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EXPONENTIALS IN LIE ALGEBRAS OF CHARACTERISTIC p

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Abstract. The relationship between the structure of a simple Lie algebra of finite characteristic and the structure of the group of its automorphisms is investigated. The results obtained are used to classify simple Lie algebras of characteristic $p > 5$ for which the largest reduced subgroup in the scheme of automorphisms is a maximal subscheme. An analogous classification theorem is proved for "simple" group schemes, i.e. schemes every normal divisor of which lying in the reduced subscheme is the kernel of some purely nonseparable isogeny. For characteristics 2 and 3, families of counterexamples are constructed to all results obtained for $p > 5$.

The fundamental question considered in this work may be formulated briefly as follows: What is the relationship between the structure of the group of automorphisms of a finite-dimensional Lie algebra over a field of characteristic p and the structure of the Lie algebra itself? Since much more is known about algebraic groups than about Lie algebras, the study of this question permits one to carry over to some extent the structural theory and classification from groups to Lie algebras.

Exponentials play the most important part in our considerations. In §2 they are used to describe Lie algebras for $p > 5$ for which the reduced group of automorphisms is isomorphic to the almost inner product of a reductive group and an arbitrary group (Theorem 2.1). In particular, it turns out that if the group of automorphisms of a simple Lie algebra G is reductive and nontrivial, then G is a Lie algebra of the classical type. For $p = 5$ we succeed in proving only a partial analog of Theorem 2.1.

Since for characteristics 2 and 3 there are considerably fewer exponentials than for $p \geq 5$, the assertion of Theorem 2.1, as one would expect, is not true for these characteristics. In §3 several families of simple finite-dimensional Lie algebras of characteristics 2 and 3 are constructed for which all the assertions of §2 are false. All these families are obtained from the single construction studied in [4] for $p > 3$. The classification obtained in [4] for $p > 3$ is extended to $p = 2$ and 3 (Theorem 3.7). We thank A. N. Rudakov, who informed us that earlier he and A. I. Kostrikin independently devised examples of families of simple finite-dimensional Lie 3-algebras (see [16]).

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In §4 the results of §2 and the method of graded algebras, developed in [4], are used to obtain a new characterization of Lie algebras of the Cartan type (Theorem 4.1). Theorem 4.1 classifies simple Lie algebras over an algebraically closed field of characteristic $p > 5$ for which the largest reduced subscheme in the scheme of all the automorphisms is maximal (modulo all the filtered Lie algebras for which the associated graded Lie algebra is a Lie algebra of Cartan type s_n or h_n).

This result is then used to study some group schemes which we call simple. The fact is that it is not possible to define a simple group scheme as a scheme without any *nontrivial normal divisors, since there is always the kernel of the Frobenius homomorphism.* Therefore it seems natural to us to consider a group scheme simple if every one of its normal divisors which lies in the reduced subscheme is contained in the kernel of some power of the Frobenius homomorphism. Obvious examples of such schemes are the simple smooth groups and also schemes of automorphisms of Lie algebras of the Cartan type. Theorem 4.7 is a step towards classifying schemes simple in the above sense. As in Theorem 4.1, the fundamental restriction in Theorem 4.7 is the fact that the largest *reduced subscheme is to be maximal.*

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§1. General observations and notation

Let $\mathbf{k}$ be an algebraically closed field of characteristic $p > 0$, let G be a Lie algebra over $\mathbf{k}$, and let $\mathcal{G}(G)$ be the largest reduced subgroup in the irreducible component of the group scheme Aut G of all automorphisms of G. In particular, let $\mathcal{G}(G)$ be the reduced irreducible affine algebraic group. By *Diff G* we shall denote the algebra of the differentiations of the algebra G, and by Lie $\mathcal{H}$, the Lie algebra of the group scheme $\mathcal{H}$. The embedding ϕ of the algebraic affine group $\mathcal{G}(G)$ into the group $\mathcal{GL}(G)$ of all nonsingular linear transformations of the space G has for its differential the mapping $d\phi$: Lie $\mathcal{G}(G) \to M(G)$ (the algebra of all linear transformations of G).

The action of the group $\mathcal{G}(G)$ by means of automorphisms on G defines the mapping $\psi: \mathcal{G}(G) \to \mathcal{G}(\mathrm{Diff}\, G)$ by the formula $\psi(a)D = \phi(a)D\phi(a)^{-1}$, $a \in \mathcal{G}(G)$, $D \in \mathrm{Diff}\ G$.

For convenience of reference, we shall enumerate several well-known and easily verifiable facts [7].

Lemma 1.1. a) $d\phi(\mathrm{Lie}\ \mathcal{G}(G)) \subset \mathrm{Diff}\ G$.

b) *If* $\mathcal{T}(t)$, $t \in \mathbf{k}$, *is an additive one-parameter subgroup (i.e. a subgroup isomorphic to the group* G_a) *of* $\mathcal{G}(G)$, *then* $d\phi(\mathrm{Lie}\ \mathcal{T}(t)) = \mathbf{k}(d\mathcal{T}/dt)(0)$.

c) *If* $\mathcal{T}(t)$, $t \in \mathbf{k}^*$, *is a multiplicative one-parameter subgroup (i.e. a subgroup isomorphic to* G_m) *of* $\mathcal{G}(G)$, *then* $d\phi(\mathrm{Lie}\ \mathcal{T}(t)) = \mathbf{k}(d\mathcal{T}/dt)(1)$.

For $D \in \mathrm{Diff}\ G$, let $E(D) = \Sigma_{m=0}^{p-1} D^m/m!$. Instead of $E(\mathrm{ad}\, g)$, we shall usually write $E(g)$. The following lemma follows from the computation carried out on Russian p. 17 in [2].

Lemma 1.2. *If* $D \in \mathrm{Diff}\ G$ *and* $D^p = 0$, *then* $E(D)$ *is an automorphism of the Lie algebra* G *if and only if*

$$\sum_{\substack{l+r\geq p\\ 0<l,r<p}} \frac{1}{l!r!}[D^l x, D^r y] = 0$$

for any $x, y \in G$. *In particular, if* $[D^l x, D^r y] = 0$ *for* $l + r \geq p$ *and for any* $x, y \in G$, *or* $D^{(p+1)/2} = 0$, *then* $E(tD)$, $t \in \mathbf{k}$, *is an additive one-parameter subgroup of* $\mathcal{G}(G)$ *and* $d\phi(\mathrm{Lie}\ E(tD)) = \mathbf{k}D$.

Denote by $\mathcal{G}_1(G)$ the subgroup of the group $\mathcal{G}(G)$ generated by all the one-parameter subgroups of the form $E(tg)$. Note that $\mathcal{G}_1(G)$ is clearly a normal divisor of the group $\mathcal{G}(G)$. However, it follows from [13] that the Lie algebra $d\phi(\mathrm{Lie}\ \mathcal{G}_1(G))$ may be different from ad G.

Example. Let $p = 3$ and let G be the factor algebra of the algebra Lie $\mathfrak{SL}(2, \mathbf{k})$ by the center. Then $\mathcal{G}_1(G)$ is of type $\mathbf{G}_2$.

Lemma 1.2 permits one to construct additive one-parameter subgroups of the group $\mathcal{G}(G)$. We shall demonstrate how to construct multiplicative one-parameter subgroups.

