed that $G(T, S)$ is exactly the subgroup $\langle T^F\rangle$ generated by all the subgroups conjugate to T by elements of F; that if F contains two maximal tori, T_1 and T_2, then $\langle T_1^F\rangle = \langle T_2^F\rangle$; and that one can choose a maximal torus T, $T \le F$, such that $F = G(T, S)N_F(T)$. The connection between Seitz's results and Lie algebras is discussed in [113]. The proofs in [150] rely essentially on the classification of finite simple groups.

The main results of [150] are rephrased in [149] and specialized to the classical cases in the geometric language. In particular, the maximal tori and their root subgroups are explicitly described for the classical groups in their natural representations. For these results, see also Kondrat′ev's survey [54].

Before Seitz's work, a description of overgroups of a nonsplit maximal torus was known only for the simplest case of a Singer cycle in the general linear group [115] (for the definition and properties of Singer cycles in the classical groups, see, e.g., [111, 112, 149, 80]). In this connection we mention also more general results of Hering [106] and Dempwolf [96] on finite linear groups containing an irreducible cyclic subgroup. Another proof for Singer cycles was recently given by Dye [98–102] in a wider context — the description of overgroups of stabilizers of spreads (analogous results were obtained by Li Shangzhi, but since he relied on the use of unipotent elements, he was forced to omit the case of tori [131–133]). Dye's proof is combinatorial-geometric; it does not depend on the assumption that $\mathrm{char}\, K \ne 2, 3$ and does not use the classification of finite simple groups.

3. Local fields. There is no other class of fields, except the finite fields, for which the description of subgroups containing nonsplit maximal tori has been accomplished to the same degree of completeness. The point is that a description of the overgroups of all maximal tori in Chevalley groups over a field K requires a good understanding of the structure of the finite extensions of K, the classification of K-forms of semisimple algebraic groups, etc. Only for certain very special classes of fields is this information readily available. It has in fact been shown [173] that many necessary facts from Steinberg's theory fail upon transition to a more general situation.

Borevich and Krupetskiĭ [5, 55, 56, 59, 60] described the subgroups of some classical groups over $\mathbb{R}$ that contain certain special tori. The problem for the compact Lie groups has been practically settled by Djoković [97], who has proved that overgroups of maximal tori are closed in the usual real topology. The situation is probably the same in other cases. Maximal tori were classified up to conjugacy by Kostant [126], Sugiura [163], and Rothschild [144]. For example, the conjugacy classes of maximal tori in a Chevalley group $G(\Phi, \mathbb{R})$ are in one-to-one correspondence with the involution classes of the Weyl group, so that numbers of classes are as follows: in E_6, 5; in E_7 and E_8, 10 each; in F_4, 8; in G_2, 4. For other $\mathbb{R}$-forms, only involutions that are products of reflections with respect to noncompact roots are admissible, so that the number of conjugacy classes of maximal tori certainly does not exceed the above values.

Krupetskiĭ [57–59, 61] considers overgroups of certain tori in some of the classical groups over non-Archimedean local fields. His work is enough to show that the answer cannot be as simple as for finite fields, since the ring of integral elements of the local field and its ideals come into play. The results are far from complete; in particular, the study of exceptional groups in this context has not yet begun. Problems of conjugacy for maximal tori have been studied by Kariyama and Morris [117, 143]. For example, it has been shown that, if the characteristic of K does not divide the order of the Weyl group $W(\Phi)$, then the number of conjugacy classes of maximal tori in a Chevalley group $G(\Phi, K)$ is finite. The same also holds without any assumption about the characteristic for tori that split over a weakly ramified extension.

4. Global fields. The case of global ground fields is, of course, even more difficult. Only two examples have been considered, and they clearly illustrate the problem. In [62], Krupetskiĭ considers subgroups containing the group of diagonal matrices in a unitary group over the division field of rational quaternions. In a recent paper [53] V. A. Koĭbaev described the overgroups of all maximal tori in $\mathrm{GL}(2, \mathbb{Q})$. The description is phrased in arithmetical terms and differs in the most striking way from the description of overgroups of split tori. Thus, for example, every nonsplit torus has a continuum of different overgroups, depending on arithmetic parameters (the set of simple ideals, defined in terms of the Legendre symbol, in intermediate subrings R, $\mathbb{Z} \le R \le \mathbb{Q}$, etc.). Most recently, Koĭbaev has generalized this description to all global fields K and minisotropic tori in $\mathrm{GL}(n, K)$ associated with cyclic Kummer extensions. This outstanding result indicates the probable nature of the answer in the general situation. Nevertheless, it is clear that a formidable amount of work will be needed to complete the description in the case of general groups and tori.

§11. Some unsolved problems

We will now state a number of unsolved problems that arise most naturally in connection with the topics discussed above. We begin with split tori.

1. Prove that, under the usual assumptions about the characteristic, the description of subgroups of $G = G(\Phi, K)$ containing $T = T(\Phi, K)$ is standard if $\Phi \neq C_l$ and $|K| \geq 7$.

