IRREDUCIBLE REPRESENTATIONS OF LIE p-ALGEBRAS

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We consider the irreducible representations of finite-dimensional Lie p-algebras over an algebraically closed field k of characteristic $p > 0$. The study of these representations was initiated in the 1940s by Zassenhaus, who described the representations of nilpotent Lie algebras [14]. Next, Chang [11] studied the representations of the Witt algebra in very great detail. After the war Zassenhaus obtained general results on the structure of the enveloping algebras of Lie p-algebras [15]. Using these results, A. N. Rudakov and I. R. Shafarevich [8] studied the structure of the set of all representations of the algebra A. These investigations were extended by A. N. Rudakov [7] who found sufficient conditions for a representation of a Lie algebra of classical type to have maximal dimension.

As in the case of characteristic zero, the problem concerning the representations is connected with the action of the Lie algebra on the space G* conjugate to it. A linear form l_V is canonically associated with each irreducible representation of G in a space V. For completely solvable Lie algebras we describe the irreducible representations entirely in terms of this form (Theorem 1). For the Lie p-algebras of classical type Theorem 2 shows that the study of representations with an arbitrary form l_V can be reduced to the case when the element of the algebra G dual to l_V (relative to the Killing form) is nilpotent. For a nilpotent element the problem is still open. But for an arbitrary l_V the connection with the action of G on G* is given by Theorem 3. The p-representations of p-algebras of Cartan type are described in Theorem 4.

Our results reflect "in miniature" the general situation in the case of characteristic zero.

§1. THE MAIN DEFINITIONS

1. Let G be a Lie algebra over a field k, and let U(G) be its universal enveloping algebra. Then (see [3], p. 207)

$$(a+b)^p = a^p + b^p + \Lambda(a, b), \quad a, b \in G, \tag{p0}$$

where $\Lambda(a, b) \in G$. A Lie algebra G is said to be a Lie p-algebra if in G there is defined a mapping $a \to a^{[p]}$, satisfying the conditions:

$$(a+b)^{[p]} = a^{[p]} + b^{[p]} + \Lambda(a, b), \tag{p1}$$

$$(\alpha a)^{[p]} = \alpha^p a^{[p]}, \quad \forall \alpha \in k, \tag{p2}$$

$$\operatorname{ad} b^{[p]} = (\operatorname{ad} b)^p. \tag{p3}$$

If a mapping $a_i \to a_i^{[b]}$ satisfying (p3) is defined on the basis $\{a_i\}$ of a Lie algebra G, then it uniquely defines the structure of a Lie p-algebra in G [3].

Let G be a finite-dimensional Lie p-algebra, and let U# be the center of U(G). If V is a simple G-module, then by Schur's Lemma $u(x) = \chi_V(u)x$ for all $u \in U\#$ and for all $x \in V$, where χ_V is a homomorphism of U# into k. As is well known (see [3]) for every $g \in G$ the element $g^p - g^{[p]}$ belongs to U#.

<u>PROPOSITION 1.1.</u> Let G be a Lie p-algebra, and let V be a simple G-module. We put $l_V(g) = (\chi_V(g^p - g^{[p]}))^{1/p}$. Then $l_V \in G^*$.

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Proof. It follows from (p0) and (p1) that $l_V(g_1 + g_2) = l_V(g_1) + l_V(g_2)$ and from (p2) that $l_V(\alpha g) = \alpha l_V(g)$.

2. Let $U^\# \subset U^\#$ be the subalgebra generated by all of the elements $g^p - g^{[p]}$, $g \in G$, and 1; let K and K_0 be the fields of the special rings of $U^\#$ and $U^\#_*$, and let $r = [K : K_0]$. Let $\mathfrak{B}$ denote the set of all simple G-modules, $\Phi : \mathfrak{B} \to \operatorname{Spec} U^\#$ be the mapping defined by the formula $\Phi(V) = \chi_V$, $V \in \mathfrak{B}$; we have $\mathfrak{B} \xrightarrow{\Phi} \operatorname{Spec} U^\# \xrightarrow{i^*} \operatorname{Spec} U^\#_*$. Then $\dim V \leq p^{(n-r)/2}$, $V \in \mathfrak{B}$ [15], and there exists an open everywhere dense set $\mathcal{M}$, Spec $U^\#$ such that $\dim V = p^{(n-r)/2}$ for $V \in \Phi^{-1}(\mathcal{M})$ and the mapping $\Phi : \Phi^{-1}(\mathcal{M}) \to \mathcal{M}$ is one-to-one. It seems plausible that $r\ r = \min_{l \in G^*} \dim G_l$ (where G_l is the stationary subalgebra in the representation contragredient to the adjoint representation). This conjecture can be verified for simple Lie algebras of classical type [7], and its validity for completely solvable Lie p-algebras follows from the results in § 2. We note that the degree of the covering i^* is equal to the degree of separability of K over K_0. For nilpotent Lie algebras, K is purely nonseparable over K_0 [15]. We can prove that this extension is separable for Lie algebras of classical type and for the Lie algebra W_n.