Lemma 1.3. *Let* G *be a Lie algebra and let* M *be a free abelian group of rank* l. *Assume that* G *has a gradation* $G = \bigoplus_{\alpha \in M} G_\alpha$. *Then the group* $\mathcal{G}(G)$ *contains an* l-*dimensional torus acting trivially on* G_0 *and acting as a scalar on* G_α.

Proof. Let $m_1, \cdots, m_l$ be a basis for M. Suppose that $d_i = (q_{i1}, \cdots, q_{il})$, $i = 1, 2, \cdots, l$, are l linearly independent integral vectors. Define the homomorphism ω_i: $\mathbf{k}^* \to \mathcal{G}(G)$ by the formula $\omega_i(t)a = t^{\Sigma r_j q_{ij}} a$ if $a \in G_\alpha$, $\alpha = \Sigma r_j n_j$. That $\omega_i(t)$ is an automorphism of G follows from the fact that the spaces G_α form a gradation of G. Clearly the images of the various ω_i commute with each other and, therefore, generate a torus. This torus is l-dimensional, since the vectors d_i are linearly independent.

Lemma 1.4. *Suppose that all the conditions of Lemma* 1.3 *are satisfied and, in addition, that the following conditions hold.*

a) *The set* $\Sigma = \{\alpha \in M\colon G_\alpha \neq 0\}$ *contains the basis* $m_1, \cdots, m_l$ *of the group* M.

b) *Let* G^+ *be the algebra generated by the spaces* G_{m_i}, $i = 1, 2, \cdots, l$. *If* $\alpha \in \Sigma$ *and* $\alpha = \Sigma r_i m_i$, $r_i \geq 0$, *then* $G_\alpha \cap G^+ \neq 0$.

c) *If* $\alpha \in \Sigma$, *then there exists a number* r *such that* $(\mathrm{ad}\ G^+)^r G_\alpha \cap G_\beta \neq 0$ *for some* $\beta \in \Sigma$, $\beta = \Sigma r_i m_i$, $r_i \geq 0$.

Let $h \in G_0$, *let* $(\mathrm{ad}\ h)^p = \mathrm{ad}\ h$, *and suppose that* ad h *acts as a scalar on all the* G_α. *Then there exists a one-dimensional subtorus* $\mathcal{T}_h$ *in* $\mathcal{G}(G)$ *such that* Lie $\mathcal{T}_h =$ $\mathbf{k}$ ad h.

Proof. Let λ_α be an eigenvalue of ad h in G_α. Since $(\mathrm{ad}\ h)^p = \mathrm{ad}\ h$, we have $\lambda_\alpha^p = \lambda_\alpha$, i.e. $\lambda_\alpha \in \mathbf{F}_p$. Choose integers q_α such that $\lambda_\alpha \equiv q_\alpha \pmod p$. Let $q_i = q_{m_i}$. Define the homomorphism $\omega\colon \mathbf{k}^* \to \mathcal{G}(G)$ by the formula $\omega(t)a = t^{\Sigma r_i q_i} a$ if $a \in G_\alpha$, $\alpha = \Sigma r_i n_i$. According to Lemma 1.3, $\omega(t)$ is an automorphism of G for all $t \in \mathbf{k}^*$. Let $\mathcal{T} = \omega(\mathbf{k}^*)$. We must show that $d\phi(\mathrm{Lie}\ \mathcal{T}) = \mathbf{k}$ ad h. Let $d\phi(\mathrm{Lie}\ \mathcal{T}) = \mathbf{k}h'$, let $h' \in \mathrm{Diff}\ G$, and let $(\operatorname{ad} h')^p = \operatorname{ad} h'$. Then h' acts as a scalar on all the G_α and its eigenvalues in G_α are equal to $\mu_\alpha \equiv \Sigma r_i q_i \pmod p$ (where $\alpha = \Sigma r_i m_i$). Clearly, $\lambda_{m_i} \mu_{m_i}$.

In accordance with condition b) and in view of the fact that h acts as a scalar on all the G_α, we have, for $\alpha \in \Sigma$, $\alpha = \Sigma r_i m_i$, $r_i \geq 0$, that $\lambda_\alpha = \Sigma r_i \lambda_{m_i}$. Since μ_α is defined by this same formula, $\lambda_\alpha = \mu_\alpha$ for all such α. Now if $\alpha \in \Sigma$, then according to c) there exist $a_i, b_i \geq 0$ $(i = 1, \cdots, l)$ such that $\alpha + \Sigma a_i m_i = \Sigma b_i m_i$ and $\lambda_\alpha = \Sigma b_i \lambda_{m_i} - \Sigma a_i \lambda_{m_i}$. From the definition of ω it once again follows that $\lambda_\alpha = \mu_\alpha$. The lemma is proved.

Remark 1.5. Let G be any Lie algebra, let $h \in G$, and let $(\operatorname{ad} h)^p = \operatorname{ad} h$. Then $\operatorname{ad} h$ can be reduced to diagonal form and its eigenvalues belong to the field $\mathbf{F}_p$. $\operatorname{ad} h$ defines in G the gradation $G = \bigoplus_{i \in \mathbf{F}_p} G_i$, where $G_i = \{g \in G: [hg] = ig\}$. Let μ_p be the group of the pth roots of unity (i.e. the group scheme with the lattice ring $\mathbf{k}[x]/(x^p - 1)$). Then we can define the monomorphism $\omega: \mu_p \to \operatorname{Aut} G$ by the formula $\omega(t)(a) = t^{\tilde{i}} a$, for all $a \in G_i$, where $\tilde{i}$ is any integer such that $i \equiv \tilde{i} \pmod p$. Obviously we shall have here that $\operatorname{Lie} \omega(\mu_p) = \mathbf{k} \operatorname{ad} h$.

Definition. The element $g \in G$ is called *semisimple* if $\operatorname{ad} g$ can be reduced to diagonal form. The subalgebra T is called *diagonalizable* if $\operatorname{ad} T$ can be reduced to diagonal form. A diagonalizable subalgebra T is called *open* if $\operatorname{ad} T \subset d\phi(\operatorname{Lie} \mathcal{G}(G))$.

Proposition 1.6. *Any two maximal diagonalizable open subalgebras T and T' of the Lie algebra G are conjugate by an element of the group $\mathcal{G}(G)$.*

Proof. Suppose that $\mathcal{T}$ is a maximal torus in $\mathcal{G}(G)$ and that $\widetilde{T} = \operatorname{Lie} \mathcal{T}$. Then the tori $\operatorname{ad} T$ and $\operatorname{ad} T'$ are conjugate to the subalgebras of $\widetilde{T}$ [5]. Since they are maximal and $\widetilde{T}$ is diagonalizable, the sum of their images in $\widetilde{T}$ coincides with each one of them. Therefore $\operatorname{ad} T$ and $\operatorname{ad} T'$ are conjugate. Since the center of G is contained in both T and T', it follows from this that T and T' are conjugate.

§2. Lie algebras with a reductive group of automorphisms for $p \geq 5$

In this section, it is convenient to use the following definition.

Definition. For $p > 3$, the Lie algebras of reductive algebraic groups and also their factor algebras by the center will be called *Lie algebras of the classical type.*

It is known (see, for example, [13]) that if $\mathcal{G}$ is an almost simple algebraic group of type $\mathbf{A}_n$, $n + 1 \not\equiv 0 \pmod p$, $\mathbf{B}_n, \cdots, \mathbf{E}_8$, then for $p > 3$ the Lie algebra $\mathcal{G}$ is simple.