We know that Seitz's result implies the bound $|K| \geq 13$. The answer is known to hold for the A_l and D_l series (see, in particular, [18, 24, 27, 30]). Problem 1 can be viewed as a specialization of the following problem of Seitz [147, Problem 1].

2. Describe the subgroups of $G = G(\Phi, K)$ containing $T = T(\Phi, K)$, on the assumption that $|K| \geq 4$.

For fields K with $|K| \leq 5$ the description is certainly nonstandard [45]. A complete answer to this problem has been obtained only for the series $\Phi = A_l$ by the author [18] and Koĭbaev [50–52] (see also the earlier [4, 46–49] where the case of the general linear group was considered). A "parastandard" description has been established for fields $\mathbb{F}_5$ and $\mathbb{F}_4$, although some new irreducible primitive subgroups containing T turn up, but all the other subgroups are constructed from them in the standard way. The situation for $\mathbb{F}_3$ is much more complex.

If the solution of problems 1 and 2 requires excessive efforts, it may be worth waiting for a complete classification of the subgroups of finite simple groups and other related groups. Vigorous efforts are underway to that effect (see, in particular, [75–78, 93, 94, 120–125, 132, 136–138, 142, 151–156, 166] and the references there). We already have a complete description of all maximal subgroups for all exceptional groups, except those of type E_7 and E_8 [76, 77, 93, 121, 122, 142, 155, 156, 166]. Solutions of problems 1 and 2 can clearly be derived from those results by brute force.

For the extended classical groups, i.e., GL_n, GSp_{2l}, SO_n, results analogous to Theorems 2 and 9 have been proved not only for fields but also for fairly large classes of rings, such as the semilocal rings (see, in particular, [3, 13, 15, 24, 27, 30, 33, 169]).

3. Prove analogs of the classification theorems for Chevalley groups over commutative semilocal rings.

This problem is still open even for SL_n (the only available result in this direction deals with discrete valuation rings [71]). There are other analogs of Theorems 2 and 9 for the classical groups, such as the classification theorems for subradical subgroups (see, in particular, [13, 15, 24, 27, 30, 33]), which are true for almost arbitrary rings. To be precise: let J be the Jacobson radical of a ring R. Let $B(\Phi, R, J)$ denote the subgroup of the Chevalley group $G = G(\Phi, R)$ generated by $B(\Phi, R)$ and all the root unipotents $x_\alpha(\xi)$, $\xi \in J$.

4. Describe the subgroups in $B(\Phi, R, J)$ that contain $T(\Phi, R)$.

Such a description would be a decisive step toward the solution of problem 3. In order to obtain a "standard" solution to problem 4, one must assume that R satisfies conditions of the following type: $R = \mathbb{Z}[R^*]$, and there exist $\varepsilon, \eta \in R^*$ such that $\varepsilon - 1, \eta - 1, \varepsilon - \eta, \varepsilon\eta - 1 \in R^*$; similar but slightly stronger conditions may also work.

It would be of great interest to obtain a uniform proof of Theorem 2, simultaneously including the cases of finite and infinite fields; as far as we know, there is no such proof even for SL_2. We point out two concrete possibilities in this direction; both may prove useful for the solution of problems 3 and 4.

5. Devise a proof of the classification theorems using not the Bruhat decomposition but the Birkhoff decomposition: $G = BWB^-$.

First, since it combines the Bruhat and Gauss decompositions, the Birkhoff decomposition bridges the gap between fields and subradical subgroups. Second, the coefficients are more closely related to the usual matrix coefficients than those of the Bruhat decomposition, thus enabling one to imitate the proofs in matrix language for the classical groups.

6. Prove the classification and conjugacy theorems using the matrix realizations of the exceptional Chevalley groups associated with modules of least dimension.

We are well acquainted with the dimensions of the minimal Weyl modules: 7 for G_2, 26 for F_4, 27 for E_6, 56 for E_7, and 248 for E_8. The exceptional groups are realized over these modules as the general isometry groups of certain multilinear forms, since the matrix coefficients of their elements satisfy especially simple equations. For G_2 and E_8 groups this was already known to Dickson in 1905; for further developments in connection with nonassociative algebras and geometry, see the references in [76–78, 88, 94, 105].

Formally speaking, the next problem is concerned with nonsplit tori, but it seems quite likely that it can be solved by the methods presented in this paper.

7. Generalize the results of this paper to subgroups of twisted Chevalley groups that contain a Cartan subgroup.

We now go on to questions relating to nonsplit tori. As already mentioned, a description of overgroups of maximal tori was obtained by Seitz [150], using the classification of finite simple groups.

8. Prove the results of [150] without using this classification and without the assumptions $\operatorname{char} K \neq 2, 3$.

One expects that in many cases the condition $|K| \geq 13$ may also be weakened or eliminated. A good deal of computational work is still needed, apparently, to specialize the results of [150]: explicitly to calculate the lattice of overgroups of maximal tori, to establish the relationships between the lattices for different tori, and so on. We mention two concrete questions related to Seitz's paper.