3. We denote by $U_l^\#$, $l \in G^*$, the subalgebra of $U^\#_*$ generated by the elements of the form $g^p - g^{[p]} - l^{(p)}(g)$, $g \in G$. We put $U_l(G) = U(G)/U^\#_* U(G)$. Let H be a p-subalgebra of a Lie p-algebra G, $\bar{l} = l|_H$ and V be an H-module such that its nucleus contains $U_l^\#(H)$. The space $V_l^G = U_l(G) \otimes_{U_{\bar{l}}(H)} V$ is given the structure of a G-module by putting $g(u \otimes x) = gu \otimes x$, $u \in U_l(G)$, $x \in V$. We say that the G-module V_l^G is induced by the H-module V. We give several obvious properties of induced modules:

a) if V is a simple G-module, $l = l_V$, H is a p-subalgebra of a Lie p-algebra G, and $V' \subset V$ is an H-submodule, then V is a factor-module of the G-module $V_l'^G$;

b) if V' is an H-module and H is a p-subalgebra of a Lie p-algebra G of codimension t, then $\dim V_l'^G = p^t \cdot \dim V'$;

c) if $H_1 \subset H_2 \subset G$ are p-subalgebras of a Lie p-algebra G, V' is an H_1-module, and $l \in G^*$, then $(V_{\bar{l}}'^{H_2})_l^G = V_l'^G$, where $\bar{l} = l|_{H_2}$.

4. We fix our notation. Let G be a Lie p-algebra, let n(G) denote the nilradical of G (that is, the maximal ideal consisting of nilpotent elements), and let C(G) denote the center of G. If H is a subalgebra of G, let N(H) [Z(H)] denote the normalizer [centralizer] of H. Let G_l, $l \in G^*$, denote the stationary subalgebra in the representation contragredient to the adjoint representation. The sign $\oplus$ symbolizes the direct sum of vector spaces.

§ 2. REPRESENTATIONS OF COMPLETELY SOLVABLE LIE p-ALGEBRAS

1. Let G be a Lie algebra, $l \in G^*$, and let H be a subalgebra such that $l([H, H]) = 0$. Since the dimension of the maximal isotropy subspace of G for the bilinear form $B(x, y) = l([x, y])$ is equal to $(\dim G + \dim G_l)/2 = a(G, l)$, we have $\dim H \leq a(G, l)$. The set of subalgebras of G for which $\dim H = a(G, l)$ is denoted by $\mathfrak{P}(G, l)$. We note that every such subalgebra contains G_l and is a p-subalgebra of a Lie p-algebra.

LEMMA 2.1. Let G be a Lie algebra, $l \in G^*$, G_1 be a subalgebra of codimension one which contains G_l, and $l_1 = l|_{G_1}$. Then $\mathfrak{P}(G_1, l_1) \subset \mathfrak{P}(G, l)$.

Proof. Let $H \in \mathfrak{P}(G_1, l_1)$. We have

$$\dim H = a(G_1, l_1) \geqslant (\dim G - 1 + \dim G_l)/2 = a(G, l) - 1/2.$$

Since the number $a(G, l)$ is an integer, we have $\dim H = a(G, l)$ and $H \in \mathfrak{P}(G, l)$.

2. A Lie algebra G is said to be completely solvable if it contains a sequence of ideals $G = G_{(0)} \supset G_{(1)} \supset \ldots \supset G_{(n)}$, for which $\dim G_{(i)} = n - i$. A subalgebra and a factor-algebra of a completely solvable Lie algebra G are completely solvable; each of its proper subalgebras is contained in a subalgebra of codimension 1 of G.

By induction on the dimension we obtain the following corollary from Lemma 2.1.

COROLLARY. For a completely solvable Lie algebra G the set $\mathfrak{P}(G, l)$ is nonempty.

PROPOSITION 2.2. Let G be a completely solvable Lie p-algebra. For a suitable choice of a p-structure in G the Lie algebra G/n(G) is commutative. The commutativity of G/n(G) implies that $G = T \oplus n(G)$ (semidirect sum), where T is any maximal torus of G. Conversely, if $G = T \oplus n(G)$, then the Lie p-algebra G is completely solvable.

Proof. The splitting off of a maximal torus was proved in [5]; the remaining assertions are obvious.

3. LEMMA 2.3. Let G be a Lie p-algebra, let G_1 be a p-subalgebra of G of codimension 1, let V be a simple G-module, $l = l_V$, let $V' \subset V$ be a simple G_1-submodule, and let $V' \neq V$. Then a) if G_1 is an ideal, then $V = V_l'^G$; b) if the algebra G is solvable, then $V = V_l'^G$.