Furthermore, if $\mathcal{G}$ is an almost simple group of type $\mathbf{A}_{lp-1}$, then we shall denote by $\mathcal{C}$ the center of $\mathcal{G}$ (in the sense of scheme theory) and by $\mathcal{C}^0$ its connected component (which is isomorphic to the group μ_{p^m} of the p^mth roots of unity). Then if $\mathcal{C}^0 \neq \{1\}$, $\operatorname{Lie} \mathcal{G}$ has a one-dimensional center $\operatorname{Lie} \mathcal{C}$. If $\mathcal{C} = 1$, then $\mathcal{G}$ is an adjoint group and $[\operatorname{Lie} \mathcal{G}, \operatorname{Lie} \mathcal{G}]$ is a simple Lie algebra $\mathbf{A}'_{lp-1}$.

The following theorem is the main result of this article.

Theorem 2.1. *Let G be a Lie algebra without a center over an algebraically closed field of characteristic $p > 5$, where $G = [G, G]$. Assume that $\mathfrak{G}(G)$ is an almost inner product, $\mathfrak{G}(G) = \mathfrak{G} \cdot \mathfrak{G}'$, where the group $\mathfrak{G}$ is reductive. Then the following assertions are true.*

a) *$\mathfrak{G}$ is semisimple.*

b) *$G = \overline{G} \oplus G'$ is a direct sum of Lie algebras, where* $\operatorname{ad}(\overline{G}) = d\phi([\operatorname{Lie}\mathfrak{G}, \operatorname{Lie}\mathfrak{G}])$, $G' = Z_G(\overline{G})$, *where* $(\operatorname{Aut}\overline{G})^0 = \mathfrak{G}$ *and* $\mathfrak{G}(G') = \mathfrak{G}'$.

We shall first prove (a).

Proposition 2.2. *Let G be a Lie algebra without a center, and let $p > 3$. Then the center of the group $\mathfrak{G}(G)$ is unipotent. In particular, the center of $\mathfrak{G}(G)$ does not contain a torus.*

Proof. Suppose that $\mathfrak{T}$ is a one-dimensional torus lying in the group $\mathfrak{G}(G)$. The group of characters of the torus $\mathfrak{T}$ is isomorphic to the group $\mathbf{Z}$. The torus $\mathfrak{T}$ acts completely reducibly on G. Suppose that $G = \bigoplus_{i \in \mathbf{Z}} G_i$ is a weighting decomposition of G with respect to $\mathfrak{T}$. We have that $[G_i G_j] \subset G_{i+j}$. Let $n = \min\{i: G_i \neq 0\}$ and $m = \max\{i: G_i \neq 0\}$. Suppose that $|n| \geq m$ (for the case $m \geq |n|$ the proof is analogous). Then $(\operatorname{ad} G_n)^3\, G \subset \bigoplus_{i=4n}^{m+3n} G_i$. This space is equal to zero, since $m + 3n < n$ (by virtue of the condition that $|n| \geq m$). Thus, by Lemma 1.2, $\mathfrak{G}(G)$ contains (for $p \geq 5$) the subgroup $E(tg)$, $t \in \mathbf{k}$, $g \in G_n$. The torus $\mathfrak{T}$ does not commute with this subgroup (since the case $G = G_0$ is impossible). The proposition is proved.

Let us turn to the proof of (b). Since G does not have a center, we may (and shall) identify G with $\operatorname{ad}(G) \subset \operatorname{Diff} G$. Let $\widetilde{G} = \operatorname{ad}(G) \cap d\phi(\operatorname{Lie}\mathfrak{G})$. Recall (§1) that the group $\mathfrak{G}$ with the help of the homomorphism $\psi: \mathfrak{G} \to \mathfrak{G}(\operatorname{Diff} G)$ acts on $\operatorname{Diff} G$. The subalgebras G and $\widetilde{G}$ are invariant with respect to this action.

Let $\mathfrak{T}$ be a maximal torus of $\mathfrak{G}$ and let X be the group of its characters. As is well known, X is a free abelian group. The action of $\mathfrak{T}$ on $\operatorname{Diff} G$ is completely reducible. The following are the weighting decompositions of the algebras $\operatorname{ad} G$ and $\widetilde{G}$ with respect to $\mathfrak{T}$:

$$\operatorname{ad} G = \bigoplus_{\alpha \in X} G_\alpha, \quad \widetilde{G} = \bigoplus_{\alpha \in X} H_\alpha.$$

Let

$$\Sigma = \{\alpha \in X : G_\alpha \neq 0\}, \quad \widetilde{\Sigma} = \{\alpha \in X : H_\alpha \neq 0\}.$$

Suppose that D and $\widetilde{D}$ are closed convex covers of the sets Σ and $\widetilde{\Sigma}$ respectively in the space $X \otimes \mathbf{R}$. Let D' be the minimal closed convex centrally symmetric set in $X \otimes \mathbf{R}$ containing D. The following relationships clearly hold: $\widetilde{D} \subset D \subset D'$.

Lemma 2.3. *Suppose that $p > 3$. Then the following assertions are true.*

(a) $\Sigma \cap \partial D' \subset \widetilde{\Sigma}$, *and* $G_\alpha \subset \widetilde{G}$ *for all* $\alpha \in \Sigma \cap \partial D'$.

(b) *$\widetilde{\Sigma}$ is a system of roots of $\mathfrak{G}$.*

(c) $\widetilde{D} = D = D'$.

Proof. The symbol ∂ will denote the boundary of a region. Let $\alpha \in \Sigma \cap \partial D'$. Then $(3\alpha + D') \cap D' = \emptyset$. As a matter of fact, suppose that M is a plane of support of the set D' at the point α. Since D' is centrally symmetric, it is clear that $3\alpha + D'$ and D' lie on different sides of M (and $3\alpha + D' \cap M = \emptyset$). Therefore $(3\alpha + D') \cap D' = \emptyset$. This means that $(\mathrm{ad}\, g)^3 = 0$ for all $g \in G_\alpha$. Hence, by Lemma 1.2, $E(tg) \subset \mathfrak{G}(G)$. Clearly $\mathfrak{T}$ normalizes $E(tg)$. Since $\mathfrak{T}$ acts nontrivially on G_α, $\mathfrak{T}$ acts nontrivially on $E(tg)$, i.e. $E(tg) \in \mathfrak{G}$. Since $\mathrm{Lie}\, E(tg) = \mathrm{k}\, \mathrm{ad}\, g$ (Lemma 1.2), it follows that $\mathrm{ad}\, g \in \widetilde{G}$, i.e. $G_\alpha \subset \widetilde{G}$ for all $\alpha \in \Sigma \cap \partial D$, and therefore $\widetilde{\Sigma} \supset \Sigma \cap \partial D'$. (a) is proved. In particular, $\widetilde{\Sigma}$ contains the roots of every simple component of the group $\mathfrak{G}$.

Assertion (b) follows from the fact that $\widetilde{G}$ is an ideal in Lie $\mathfrak{G}$ containing the nontrivial root subspace of every simple component of Lie $\mathfrak{G}$.

We shall prove (c). Since the region $\widetilde{D}$ is invariant with respect to the Weil group of $\mathfrak{G}$, it is automatically centrally symmetric. Since D' is completely determined by the vectors from $\Sigma \cap \partial D'$, since these vectors lie in $\widetilde{\Sigma}$, and since $\widetilde{D}$ is centrally symmetric, we must have $\widetilde{D} = D'$, as was required.

