REMARKS ON SOME ALGEBRAIC GROUPS

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Let k be a field, char k $\neq$ 2, K a quadratic extension of the field, and G a semisimple algebraic group which is defined over k and decomposable over K. A maximum torus $T \subset G$ is said to be <u>allowed</u> if it is defined over k, anisotropic over k, and decomposable over K. Let Σ be a system of roots in G relative to T; if $\alpha \in \Sigma$, we shall denote by G_α a three-dimensional simple subgroup generated by the root subgroups N_α and $N_{-\alpha}$, and normalizable by the torus T. The Galois group $\Gamma(K/k) = \{1, \sigma\}$ acts on the group of characters X(T) of the allowed torus T.

<u>LEMMA.</u> Let T be an allowed torus in G. Hence a) $\sigma\alpha = -\alpha$, $\forall \alpha \in X(T)$; b) the subgroups G_α, $\alpha \in \Sigma$ are defined over k.

To the subgroups G_α it is possible to assign central quaternion algebras $\mathfrak{D}_\alpha$. We shall denote by Nrd a homomorphism of a reduced norm of the algebra $\mathfrak{D}_\alpha$ into its center. The algebras $\mathfrak{D}_\alpha$ are cyclic algebras: $\mathfrak{D}_\alpha \cong (K, \lambda_\alpha)$, $\lambda_\alpha \in k^* \bmod N_{K/k}(K^*)$; in this case we shall say that the group G <u>represents an array</u> $\{\lambda_\alpha\}_{\alpha \in \Sigma}$ with respect to the torus T. The principal question of interest to us is as follows: How is it possible to distinguish arrays that are represented by the group G with respect to two distinct allowed tori?

An allowed torus T' is said to be <u>associated with respect to $\alpha \in \Sigma$ to the torus</u> T if $T' \subset G_\alpha T$, where $\alpha \in \Sigma$, $\mathrm{rg}_k G_\alpha = 0$. It is easy to prove the following result:

<u>THEOREM.</u> If $\mathrm{rg}_k G = 0$, the group G will contain allowed tori and any two allowed tori can be obtained from each other by a finite number of transitions to associated tori.

By a simple analysis of type-A_1 groups it is possible to obtain

<u>PROPOSITION 1.</u> If an allowed torus T' is associated with an allowed torus T with respect to $\beta \in \Sigma$ and the group G represents the arrays $\{\lambda_\alpha\}$ and $\{\lambda'_\alpha\}$ with respect to the tori T and T', then $\lambda_\alpha = \nu^{[\alpha,\beta]} \cdot \lambda_\alpha$, where $\nu \in \mathrm{Nrd}\,\mathfrak{D}_{\beta,k}$. Conversely, if $\nu \in \mathrm{Nrd}\,\mathfrak{D}_{\beta,k}$, then the arrays λ_α and $\{\nu^{[\alpha,\beta]} \cdot \lambda_\alpha\}$ will be represented by the group G with respect to associated tori; $[\alpha, \beta] = 2(\alpha, \beta) \cdot (\beta, \beta)^{-1}$.

We can also prove

<u>PROPOSITION 2.</u> If $\mathrm{rg}_k G > 0$ and the group G contains an allowed torus, then it will contain an allowed torus T such that $\lambda_\alpha \in N_{K/k}(K^*)$ for at least one $\alpha \in \Sigma$ (i.e., the group G_α is decomposable).

Over special fields it is possible to obtain with the aid of our results interesting corollaries.

<u>COROLLARY 1.</u> If $\mathrm{Nrd}\,\mathfrak{D}_k = k$ for any central quaternion algebra $\mathfrak{D}/k$ and $\mathrm{rg}_k G = 0$, then G will be of type A_1.

This is a direct consequence of Proposition 1.

<u>COROLLARY 2.</u> If $\mathrm{Nrd}\,\mathfrak{D}_k = N_{K/k}(K)$ for any algebra $\mathfrak{D} = (K, \lambda)$, $\lambda \in k^*$, and $\mathrm{rg}_k G = 0$, then any two allowed tori will be conjugate in G_k.

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Indeed, by virtue of our condition the properties of associativity with respect to $\alpha \in \Sigma$ and conjugateness in $G_{\alpha, k}$ coincide. Our assertion follows from the theorem. Let us note that the field of real numbers satisfies our conditions; therefore, Corollary 2 is a generalization of E. Cartan's well-known theorem on the conjugateness of maximum tori in a compact Lie group.

It is also possible to prove the following simple result:

PROPOSITION 3. If T and T' are two allowed tori in G and the group G represents with respect to T and T' the same arrays, then the tori T and T' will be conjugate in the group $(\mathrm{Aut}\ G)_k$.

Results, similar to those presented above, can be obtained for groups containing maximum tori that are defined over k and decomposable over a Galois extension of the Galois group Z_p.

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