Proof. We prove a) first. It follows from results in [14] that $\dim V = p^t \cdot \dim V'$, $t > 0$. Since V is a factor-module of the G-module $V_l'^G$ we have $\dim V \leq p \dim V'$; hence $V = V_l'^G$. Hence a) is proved. From it we obtain by induction on dim G that if a Lie algebra G is solvable then $\dim V = p^t$. Assertion b) obviously follows from here.

Let $v(H, \tilde{l})$ denote the one-dimensional H-module defined by the mapping $h \to \tilde{l}(h)$, $\tilde{l} \in H^*$, $h \in H$, $\tilde{l}([H, H]) = 0$.

THEOREM 1. Let G be a completely solvable Lie p-algebra, $\tilde{l}, l \in G^*$, $H \in \mathfrak{P}(G, \tilde{l})$ and $l(h) = \tilde{l}(h) - \tilde{l}^{1/p}(h^{[p]})$ for all $h \in H$. Then

a) the G-module $(v(H, \tilde{l}))_l^G$ is simple;

b) every simple G-module V is isomorphic to one of the modules $(v(H, \tilde{l}))_l^G$ $(l = l_V)$.

Proof. We prove b) by induction on dim G. Let V be a simple G-module. We remark that if G_1 is a p-subalgebra of G of codimension one, and $V' \subset V$ is a simple G_1-submodule, then by the induction hypothesis there exist $\tilde{l}_1 \in G_1^*$ and $H \in \mathfrak{P}(G_1, \tilde{l}_1)$ for which the H-module $v(H, \tilde{l}_1)$ induces the G_1-module V'. If $V' \neq V$ and there is an $\tilde{l} \in G^*$ such that $\tilde{l}_1 = l|_{G_1}$ and $G_l \subset G_1$, then by Lemmas 2.1 and 2.3 $H \in \mathfrak{P}(G, \tilde{l})$ and the H-module $v(H, \tilde{l})$ induces the G-module V.

Let G_0 be the annihilator of the G-module V, $\bar{G} = G/G_0$, and $C = C(\bar{G})$. Then $\dim C \leq 1$. We can obviously assume that $\dim V > 1$, and consequently, that $\bar{G} \neq C$. We consider three separate cases.

CASE 1. There is a one-dimensional ideal $ky \subset \bar{G}$ not lying in C. Then it is obvious that for some $x \in \bar{G}$ we have $\bar{G} = Z(ky) \oplus kx$ and $[x, y] = y$. Let $\tilde{y}$ be an image of y in G, $G_1 = \{g \in G : [g, \tilde{y}] \in G_0\}$ and let V' be the simple G_1-module which is induced by the H-module $v(H, \tilde{l}_1)$. It is obvious that G_1 is a p-subalgebra of G of codimension one and that $G_{\tilde{l}} \subset G_1$ for any $\tilde{l} \in G^*$ for which $\tilde{l}|_{G_1} = \tilde{l}_1$. If $V' = V$ then $y \in C$, hence $y(v) = \lambda v$ for all $v \in V$; in particular $[x, y] = 0$ contradicting the choice of x, y.

CASE 2. Dim C = 1 and $C(\bar{G}/C) \neq 0$. Let $z \in C$, $z \neq 0$. It is obvious that there are elements $x, y \in \bar{G}$ for which $\bar{G} = Z(ky) \oplus kx$ and $[x, y] = z$. The subsequent reasoning is the same as in Case 1.

CASE 3. Dim C = 1, $C(\bar{G}/C) = 0$ and C is the only one-dimensional ideal in $\bar{G}$. Let $z \in C$, $z \neq 0$, $k\bar{y}$ be a one-dimensional ideal in $\bar{G}/C$, and y be an inverse-image of $\bar{y}$ in $\bar{G}$ such that $y(v) = 0$ for some $v \in V$, $v \neq 0$. It is obvious that there is an element $x \in \bar{G}$ such that $\bar{G} = N(ky) \oplus kx$ and $[x, y] = z$. We put $G_1 = \{g \in G : [g, y] \in G_0 \oplus ky\}$. Let $V' \subset V$ be a simple G_1-submodule. We need to prove that $V' \neq V$. Let us assume otherwise: $V' = V$. We have $G_1/G_0 = N(ky)$ and $y(v) = 0$ for some $v \in V$, $v \neq 0$. Then V is a simple G_1-module, therefore every vector $v_1 \in V$ is a linear combination of vectors of the form $g_1 \ldots g_k(v)$, where $g_i \in N(ky)$. By induction we have

$$yg_1 \ldots g_k(v) = [yg_1]g_2 \ldots g_k(v) + g_1yg_2 \ldots g_k(v) = 0.$$

Hence $y(V) = 0$ which contradicts the choice of y. This proves b), and a) is proved similarly.

LEMMA 2.4. Let G be a Lie algebra and V be a simple G-module for which all the elements from [G, G] are nilpotent. Then dim V = 1.