Lemma 2.4. *Suppose that* $p > 5$. *Then* $\widetilde{\Sigma} = \Sigma$ *and* $G_\alpha = H_\alpha$ *for any* $\alpha \in \Sigma \backslash 0$.

Proof. It is sufficient to consider the case for which $\widetilde{\Sigma}$ is a connected system of roots. If $\widetilde{\Sigma}$ is a system of roots of type $\mathbf{G}_2$, then, as is easy to see, $\widetilde{D} \cap X = \widetilde{\Sigma}$, and consequently, by Lemma 2.3, $\Sigma = \widetilde{\Sigma}$. Moreover, if Σ is of type $\mathbf{G}_2$, then $\alpha + 4\beta \notin \Sigma$ for any $\alpha, \beta \in \Sigma$. Therefore $(\mathrm{ad}\, g)^4 = 0$ for all $g \in G_\alpha$, $\alpha \in \Sigma \backslash 0$, and, by Lemma 1.2, $G_\alpha = H_\alpha$ for all $\alpha \in \Sigma$.

If $\widetilde{\Sigma}$ is a system of roots not of type $\mathbf{G}_2$, then, as is known, the number of modules of all the coordinates of those roots in the basis consisting of the fundamental weights does not exceed two. By Lemma 2.3 this is true for the vectors of the system Σ. Therefore $\beta + \gamma + a\alpha \notin \Sigma$ for $a \geq 7$, $\alpha, \beta, \gamma \in \Sigma$, $\alpha \neq 0$, and consequently $[(\mathrm{ad}\, g)^r x, (\mathrm{ad}\, g)^l y] = 0$, $l + r \geq 7$, for all $g \in G_\alpha$, $x \in G_\beta$ and $y \in G_\gamma$. Applying Lemma 1.2, we have that $\widetilde{\Sigma} = \Sigma$ and $G_\alpha = H_\alpha$ for all $\alpha \in \Sigma \backslash 0$.

Proof of Theorem 2.1. Denote by $\overline{G}$ the subalgebra in G generated by the space $\bigoplus_{\alpha \neq 0} G_\alpha$. Since $G = \bigoplus_{\alpha \neq 0} G_\alpha \oplus G_0$ and $[G_0, G_\alpha] < G_\alpha$ for all $\alpha \in \Sigma$, $\overline{G}$ is an ideal in G. Suppose that $G' = Z_G(\overline{G})$. Clearly $G' \subset G_0$. The intersection $C = \overline{G} \cap G'$ lies in the center of $\overline{G}$. Therefore $C \subset \mathrm{Lie}\, \mathfrak{T}$, i.e. $C \subset G_0$. Since $\mathfrak{T}$ acts trivially on G_0, C lies in the center of G_0 and consequently in the center of G'. Therefore C lies in the center of G, i.e. $C = 0$. Thus $\overline{G} \cap G' = 0$.

We shall show that $G = G' \oplus \overline{G}$. In accordance with what has been said above, it is sufficient that $G = G' + \overline{G}$. For the proof we shall use the condition $[G, G] = G$. Since $\overline{G}$ is an ideal in $\widetilde{G}$, it follows from the description of the Lie algebra of the differentiations of a Lie algebra of the classical type. (See, for example, the Corollary to Lemma 3.4 in §3) that $[\mathrm{Diff}\, \overline{G}, \mathrm{Diff}\, \overline{G}] \subset \mathrm{ad}\, (\overline{G})$. Let $g_1, g_2 \in G$. We have $\mathrm{ad}[g_1, g_2]|\overline{G} \subset \mathrm{ad}(\overline{G})$, i.e. $\mathrm{ad}[g_1, g_2] = \mathrm{ad}\, g_0$, $g_0 \in \overline{G}$. Hence $[g_1, g_2] - g_0 \in G'$, i.e. $[g_1, g_2] \in G' + \overline{G}$, i.e. $[G, G] \subset G' + \overline{G}$, as required. The remaining assertions of the theorem are now obtained automatically.

The assertion formulated below is a modification of Theorem 2.1.

Proposition 2.5. *Suppose that* $p > 5$ *and that* G *is a Lie algebra without a center,* $\mathcal{G}(G) = \mathcal{G} \cdot \mathcal{G}'$, $[\mathcal{G}, \mathcal{G}'] = 1$, *and* $\mathcal{G}$ *reductive. If* $d\phi(\mathrm{Lie}\ \mathcal{G}) \subset \mathrm{ad}\ (G)$, *then* $G = G' \oplus d\phi(\mathrm{Lie}\ \mathcal{G})$ *is a direct sum of Lie algebras.*

Proof. We shall accept the notation and the agreements of the proof of Theorem 2.1. Since $d\phi(\mathrm{Lie}\ \mathcal{G}) \subset \mathrm{ad}\ (G)$, it follows that $\tilde{G} = \mathrm{Lie}\ \mathcal{G}$. For $g \in G_0$ we have $\mathrm{ad}\ g|_{G_\alpha} = \lambda_\alpha(g)E$. Let $T = \mathrm{Lie}\ \mathcal{T} \subset \tilde{G}$, let Δ be a system of simple roots in Σ, and let ϕ_α, $\alpha \in \Delta$, be a dual basis. We shall formulate condition a). Suppose that c_α $(\alpha \in \Delta)$ are any elements of the field **k**. There exists a $t \in T$ such that $\lambda_\alpha(t) = c_\alpha$ for all $\alpha \in \Delta$.

As is well known, this condition is fulfilled if $\tilde{G}$ is a Lie algebra of an adjoint group. The only case in which $\tilde{G}$ may be not a Lie algebra of an adjoint group is the case in which $\mathcal{G}$ is of type A_{lp-1} and $\mathcal{C}^0 \neq 1$. In this case Lie $\mathcal{G}$ has center $C = \mathrm{Lie}\ \mathcal{C}^0$, $C \subset T$, i.e. $[C, G_0] = 0$. Since $\Sigma = \tilde{\Sigma}$ (Lemma 2.4 did not make use of the condition $[G, G] = G$), we have $[C, G_\alpha] = 0$, i.e. C is the center in G, i.e. $C = 0$. This proves a).

Now we shall prove the proposition. Let $g \in G_0$. Choose a $t(g) \in T$ such that $\lambda_\alpha(g) = \lambda_\alpha(t(g))$ for all $\alpha \in \Delta$. Then $\lambda_\alpha(g) = \lambda_\alpha(tg)$ for all $\alpha \in \Sigma$, and therefore $g - t(g) \in Z_G(\tilde{G})$. Since $G = \tilde{G} + G_0$, it follows that $G = \tilde{G} \oplus Z_G(\tilde{G})$, as required.

Corollary 2.6. *Let* $p > 5$ *and let* G *be a Lie algebra without a center. Assume that* $G = [G, G]$ *and that* $\tilde{G}$ *is not a Lie algebra of the classical type and cannot be decomposed into a direct sum of two Lie algebras. Let* $\mathcal{N}$ *be an unipotent radical in* $\mathcal{G}(G)$. *Then the following assertions are true.*

(a) *If* $\mathcal{G}(G) \neq 1$, *then* $\mathcal{N} \neq 1$.

(b) $\mathcal{L}_{\mathcal{G}(G)}(\mathcal{N}) \subset \mathcal{N}$.