9. Classify the maximal subgroups of $G(\Phi, F_q)$ that contain a maximal tori.

10. Classify the pronormal maximal tori in $G(\Phi, F_q)$.

One cannot expect Seitz's results to carry over completely to the case of infinite fields. However, the surprising parallel between the properties of solvable subgroups containing a maximal torus and Suprunenko's theory of maximally solvable linear groups [68] raises hopes for a fairly explicit solution of the following problem for any field.

11. Describe the solvable subgroups of $G(\Phi, K)$ that contain a maximal torus.

By removing the solvability condition, one may hope for such a description only for very special classes of fields.

12. Describe all the subgroups of $G(\Phi, K)$ that contain a maximal torus, when K is either local or global.

Another interesting problem is to describe the subgroups of G that contain only part of a maximal torus. Of course, the part should not be too small, if one is to expect a meaningful description. Several such problems for finite ground fields were stated in [54, 147, 150]. We point out one more arithmetical realization of this problem.

13. Let R be the ring of integral elements of a complete valued field K. Describe the subgroups of $G(\Phi, K)$ that contain $T(\Phi, R)$.

This problem was solved for GL_n by the author and Khamdan [36, 71]. A complete solution would be, in particular, a very broad generalization of results by Iwahori, Matsumoto, Bruhat, and Tits concerning the description of parahoric subgroups.

Another natural realization of the general problem is: what can be said about subgroups containing a not necessarily maximal torus? The following version of this problem was suggested by Berman in 1977.

14. Let S be a torus in G. Is it true that, knowing all the subgroups of $N_G(S)/S$, one can describe the subgroups of G that contain S in combinatorial terms (relative root system, etc.)?

This question is, of course, meaningful not only for Chevalley groups but also for other kinds of reductive groups. As always, when speaking about reductive groups, tori, etc., one means their groups of points over some not necessarily algebraically closed field K, and one is interested in all the subgroups, not merely the algebraic ones. In order to demonstrate the difficulty of problem 14, we mention the following very special "subquestion," which itself is an unsolved problem.

15. Let S be a maximal split subtorus (i.e., maximal with respect to all split subtori) in G. Describe all subgroups of G that contain subtori $C_G(S)$.

This version of the problem eliminates all the difficulties due to the need to consider subgroups lying in the anisotropic kernel; hence this problem is closely related to our paper. Nevertheless, even problem 15 cannot be solved by directly applying our methods; it requires additional considerations. Some practical results of relevance for isotropic orthogonal groups have been obtained by Golubovsky.

During more than ten years of work on these problems, I have enjoyed the help and support of many of my colleagues; it would be difficult to thank all of them. Nevertheless, I must mention at least a few names. Z. I. Borevich, A. I. Kostrikin, and D. K. Faddeev actively encouraged me to work on the problem. Innumerable conversations and lengthy correspondence with B. B. Venkov, E. B. Vinberg, A. E. Zalesskiĭ, G. M. Seitz, V. A. Koĭbaev, and B. N. Cooperstein have been very useful. They and many others, including M. Aschbacher, R. G. Dye, O. King, P. Kleidman, M. U. Liebeck, and Li Shangzhi, have kept me abreast of developments, often sending me their papers long before publication. Without all this, it would have been impossible either to obtain the results presented here or to write this survey. I am deeply grateful to all of them.

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Translated by N. ZOBIN

Amer. Math. Soc. Transl.
(2) Vol. **155**, 1993

On the Boundary Integral Equation of the Neumann Problem in a Domain with a Peak

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ABSTRACT. The integral equations generated by the Neumann problem for the Laplace equation are considered on a plane contour. We assume that there is an inward or an outward cusp at this contour. Theorems on the solvability and the number of solutions to this integral equation are proved. Asymptotic formulae for solutions near the cusp are also obtained.
Bibliography: 9 titles.

In this article we study integral equations of the Neumann problem for the Laplace operator in a plane domain with inward or outward peaks at the boundary. We prove theorems on the unique solvability and asymptotics of solutions near peaks.

The classical method for solving boundary value problems is their reduction to boundary integral equations by using potentials. In the case of the Dirichlet and Neumann problems this procedure leads to equations which were studied traditionally by methods of the theory of Fredholm integral operators. However, as was shown by J. Radon in 1919 [1], if a plane domain has cusps at the boundary, then the Fredholm radius of the integral operator, generated by the double layer potential acting in the space of continuous functions, equals 1. Thus a direct application of the Fredholm theory is impossible if the contour contains cusps. In the present paper we use another approach, proposed by one of the authors several years ago [2, 3]. It enables one to get information about the inverse operators of boundary integral equations by applying theorems on inverse operators of auxiliary boundary value problems.

Let Ω be a plane simply-connected domain with piecewise smooth boundary Γ that has a unique peak at the origin. Suppose that near the point 0 either the domain Ω or the complementary domain Ω^c is given by the

1991 *Mathematics Subject Classification*. Primary 31A10, 31A35.