Proof. Let V_0 be the nucleus of the [G, G]-module V and $\bar{G} = G/[G, G]$. By a theorem of Engel, $V_0 \neq 0$. It is obvious that V_0 is a submodule of the G-module V, hence $V_0 = V$ and we can regard V as a $\bar{G}$-module. Since the Lie algebra $\bar{G}$ is commutative and the $\bar{G}$-module V is simple, we have dim V = 1.

Let $G = T \oplus n(G)$, where T denotes the maximal torus of the Lie p-algebra G, and $(n(G))^{[p]^{k+1}} = 0$. Let V be a simple G-module and $l = l_V$. Since $(g^p - g^{[p]})v = l^p(g)v$ for all $v \in V$, as is easily seen, the endomorphism $g \in n(G)$ has a unique eigenvalue $\lambda_V(g) = l(g) + l^{p^{-1}}(g^{[p]}) + \ldots + l^{p^{-k}}(g^{[p]^k})$.

PROPOSITION 2.5. Let $G = T \oplus n(G)$, V be a simple G-module, $l = l_V$, and $\lambda = \lambda_V$. Let H be the subalgebra of the highest dimension for which $\lambda([H, H]) = 0$. Then every single submodule of the H-module V is one-dimensional and induces V.

Proof. By Theorem 1 the G-module V is induced by a one-dimensional submodule of some subalgebra $\tilde{H}$. It is obvious that $\lambda([\tilde{H}, \tilde{H}]) = 0$. Next if $\lambda([H, H]) = 0$, by Lemma 2.4 every simple submodule of the H-module V is one-dimensional; hence $\dim \tilde{H} \geq \dim H$.

COROLLARY 2.6. Let $G = T \oplus n(G)$, and let V', V" be simple G-modules for which $l_{V'} = l_{V''}$. Then $\dim V' = \dim V''$; if $T = 0$ then the G-modules V' and V" are isomorphic.

Remark. In a somewhat different form a classification of the irreducible representations of nilpotent Lie algebras over k is derived in [14]. For the characteristic zero case results similar to Theorem 1 are well known.

§3. REPRESENTATIONS OF LIE ALGEBRAS OF CLASSICAL TYPE

1. Let G be a simple Lie algebra over k of classical type with the nondegenerate, invariant, symmetric bilinear form F considered in [4, 9]. This form determines an isomorphism $F : G^* \to G$.

Remark. Apparently the results we have obtained can be extended to any factor-algebras of the algebras Lie $\mathcal{G}$, where $\mathcal{G}$ is a smooth semisimple algebraic group.

Let $\bar{\mathcal{G}} = \operatorname{Aut} G$ and let $\mathcal{G}$ be a simply connected covering of $\bar{\mathcal{G}}$. As is well known G is a factor-algebra of the algebra Lie $\mathcal{G}$; let Φ : Lie $\mathcal{G} \to G$ be the corresponding projection. If $\mathcal{B}$ is a Borel subgroup of $\mathcal{G}$ the subalgebra $\Phi(\mathrm{Lie}\,\mathcal{B}) \subset G$ is called a Borel subalgebra of G. Then G is the union of its Borel subalgebras [10]. Let $l \in G^*$, $q \in F_G(l)$, and $B \subset G$ be a Borel subalgebra such that $q \in B$. Then $l|_{[B, B]} = 0$, and by Lemma 2.4 in each G-module V for which $l_V = l$ the subalgebra B has an eigenvector.

2. Let $q = q_s + q_n$ denote the decomposition of q into semi-simple and nilpotent parts, $G' = Z_G(q_s)$, T_0 be the center of G' and $G_0 = [G', G']$.

We choose a Borel subalgebra B_0 of G' such that $q \in B_0$. Let B be a Borel subalgebra of G which contains B_0. Then $P = B + G_0$ is a parabolic subalgebra of G. Let T be the maximal torus of G, $B \supset T \supset T_0$, N be the nilradical of P, Σ be the system of roots of G relative to T, Σ^+ be the system of positive roots in Σ defined by B, $\Sigma_0 = \{\alpha : e_\alpha \in G_0\}$, $\Sigma' = \Sigma^+ \setminus \Sigma_0$, $\Sigma'' = -\Sigma'$, and $G_0 = ke_\alpha + kh_\alpha + ke_{-\alpha}$. In view of our choice of B and G_0 we have

$$l(e_\alpha) = 0, \quad \forall \alpha \in \Sigma^+ \cup \Sigma''. \qquad (*)$$

THEOREM 2. Let V be a simple $U_l(G)$-module and $V' \subset V$ be a simple P-submodule. Then $V = V_l'^G$. In particular, $\dim V = p^{\dim N} \cdot \dim V'$, and the dimension of any $U_l(G)$-module is divisible by $p^{\dim N}$.