Proof. Property (a) is an immediate consequence of Theorem 2.1. We shall prove (b). Let $\mathcal{G} = \mathcal{G}(G)$ and let $\mathcal{H} = \mathcal{L}_{\mathcal{G}}(\mathcal{N})$. The unipotent radical $\mathcal{N}'$ of the group $\mathcal{H}$ lies, by construction, in the center of $\mathcal{H}$. Now we shall need two lemmas.

Lemma 2.7. *Let* $\mathcal{H}$ *be an algebraic group whose unipotent radical* $\mathcal{N}'$ *lies in the center. Then* $\mathcal{H} = \mathcal{H}' \times \mathcal{N}'$, *where* $\mathcal{H}'$ *is a reductive group.*

Proof. Following Humphreys' example [9], we shall consider $\mathcal{H}$ as an extension of the reductive group $\mathcal{H}' = \mathcal{H}/\mathcal{N}'$ by means of the periodic group $\mathcal{N}'$ (of period p^l for some suitable l). In accordance with Steinberg's results ([12], 3.2, 5.1), we must have $\mathcal{N}' \cap [\mathcal{H}, \mathcal{H}] = 1$. Assuming that $\mathcal{H}' = [\mathcal{H}, \mathcal{H}]$, we obtain our assertion.

Lemma 2.8. *Let* $\mathcal{G}$ *be an algebraic group, let* $\mathcal{N}$ *be an unipotent radical of* $\mathcal{G}$, *and let* $\mathcal{H}'$ *be a reductive group in* $\mathcal{G}$ *centralizing* $\mathcal{N}$. *Then* $\mathcal{G} = \mathcal{H}' \cdot \mathcal{G}'$ *is an almost inner product of* $\mathcal{H}'$ *and some subgroup of* $\mathcal{G}'$.

Proof. Let $\omega: \mathcal{G} \to \mathcal{G}/\mathcal{N}$ be the natural projection. Then $\omega(\mathcal{G}) = \omega(\mathcal{H}') \times \overline{\mathcal{G}}$ is an almost inner product. Let $\mathcal{G}' = \omega^{-1}(\overline{\mathcal{G}})$, let $\mathcal{B}'$ be any Borel group in $\mathcal{G}'$, and let h be a semisimple element of $\mathcal{H}'$. Then h acts trivially on $\mathcal{B}'/\mathcal{N} = \omega(\mathcal{B}')$ and on $\mathcal{N}$. From this, in view of [6], pp. 4–13, it follows that $[h, \mathcal{B}'] = 1$, i.e. $[\mathcal{B}', \mathcal{H}'] = 1$. Since any element of $\mathcal{G}'$ is contained in some suitable Borel group, it follows from this that $[\mathcal{G}', h] = 1$. Recalling that $\mathcal{H}'$ is generated by its semisimple elements, we have $[\mathcal{G}', \mathcal{H}'] = 1$, i.e. $\mathcal{G} = \mathcal{G}' \times \mathcal{H}'$, as required.

Corollary 2.6 is obtained by successive application of Lemmas 2.7, 2.8 and Theorem 2.1.

Corollary 2.9. *Suppose that G is a Lie algebra without a center, $[G, G] = G$. Let $p > 5$ and let* ad $(G) \cap d\phi(\text{Lie } \mathcal{G}(G)) = \widetilde{G} \oplus \widetilde{G}'$, *where $\widetilde{G}$ is a Lie algebra of the classical type. Then $G = \widetilde{G} \oplus G'$.*

Proof. Suppose that $\mathcal{N}$ is an unipotent radical in $\mathcal{G}(G)$. Then $N = \text{Lie } \mathcal{N}$ is a nilradical in Lie $\mathcal{G}(G)$ [5]. On the other hand, ad (G) is an ideal of Diff G, i.e. $\widetilde{G} \oplus \widetilde{G}'$ is an ideal of Lie $\mathcal{G}(G)$. Therefore $[\widetilde{G} \oplus \widetilde{G}', N] \subset N \cap \widetilde{G}' \subset \widetilde{G}'$. Let $N' = N \cap \widetilde{G}'$. We are given that $[\widetilde{G}, N'] = 0$. We have shown that $\widetilde{G}$ acts trivially on N/N'. If $g \in \widetilde{G}$ is a semisimple element, it follows from this that $[g, N] = 0$. Since $\widetilde{G}$ is generated by semisimple elements, $[\widetilde{G}, N] = 0$. Hence $d\phi(\text{Lie } \mathcal{G}(G)) = \widetilde{G} \oplus N$. Suppose that $\mathcal{T}$ is a subtorus in $\mathcal{G}(G)$ such that $d\phi(\text{Lie } \mathcal{T})$ is a maximal subtorus in $\widetilde{G}$ (the existence of $\mathcal{T}$ follows from the results of [5]). Suppose that $\widetilde{\mathcal{G}}$ is a subgroup of $\mathcal{G}(G)$ generated by all the tori $g\mathcal{T}g^{-1}$, $g \in \mathcal{G}(G)$. Clearly Lie $\widetilde{\mathcal{G}} \supset \widetilde{G}$ and $\mathcal{T}$ is a maximal torus in $\widetilde{\mathcal{G}}$. We have $d\phi(\text{Lie } \widetilde{\mathcal{G}}) = \widetilde{G} \oplus \widetilde{N}$, $\widetilde{N} \subset N$. From this it follows, again by [9] (see also Lemmas 2.7 and 2.8), that Lie $\widetilde{\mathcal{G}} \simeq \widetilde{G}$, i.e. $\widetilde{\mathcal{G}}$ is a semisimple group which is an almost inner factor in $\mathcal{G}(G)$. Our assertion now follows from Theorem 2.1.

The following assertion will be needed in §4.

Corollary 2.10. *Let G be a Lie algebra without a center, let $\mathcal{N}$ be an unipotent radical in $\mathcal{G}(G)$, let $G_0 = d\phi(\text{Lie } \mathcal{G}(G)) \cap$ ad (G), and suppose that G_0 does not contain any ideals of G. Then for $p > 5$ the following assertions are true.*

(a) *If $G_0 =$ ad (G), then G is a Lie algebra of the classical type.*

(b) *If $d\phi(\text{Lie } \mathcal{N}) \cap G_0 = 0$, then G is a Lie algebra of the classical type.*

(c) $Z_{G_0}(\text{Lie } \mathcal{N} \cap \text{ad } (G)) \subset \text{Lie } \mathcal{N}$.

Proof. The proof of (a) does not, in general, make use of Theorem 2.1. If $G_0 =$ ad (G) and G is simple, then G_0 is a simple ideal of Lie $\mathcal{G}(G)$. After the factorization of Lie $\mathcal{G}(G)$ by a nilpotent radical, G_0 is mapped isomorphically onto a simple ideal of a Lie algebra of the classical type. Consequently G_0 is also a Lie algebra of the classical type.

We now prove (b). If $d\phi(\text{Lie } \mathcal{N}) \cap G_0 = 0$, then G_0 is an ideal of Lie $\mathcal{G}(G)$ which does not intersect Lie $\mathcal{N}$. From this it follows that G_0 is a Lie algebra of the classical type. This actually was used in the proof of Theorem 2.1 to establish that

[Lie $\mathfrak{G}(G)$, Lie $\mathfrak{G}(G)$] is an ideal of the Lie algebra ad(G) (lying in G_0). Note that the restriction $G = [G, G]$ was not used here. Consequently ad $(G) = G_0 =$ [Lie $\mathfrak{G}(G)$, Lie $\mathfrak{G}(G)$] and G is a Lie algebra of the classical type.