LEMMA 3.1. Let Δ denote the system of simple roots in Σ^+. There is an indexing of the roots in Σ'' : $\Sigma'' = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$, such that if $\Sigma_1 = \Sigma^+$, $\Delta_1 = \Delta$, $\Sigma_{i+1} = s_{\alpha_i}\Sigma_i$, and $\Delta_{i+1} = s_{\alpha_i}\Delta_i$, then Σ_i is the system of positive roots in Σ, Δ_i is the system of simple roots in Σ_i and $-\alpha_i \in \Delta_i$.

Proof.(communicated to us by E. B. Vinbert). Let $\mathbf{R}^m = \mathbf{R} \cdot \Sigma$, X_α be the hyperplane orthogonal to the root α, and C', C" be the Weyl chambers in $\mathbf{R}^m$ corresponding to the systems of positive roots $\Sigma' \cup \Sigma_0^+$, $\Sigma'' \cup \Sigma_0^+$. We take points $y' \in C'$, $y'' \in C''$ such that the straight line $\mathcal{D}$ through y', y" does not pass through $X_\alpha \cap X_\beta$, $\alpha, \beta \in \Sigma$. (This is possible since a chamber is an open set.) Let $\mathcal{D}_0$ denote the segment of $\mathcal{D}$ between y' and y". When we move along $\mathcal{D}_0$ from y' to y" we can write down and successively enumerate the negative roots corresponding to the planes X_α which we cross. Let them $\beta_1, \ldots, \beta_m$. Since $\mathcal{D}_0$ meets a plane at not more than one point we have $\beta_i \neq \beta_j$ for $i \neq j$. It is obvious that $\{\beta_1, \ldots, \beta_m\} \supset \Sigma'$. Next let β_i be a root such that the chambers C', C" lie on different sides of the plane X_{β_i}. But all the negative roots with this property lie in Σ'', that is, $\beta_i \in \Sigma'$ for all i. Thus the lemma is proved.

3. Let B_i denote the Borel subalgebra of G corresponding to Σ_i in Lemma 3.1. We put $P_i = B_i + ke_{\alpha_i}$, $U_i = U_l(P_i)$, $U_i' = U_l(B_i)$, $N_i = \bigoplus_{\alpha \in \Sigma_i,\, \alpha \neq -\alpha_i} ke_\alpha$ (the nilradical of P_i), $G_i = G_{\alpha_i}$, $e_i = e_{\alpha_i}$, and $f_i = e_{-\alpha_i}$. Since the G-module V is simple, V is the factor-module of the G-module $W = V_l'^G$ with respect to some maximal G-submodule $\tilde{W}$. We put $W_1 = 1 \otimes V' \subset W$, $W_{i+1} = U_i \otimes_{U_i'} W_i$. This definition is correct since $B_{i+1} \subset P_i$ and therefore W_i can be regarded as a B_{i+1}-module. We assume that W_i is embedded in W. Let M_i be a simple P_i-submodule of $\tilde{W} \cap W_i$ ($M_i = 0$ if $\tilde{W} \cap W_i = 0$, and $M_i \neq 0$ if $\tilde{W} \cap W_i \neq 0$). We put $\bar{W}_i = \{w \in W_i : N_{i-1}w = 0\}$. Let us note the following facts:

(a) $l(e_\alpha) = 0$ for all $\alpha \in \Sigma_i$ (see (*)), $l(h_\alpha) \neq 0$ for all $\alpha \in \Sigma''$ (because if $l(h_\alpha) = l(e \pm \alpha) = 0$ then $\alpha \in \Sigma_0$);

(b) $N_{i-1}M_i = 0$, M_i is a simple G_i-module (because $l(e_\alpha) = 0$ for all $e_\alpha \in N_{i+1}$);

(c) $f_i^m M_i \neq 0$ for $m \leq p-1$, $e_i^m M_i \neq 0$ for $m \leq p-1$ (this is a property of any simple G_i-module with a form l for which $l(e_i) = l(f_i) = 0$, $l(h_{\alpha_i}) \neq 0$);

(d) $W_{i+1} = \bigoplus_{0 \leq j \leq p-1} e_i^j W_i$, $j = 0, 1, \ldots, p-1$ (by the definition of the tensor product);

(e) $e_\alpha e_\beta^m = e_\beta^m e_\alpha + \sum_{j=0}^{m-1} \binom{m}{j} (-1)^j e_\beta^j ((\mathrm{ad}\, e_\beta)^j (e_\alpha))$ ([3], p. 49).

LEMMA 3.2. If $M_i = 0$ then $M_{i+1} \subset \bigoplus_{0 \leq j \leq p-1} e_i^j \overline{W}_i$.