Finally, we prove (c). If Z_{G_0}(Lie $\mathfrak{N} \cap$ ad $(G)) \not\subset$ Lie $\mathfrak{N}$, then the method of proof of Corollary 2.6 leads to the conclusion that there is an ideal of G in G_0 isomorphic to a Lie algebra of the classical type. The assertion is proved.

We shall now formulate the best approximation of Theorem 2.1 for $p = 5$ which we have obtained.

Theorem 2.11. *Let G and $\mathfrak{G}(G)$ satisfy the conditions of Theorem* 2.1, *and let* $p = 5$. *Then the following assertions are true.*

(a) *The group $\mathfrak{G}$ is semisimple.*

(b) *If $\mathfrak{G}$ does not contain a component of type* $\mathbf{C}_n$, $n \geq 1$, *then G satisfies the conclusions of Theorem* 2.1.

(c) *If $\mathfrak{G}$ is an adjoint group, then G satisfies the conclusions of Theorem* 2.1.

Property (a) was proved in (2.2). First we shall assume that $\mathfrak{G}$ is an almost simple group. It is clear that we can assume this without loss of generality. We shall prove the analog of Lemma 2.4 (since this is the only place in the proof of Theorem 2.1 where the condition that $p > 5$ was used).

Lemma 2.12. *If $\mathfrak{G}$ is of type* $\mathbf{G}_2$ *or* $\mathbf{A}_2$ *and* $p = 5$, *then* $\Sigma = \tilde{\Sigma}$ *and* $G_\alpha = H_\alpha$ *for all* $\alpha \in \Sigma$.

Proof. If $\tilde{\Sigma}$ is of type $\mathbf{G}_2$, one can immediately verify that $\Sigma \cap \tilde{\mathcal{D}} = \tilde{\Sigma}$ (since any weight lying in $\tilde{\mathcal{D}}$ is a root). It can also immediately be verified that if $\alpha, \beta, \gamma \in \tilde{\Sigma}$, then $\alpha + \beta + a\gamma \notin \tilde{\Sigma}$ for all $a \geq 5$. From this it follows, by Lemma 1.2, that $G_\gamma \subset \tilde{G}$, i.e. $G_\gamma = H_\gamma$ for all $\gamma \in \Sigma\backslash 0$, as required. If $\tilde{\Sigma}$ is of type $\mathbf{A}_2$, then $X \cap \tilde{\mathcal{D}}$ is a system of roots of $\mathbf{G}_2$, and consequently the lemma is also true for $\mathbf{A}_2$.

Lemma 2.13. *If* $p = 5$, *then* $G_\alpha = H_\alpha$ *for all* $\alpha \in \tilde{\Sigma}$.

Proof. By virtue of 2.12, we may assume that $\tilde{\Sigma}$ is not of type $\mathbf{G}_2$. Then all the roots from $\tilde{\Sigma}$ lie on $\partial\tilde{\mathcal{D}}$. (If all the roots in $\tilde{\Sigma}$ are of the same length, they are all vertices of the polyhedron $\tilde{\mathcal{D}}$. If $\tilde{\Sigma}$ is of type $\mathbf{B}_n$, $\mathbf{C}_n$, or $\mathbf{F}_4$, then the long roots are vertices of the polyhedron $\tilde{\mathcal{D}}$ and the short ones lie on the boundaries.) From this and from Lemma 2.3 (a), Lemma 2.13 follows.

Lemma 2.14. *If $\tilde{\Sigma}$ is not of type* $\mathbf{A}_2$, $\mathbf{G}_2$, *or* $\mathbf{C}_n$, $n \geq 1$, *then for any nonzero weights* $\lambda, \mu \in \Sigma$ *it is true that* $\lambda + 3\mu \notin \Sigma$. *If $\tilde{\Sigma}$ is of type* $\mathbf{C}_n$ *and* $\lambda + 3\mu \in \Sigma$, *then* λ *and* μ *are proportional to the highest root of* $\mathbf{C}_n$ *or to its conjugate with respect to the Weil group.*

Proof. We shall assume that $\tilde{\Sigma}$ is different from $\mathbf{G}_2$. Suppose that $\lambda_1, \cdots, \lambda_n$ is a system of fundamental weights, the dual of the system of simple roots. Let $\lambda = \Sigma k_i\lambda_i$ and $\mu = \Sigma l_i\lambda_i$. Since for $\lambda \in \tilde{\Sigma}$ we have $|k_i| \leq 2$ for all i, it follows from

Lemma 2.3 that $|k_i| \leq 2$ for all i and if $|k_i| = 2$ for some i, then $\lambda \in \widetilde{\Sigma} \cap \partial\widetilde{D}$. Moreover, if $\lambda + 3\mu \in \Sigma$, then $|l_i| \leq 1$ for all i. Note that any root from $\widetilde{\Sigma}$ is taken by the Weil group into the root $\theta = \Sigma s_i\lambda_i$, for which all the $s_i \geq 0$. For all systems $\widetilde{\Sigma}$ this root is equal to one of the fundamental weights λ_m except for the highest root of A_n, which is equal to $\lambda_1 + \lambda_n$, and the highest root of C_n, which is equal to $2\lambda_1$. We shall consider the two cases separately.

Case I. λ lies in the interior of $\widetilde{D}$, and consequently $|k_i| \leq 1$ for all i. Then, clearly, if $\lambda + 3\mu \in \Sigma$, then $\lambda + 3\mu \in \widetilde{\Sigma} \cap \partial D$. Using the Weil group, we may assume that $\lambda + 3\mu = \theta$. If $\theta = \lambda_m$ and $\theta - 3\mu \in \Sigma$, we clearly have $\mu = \lambda_m$ and $\theta - 3\mu = -2\lambda_m \in \Sigma$, which is not possible in view of our assumption that λ lies in the interior of $\widetilde{D}$. If $\theta = \lambda_1 + \lambda_n$ (case A_n), then $\mu = \lambda_1$ or λ_n, which again contradicts our assumption. If $\theta = 2\lambda_1$ is the highest root of C_n, clearly $\mu = \lambda_1$ and $\lambda = -\lambda_1$.

Case II. $\lambda \in \widetilde{\Sigma}$ and again we may assume that $\lambda = \theta$. In all cases where θ is not the highest root of C_n, we have $\theta + 3\mu = \alpha \in \widetilde{\Sigma}$. Once again, this is impossible when $\theta = \lambda_m$ and θ is not proportional to the highest root of C_n. If $\theta = \lambda_1 + \lambda_n$, then, as can easily be seen, this is possible only in the case of A_2.

The lemma is proved.

Assertion (b) of Theorem 2.11 follows from Lemmas 2.12 and 2.14, taking into account Lemma 1.2.

Lemma 2.15. *Assertion* 2.11 (c) *is true.*

The proof may be obtained immediately upon observing that in the case in which $\mathfrak{G}$ is an adjoint group we have $X = \mathbf{Z}\widetilde{\Sigma}$ and $X \cap \widetilde{D} = \widetilde{\Sigma}$. On the other hand, this assertion follows form (2.12)–(2.14) and from the fact that an adjoint group of type C_n does not have a representation with weight λ_1.

Remark. Analogous considerations show that Lemma 2.3 holds for $p = 3$. For $p = 2$, counterexamples will be constructed in §3. We will also construct there counterexamples to Lemma 2.4 for $p = 3$.

§3. Contragredient Lie algebras for characteristics 2 and 3

From the results of [4], §2, it is easy to prove that for $p > 3$ every simple finite-dimensional contragredient Lie algebra is isomorphic to one of the simple Lie algebras of the classical type. We shall show that for $p = 2$ and 3 the picture changes sharply. There exist families of simple finite-dimensional contragredient Lie algebras such that the groups of automorphisms of all these Lie algebras are reductive. In particular, it follows from this that for $p = 2$ and 3, Theorem 2.1 is not true. For $p = 5$, the question remains open.

Let us recall the definition of a contragredient Lie algebra.

Suppose that $A = (a_{ij})$, $i, j \in I = \{1, 2, \cdots n\}$, is a matrix with elements from the field $\mathbf{k}$. Denote by $\widetilde{G}(A)$ the Lie algebra over $\mathbf{k}$ with generators e_i, f_i and h_i, $i \in I$, and the following defining relations $(i, j \in I)$:

$$[e_i f_j] = \delta_{ij} h_i, \quad [h_i h_j] = 0, \quad [h_i e_j] = a_{ij} e_j, \quad [h_i f_j] = -a_{ij} f_j.$$

Letting $\deg e_i = 1$, $\deg f_i = -1$ and $\deg h_i = 0$, $i \in I$, we transform $\tilde{G}(A)$ into a graded Lie algebra, $\tilde{G}(A) = \bigoplus_{i \in Z} \tilde{G}_i$. Let $J(A)$ be a maximal homogeneous ideal in $\tilde{G}(A)$ such that $J(A) \cap (\tilde{G}_{-1} \oplus \tilde{G}_1) = 0$ (such an ideal is unique). The Lie algebra $G(A) = \tilde{G}(A)/J(A)$ is called a contragredient Lie algebra and the matrix A is its Cartan matrix. Since in changing h_i to ch_i and f_i to cf_i, $c \in \mathbf{k}^*$, the ith row of the matrix A is multiplied by c, the contragredient Lie algebras associated with Cartan matrices with proportional corresponding rows are isomorphic.

If the matrix A can be obtained form the matrix $\tilde{A}$ by multiplying any rows by nonzero numbers and by renumbering the indices, then the matrices A and $\tilde{A}$ will be called equivalent. Contragredient Lie algebras with equivalent Cartan matrices are isomorphic.

For $p > 3$, $\mathbf{A}_n$, $n + 1 \not\equiv 0 \pmod p$, $\mathbf{A}'_{lp-1}$, $\mathbf{B}_n, \cdots, \mathbf{E}_8$ are examples of simple finite-dimensional contragredient Lie algebras. For $p = 3$, all these Lie algebras are simple finite-dimensional contragredient Lie algebras except $\mathbf{E}_6$ and $\mathbf{G}_2$. The Lie algebra $\mathbf{E}_6$ contains a one-dimensional center and the factor algebra $\mathbf{E}'_6$ of $\mathbf{E}_6$ by the center is a simple finite-dimensional contragredient Lie algebra. $\mathbf{G}_2$ contains a unique maximal ideal $\mathbf{A}_2$. For $p = 2$, the Lie algebras $\mathbf{A}_n$, $n + 1 \not\equiv 0 \pmod 2$, $\mathbf{A}'_{2l-1}$, $\mathbf{F}_6$ and $\mathbf{E}_8$, and also the factor algebras of the Lie algebras $\mathbf{D}_{2n+1}$ and $\mathbf{E}_7$ by the one-dimensional centers $\mathbf{D}'_{2n+1}$ and $\mathbf{E}'_7$, respectively, and of $\mathbf{D}_{2n}$ by the two-dimensional center $\mathbf{D}'_{2n}$, are simple finite-dimensional contragredient Lie algebras. Furthermore, $\mathbf{F}_4$ and $\mathbf{C}_n$ contain unique maximal ideals $\mathbf{D}_4$ and $\mathbf{D}_n$, respectively, and $\mathbf{B}_n$ contains a unique maximal ideal the factor algebra by which is $\mathbf{D}'_n$. All the above-mentioned simple Lie algebras for characteristics 2 and 3 will be called simple Lie algebras of the classical type in these characteristics.

Suppose that $G(A) = \bigoplus_{i \in Z} G_i$ is an induced gradation in $G(A)$.

Let $\mathfrak{G}(A) = \mathfrak{G}(G(A))$, $D(A) = \operatorname{Diff} G(A)$, and $P(A) = \{D \in D(A) : D(e_i) = a_i e_i,\ D(f_i) = -a_i f_i,\ D(h_i) = 0 \text{ for all } i \in I\}$.

Lemma 3.1. *If $D \in P(A)$ and $D^p = D$, then there exists a multiplicative one-parameter group $\mathfrak{T}(t)$ in $\mathfrak{G}(A)$ such that $(d\mathfrak{T}/dt)(1) = D$.*

Proof (compare with 1.4). Since $D^p = D$, we have $D(e_i) = k_i e_i$ and $D(f_i) = -k_i f_i$, where $k_i \in \mathbf{F}_p$, $i \in I$. Let $\tilde{k}_i \in \mathbf{Z}$ be any preimage of k_i under the homomorphism $\mathbf{Z} \to \mathbf{F}_p$. For $t \in \mathbf{k}^*$, let

$$\mathcal{T}(t) e_i = t^{\tilde{k}_i} e_i, \quad \mathcal{T}(t) f_i = t^{-\tilde{k}_i} f_i, \quad \mathcal{T}(t)|_{G_0} = \mathrm{id}.$$

This automorphism of the local part $G_{-1} \oplus G_0 \oplus G_1$ of the graded Lie algebra $G(A)$ may be extended to an automorphism of $G(A)$ (see [3], Chapter I, §2), which will be the one required.

Clearly there exists a basis of the Lie algebra $P(A)$ consisting of the elements $D_1, \cdots D_k$, for which $D_i^p = D_i$. The multiplicative one-parameter subgroups corresponding to all these elements generate a torus in the group $\mathcal{G}(A)$, which we shall denote by $\mathcal{T}(A)$. We have

$$d\varphi(\operatorname{Lie}\mathcal{T}(A)) = P(A) \supset \operatorname{ad}(G_0).$$

Lemma 3.2. *If there exists an isomorphism $\Psi: G(A) \to G(\tilde{A})$ and the group $\mathcal{G}(A) = \mathcal{G}(\tilde{A})$ is finite dimensional, then there exists an isomorphism $\Phi: G(A) \to G(\tilde{A})$ which takes every weight space in $G(A)$ with respect to $\mathcal{T}(A)$ into a weight space in $G(\tilde{A})$ with respect to $\mathcal{T}(\tilde{A})$.*

Proof. As can easily be seen, the factor algebra of the Lie algebra $G(A)$ by the center is a contragredient Lie algebra without center, and it is sufficient to prove the lemma for the latter. Therefore we may assume that the center of $G(A)$ is trivial. $\mathcal{T}(A)$ is a maximal torus of the group $\mathcal{G}(A)$, since every torus containing $\mathcal{T}(A)$ preserves the weighting decomposition with respect to $\mathcal{T}(A)$ and, since the center of $G(A)$ is trivial, it must obviously coincide with $\mathcal{T}(A)$. Since all the maximal tori in an algebraic group over an algebraically closed field are conjugates, $\mathcal{T}(A) = \omega\mathcal{T}(\tilde{A})\omega^{-1}$ for some $\omega \in \mathcal{G}(A)$. Then the element $\Phi = \omega\Psi$ will clearly be the one required.