Proof. Let $x = \sum e_i^j x_j \in M_{i+1}$, $x_j \in W_i$. Then $e_i^m x \in M_{i+1}$ for all $m \leq p-1$. By (b) we have $N_i(e_i^m x) = 0$ for all m. Let $x = x_0 + e_i x_1 + \ldots + e_i^t x_t$. We put $y_m = e_t^{p-t+m-1} x$ for all $m = 0, 1, \ldots, t$. Then $y_m = e_t^{p-t+m-1} x_0 + \ldots + e_i^{p-1} x_{t-m}$. We show that for the vector $y = \sum_{0 \leq j \leq p-1} e_i^j z_j$, $z_j \in W_i$, $z_{p-1} \neq 0$, the condition $N_i y = 0$ means that $z_{p-1} \in \overline{W}_i$. By applying this assertion to all the vectors y_m, $m = 0, 1, \ldots, t$ we obtain $x_{t-m} \in \overline{W}_i$, as required. Let $e_\alpha \in N_i$. We have $0 = e_\alpha y = \sum_{0 \leq j \leq p-1} e_i^j e_\alpha z_j + \sum_{0 \leq j \leq p-1} [e_\alpha, e_i^j] z_j$. We note that $e_i^{p-1} e_\alpha z_{p-1} \in e_i^{p-1} W_i$, and in view of (e) and since $(\mathrm{ad}\, e_i)^m e_\alpha \in N_i$ for all m, all the remaining terms in our sum lie in $\bigoplus_{0 \leq j \leq p-2} e_i^j W_i$. Therefore, in view of (d) (and by the condition $e_\alpha y = 0$) we must have $e_i^{p-1} e_\alpha z_{p-1} = 0$, that is $e_\alpha z_{p-1} = 0$. Thus the lemma is proved, then $M_{i+1} = 0$.

LEMMA 3.3. If $M_i = 0$ and $f_i \overline{W}_i = 0$.

Proof. In view of Lemma 3.2 we have $M_{i+1} \subseteq U_i \overline{W}_i$. By (c) there exists an $x \in M_{i+1}$ for which $f_i^m x \neq 0$ for all $m \leq p-1$. Let $x = \sum_{0 \leq j \leq t} e_i^j x_j$, $x_j \in \overline{W}_i$, $x_t \neq 0$, $j = 0, 1, \ldots, t < p$. By considering $f_i x$ and taking account of (e) and the condition $f_i x_t = 0$, we obtain $f_i x \in \bigoplus_{0 \leq j \leq t-1} e_i^j \overline{W}_i$. Since $f_i^m x \neq 0$ for all $m \leq p-1$, by applying f_i to x exactly t times we obtain, as above, that $f_i^t x = y \in \overline{W}_i$. Since $y \neq 0$ (because $t < p$) we have $M_{i+1} \cap \overline{W}_i \neq 0$, that is, $M_i \neq 0$. Thus, we arrive at a contradiction.

LEMMA 3.4. Let $K_{i+1} = \{w \in W_{i+1} : e_i x = 0\}$. Then $K_{i+1} = e_i^{p-1} W_i$.

Proof. We have $W_{i+1} = \bigoplus_{0 \leq j \leq p-1} e_i^j W_i$. Since $l(e_i) = 0$, $e_i^{p-1} W_i \subset K_{i+1}$. On the other hand if $0 \neq x \in$ $x \in K_{i+1} \cap \bigoplus_{0 \leq j \leq p-2} e_i^j W_i$ by (d) $e_i x \neq 0$. Thus the lemma is proved $W_{i+1} = 0$.

LEMMA 3.5. $\overline{W}_{i+1} \subseteq e_i^{p-1} e_{i-1}^{p-1} \ldots e_1^{p-1} \overline{W}_1$; $f_{i+1} = 0$.

Proof. We put $q_i = e_i^{p-1} e_{i-1}^{p-1} \ldots e_1^{p-1}$. As for (d) it follows from the definition of the tensor product that

$$(\mathrm{d}') \qquad W_{i+1} = \bigoplus_{0 \leq m_j \leq p-1} e_i^{m_i} e_{i-1}^{m_{i-1}} \ldots e_1^{m_1} \cdot W_1.$$

We put $W'_{i+1} = \bigoplus_{0 \leq m_j \leq p-1, \Sigma m_j < i(p-1)} e_i^{m_i} \ldots e_1^{m_1} W_1$. By Lemma 3.4 $\overline{W}_{i+1} \subseteq K_{i+1}$. Let $x \in \overline{W}_{i+1}$, $x = e_i^{p-1} y$, $y \in W_i$. Then $0 = e_{i-1} x = e_i^{p-1}(e_{i-1} y) + [e_{i-1}, e_i^{p-1}] y$. Since by (e) we have $[e_{i-1}, e_i^{p-1}] y \in W'_{i+1}$, we obtain [by (d')] $e_{i-1} y = 0$, that is, $x \in e_i^{p-1} \cdot e_{i-1}^{p-1} W_{i-1}$. When we apply $e_{i-2}, \ldots, e_1$ in succession to x we obtain that $\overline{W}_{i+1} \subset q_i W_1$.

Now let $\alpha \in \Sigma_0^+ \subset \Sigma_{i+1}$, $x = q_i y$, $y \in W_1$. Again $e_\alpha x \equiv q_i(e_\alpha y) \bmod W'_{i+1}$. Since $\Sigma_0^+ \subset \Sigma_m$ for all m, we have $e_\alpha \in N_i$. Therefore, if $x \in \overline{W}_{i+1}$, then $e_\alpha x = 0$ for all $\alpha \in \Sigma_0^+$. Hence $e_\alpha y = 0$ for all $\alpha \in \Sigma_0^+$. We now note that $e_\gamma W_1 = 0$ for all $\gamma \in \Sigma'$. Therefore, $\{y \in W_1 : e_\alpha y = 0 \text{ for all } \alpha \in \Sigma_0^+\} = \overline{W}_1$. Thus $\overline{W}_{i+1} \subset q_i \overline{W}_1$; that is, we have proved the first assertion.

Next suppose that $x = q_i y \in \overline{W}_{i+1}$, $y \in \overline{W}_1$. We have $-\alpha_{i+1} \in \Sigma_i \cap \Sigma^+$ and $f_{i+1} x \equiv q_i(f_{i+1} y) \bmod W'_{i+1}$. Since $f_{i+1} \overline{W}_{i+1} \subset \overline{W}_{i+1}$, $\overline{W}_{i+1} \subseteq q_i \overline{W}_1$, $f_{i+1} \overline{W}_1 = 0$, we have $f_{i+1} x = 0$ as required.

4. PROOF OF THEOREM 2. Since $\widetilde{W}$ is a maximal G-submodule of W we have $M_1 = W_1 \cap \widetilde{W} = 0$ (because W_1 is a simple P-module). We assume that $W_{t-1} \cap \widetilde{W} = 0$, that is, $M_{t-1} = 0$. We show that $M_t = 0$. By Lemma 3.5 $f_{t-1} \overline{W}_{t-1} = 0$, whence by Lemma 3.3 we have $M_t = 0$ as required.

Remark. The method of proving Theorem 2 (the extension of the properties of the algebras A_1 to arbitrary algebras of classical type) is similar to the method employed in [1, 2] (p. 443), and [13] (p. 123).

5. THEOREM 3. Let V be a simple G-module and $l_V \neq 0$. We put $t = \min_{l \in G^* \setminus \{0\}} (\dim G - \dim G_l)/2$. Then dim V is divisible by p^t.

Proof. Let e_α be a root vector. Since the Lie algebra G is simple, the linear hull of the orbit of e_α under the action of the adjoint group coincides with G. Therefore, there is a Borel subalgebra B and an ordering of its roots such that $l_V(e_\theta) \neq 0$, where θ is the highest root. Let V_1 be a simple submodule of the B-module V. By Theorem 1 there exist $\tilde{l} \in B^*$ and $H \in \mathfrak{P}(B, \tilde{l})$ for which the B-module V_1 is induced by a one-dimensional H-submodule. Therefore, $\dim V_1 = p^{t_1}$, where $t_1 = (\dim B - \dim B_{\tilde{l}})/2$. In addition $l(e_\theta) \neq 0$, for otherwise we would have $e_\theta \in B_{\tilde{l}} \subset H$ and $l(e_\theta) = \tilde{l}(e_\theta) \neq 0$, which is impossible.

Let $l' \in G^*$ be a form such that $l'(e_\alpha) = 0$ for $\alpha \neq \theta$, $l'(T) = 0$ and $l'(e_\theta) \neq 0$. Let $\tilde{l}' = l'|_B$. If $g = \sum_{\alpha \in M} e_\alpha \in B_{\tilde{l}}$, then it is obvious that $e_\alpha \in B_{\tilde{l}'}$ for the smallest root α from M. Therefore, $\dim B_{\tilde{l}} \leq \dim B_{\tilde{l}'}$. We have

$$2t_1 \geqslant (\dim B - \dim B_{\tilde{l}'}) = (\dim G - \dim G_{l'}) \geqslant 2t.$$

Thus dim V_1 is divisible by p^t. But by Corollary 2.6 all the composition factors of the B-module V have the same dimension. Hence dim V is divisible by p^t.

Remark. It appears that for every simple G-module V dim V is divisible by $p^{(\dim \Omega)/2}$, where $\Omega = \mathcal{G}(l_V)$. Theorems 2 and 3 support this conjecture.

§4. p-REPRESENTATIONS OF LIE ALGEBRAS OF CARTAN TYPE

An irreducible representation of a Lie p-algebra G is called a p-representation if $l_V = 0$. Everywhere in this section the word "module" means the module of a p-representation.

1. Let $G = \bigoplus_{i \geqslant m} G_i$ be a Lie p-algebra of Cartan type with the natural gradation (see [5, 6]). It is known that

(a) If $G \neq W_1$, then G_{-1} and G_1 generate G, and G_1 generates $\bigoplus_{i>0} G_i$.

(b) G_{-1} and G_1 consist of nilpotent elements.

2. Let V be a simple G-module. We put $G^- = \bigoplus_{i \leqslant -1} G_i$, $G^+ = \bigoplus_{i \geqslant 1} G_i$, $V^\pm = \{v \in V : G^\pm v = 0\}$. It is obvious that V^+ and V^- are submodules of the G_0-module V.

THEOREM 4. a) V^+ and V^- are simple G_0-modules.

b) For any simple G_0-module V' there are simple G-modules V_1 and V_2 such that the G_0-modules V_1^+ and V_2^- are isomorphic to V'.

c) If V_1 and V_2 are simple G-modules, and the G_0-modules V_1^+ and V_2^+ (respectively V_1^- and V_2^-) are isomorphic, then the G-modules V_1 and V_2 are isomorphic.

Proof. We restrict ourselves to the "−" case. We show first that $W = V^- \cap G_1 V$ is equal to 0. Let $U_0'(G_1) \subset U_0(G)$ be the subalgebra generated by G_1, and let $U_0(G_1) = U_0'(G) \oplus k \cdot 1$. We have that $U_0(G_1)W \subset U_0'(G_1)V \neq V$ since $U_0'(G_1)$ is nilpotent (Property (b)). The space $U_0(G_0)W$ is a G_{-1}-submodule since $G_{-1}W = 0$; it is obviously also a G_0- and a G_1-submodule. We obtain from (a) that $U_0(G_1)W$ is a G-submodule. Since $U_0(G_1)W \neq V$, and because V is a simple G-module we obtain that $U_0(G_1)W = 0$ and in particular that $W = 0$.

We put $V_0 = V^-$, $V_{i+1} = G_1 V_i$. By reasoning as above we obtain that the sum of all the spaces V_i is V. We show by induction that this sum is direct. We assume otherwise: $V_{m+1} \cap \bigoplus_{0 \leqslant i \leqslant m} V_i \ni v \neq 0$. Then $G_{-1}v \neq 0$ since $W = 0$. But $G_{-1}v \subset V_m \cap \bigoplus_{i \leqslant 0 \leqslant m-1} V_i$ which is impossible. Thus $V = \bigoplus_{i \geqslant 0} V_i$. It follows from here that the G_0-module V_0 is simple, for if $\bar{V}_0$ is a proper submodule of the G_0-module V_0, it is obvious that $U_0(G_1)\bar{V}_0$ is a proper submodule of the G-module V.

To prove b) we define the action of $G^- \oplus G_0$ on V' by putting $G^-V' = 0$. Then the desired G-module is a factor-module of $V_0'^G$. Every simple G-module V can be obtained in this way. Since $V = \bigoplus_{i \geq 0} V_i$, the kernel of the mapping $\varphi : V_0'^G \to V$ is homogeneous with respect to the decomposition $V_0'^G$ $V_0'^G = \bigoplus_{i \geq 0} G_1^i V'$; in view of the simplicity of the G_0-module V' this kernel does not intersect V'. Thus the kernel of the mapping φ is defined uniquely as the sum of the homogeneous submodules of the G-module $V_0'G$, and c) is proved.

We obtain the following corollary from Theorem 4 and Lemma 2.4.

COROLLARY 1. Let B be a Borel subalgebra of $G_0 \subset G$, $B \supset T$, where T is the maximal torus of G_0, and let h_i be a basis of T for which $h_i^{[p]} = h_i$. We put $B^{\pm} = G^{\pm} \oplus B$. Then every simple G-module is isomorphic to a factor-module of a G-module induced by a one-dimensional B^--module (respectively B^+-module), and is determined uniquely by the latter, that is, there is a one-to-one correspondence between the p-representations of the Lie algebra G and the linear forms $\lambda_+ \in T^*$ for which $\lambda_+(h_i) \in F_p$ (respectively λ_-).

A similar description of the p-representations of a Lie algebra of classical type was obtained in [12].

COROLLARY 2. Let V and V* be contragredient simple G-modules. Then the G_0-modules $(V^+)^*$ and $(V^*)^-$ are isomorphic. There exists a non-degenerate, invariant, bilinear form on V if and only if the G_0-modules $(V^+)^*$ and V^- are isomorphic.

Proof. Let $V' = G_1V$, $\bar{V}^* = \{v \in V^* : v(V') = 0\}$. Then $\bar{V}^* = (V^*)^+$, and consequently, there is a non-degenerate pairing between $V^- \cong V/V'$ and $(V^*)^+$; this proves the first assertion. The second assertion is equivalent to the isomorphism condition on the G-modules V* and V, hence it follows from the first assertion and from Theorem 4c).

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