Lemma 3.3. *Let $G(A) = \bigoplus_{i=-m}^{m} G_i$ be a finite-dimensional contragredient Lie algebra. For the Lie algebra $G(A)$ to be simple, it is necessary and sufficient for the matrix A to have the following property:*

(m) *For any $i, j \in I$, there exists a sequence $i_1, \cdots, i_r \in I$ for which*

$$a_{ii_1}a_{i_1i_2}\ldots a_{i_rj} \neq 0.$$

Proof. If condition (m) is not satisfied for some $i, j \in I$, then, as can easily be seen, the ideal generated by the element e_i does not contain e_j. Therefore condition (m) is necessary. We shall prove that it is sufficient. Let J be a nonzero ideal in $G(A)$ and let $g = \sum_{i\geq r} g_i$ be a decomposition of the nonzero element $g \in J$ into homogeneous components, where r is the largest number for all the nonzero $g \in J$. Then $[g_r G_1] = 0$, and consequently the space

$$\bigoplus_{i,j\geqslant 0} (\operatorname{ad} G_{-1})^i (\operatorname{ad} G_0)^j g_r$$

is a nonzero homogeneous ideal in $G(A)$. Therefore $r = m$ and g is a homogeneous element. The ideal generated by the element g_r is homogeneous and is contained in J. Therefore from the definition of a contragredient Lie algebra it follows that $J \cap (G_{-1} \oplus G_1) \neq 0$. From this it clearly follows that e_i (or f_i) $\in J$ for some $i \in I$. It can easily be seen that condition (m) now implies that $e_i, f_i \in J$ for all $i \in I$, and consequently that $J = G(A)$. The lemma is proved.

The following is a weight decomposition of the Lie algebra $G(A)$ with respect to $\mathcal{T}(A)$:

$$G(A) = \bigoplus_{\alpha\in X} G_\alpha$$

and with respect to ad G_0 its weight decomposition is

$$G(A) = \bigoplus_{\alpha \in G_0^*} G_{(\alpha)}.$$

Lemma 3.4. *Let $G(A)$ be a contragredient Lie algebra having the following properties:*

a) $\dim G_{(\alpha)} = 1$ *for* $\alpha \neq 0$ *and* $p \geq 3$, *and* $\dim G_{(\alpha)} = 2$ *for* $\alpha \neq 0$ *and* $p = 2$.

b) *For* $p = 2$ *and for any* $i \in I$, *there exists a* $j \in I$ *such that*

$$\begin{pmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{pmatrix} = \begin{pmatrix} 0 & c_1 \\ c_2 & 0 \end{pmatrix}, \text{ where } c_1, c_2 \neq 0.$$

Then $D(A) = \operatorname{ad}(G(A)) + P(A)$. *In particular, if in addition to this* $\det A \neq 0$, *then all the differentiations of the Lie algebra* $G(A)$ *are inner differentiations.*

Proof. Since the space $\operatorname{ad}(G_0)$ lies in $P(A)$, it consists of semisimple elements. Therefore $D(A)$ contains in addition to $\operatorname{ad}(G(A))$ a subspace V which is invariant with respect to the adjoint representation of $\operatorname{ad}(G_0)$ in $D(A)$. Since $\operatorname{ad}(G(A))$ is an ideal in $D(A)$, $[\operatorname{ad}(G_0), V] = 0$. Consequently every space $G_{(\alpha)}$ is invariant with respect to the differentiations from V. Therefore, by virtue of condition a), $V \subset P(A)$ for $p \geq 3$. If $p = 2$, then by condition b) the subalgebra H in $G(A)$ generated by the elements e_i, e_j, f_i and f_j is isomorphic to $\mathbf{A}_2$, where, by virtue of what we said above, $D(H) \subset H$ for any $D \in V$. By direct computations in $\mathbf{A}_2$, it is now easy to obtain that once again $V \subset P(A)$.

Corollary (compare with [13]). *In the simple Lie algebras of the classical type, all the differentiations are inner with the exception of the Lie algebras* $\mathbf{A}'_{lp-1}$ *for any* $p > 0$, $\mathbf{E}'_6$ *for* $p = 3$, *and* $\mathbf{D}'_n$ *and* $\mathbf{E}'_7$ *for* $p = 2$. *The Lie algebras* $\mathbf{A}'_{lp-1}$ *with the exception of* $\mathbf{A}'_1$ *for* $p = 2$ *and* $\mathbf{A}'_2$ *for* $p = 3$, *and also* $\mathbf{E}'_6$ *for* $p = 3$ *and* $\mathbf{E}'_7$ *for* $p = 2$, *are ideals of codimensionality* 1 *in the Lie algebra of differentiations. The Lie algebra of differentiations of the Lie algebra* $\mathbf{A}'_2$ *for* $p = 3$ *is* $\mathbf{G}_2$, *of the Lie algebra* $\mathbf{D}'_n$ *for* $p = 2$ *and* $n \neq 4$ *is* $\mathbf{C}_n$, *and of the Lie algebra* $\mathbf{D}'_4$ *for* $p = 2$ *is* $\mathbf{F}_4$. *If* $G_1, \cdots, G_k$ *are any of the above-mentioned Lie algebras, then* $\operatorname{Diff}(\bigoplus_{i=1}^k G_i) = \bigoplus_{i=1}^k \operatorname{Diff} G_i$.

Lemma 3.5. *Suppose that* $G(A)$ *is a finite-dimensional contragredient Lie algebra satisfying conditions* a) *and* b) *of Lemma* 3.4, *and suppose, moreover, that the matrix* A *possesses the following property.*

(M) *From* $a_{ij} = 0$ *it follows that* $a_{ji} = 0$; *and for any set* $i_1, \cdots, i_r \in I$,

$$a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_r i_1} = a_{i_2 i_1} \ldots a_{i_1 i_r}.$$

Then the group $\mathcal{G}(A)$ *is reductive.*

Proof. Let $\mathfrak{N}$ be an unipotent radical of the group $\mathcal{G}(A)$ and let Z be the center of the Lie algebra Lie $\mathfrak{N}$.

The reductiveness of the group $\mathcal{G}(A)$ means that $\mathfrak{N} = 1$, which is clearly equivalent to the equation $d\phi(Z) = 0$. By Lemma 3.4, $d\phi(Z) \subset \mathrm{ad}(G(A))$. By virtue of condition a) of Lemma 3.4, all the weight spaces G_α, $\alpha \neq 0$, of the Lie algebra $G(A)$ with respect to the torus $\mathfrak{T}(A)$ are one dimensional. The subalgebra $d\phi(Z)$ in $G(A)$ is clearly homogeneous with respect to this weight decomposition. Assume that $d\phi(Z) \neq 0$. Then $d\phi(Z)$ contains a nonzero element $g \in G_\alpha$, $\alpha \neq 0$. Since there exists an automorphism σ of the Lie algebra $G(A)$ for which $\sigma(e_i) = f_i$ and $\sigma(f_i) = e_i$, it follows that $d\phi(Z)$ also contains $\sigma(g) \in G_{-\alpha}$.

Condition (M) ensures the existence on $G(A)$ of an invariant bilinear form (,) for which the coupling of the spaces $G_{-\alpha}$ and G_α is nonsingular, $[e_\alpha, e_{-\alpha}] = (e_\alpha, e_{-\alpha})h_\alpha$, where $e_\alpha \in G_\alpha$, $e_{-\alpha} \in G_{-\alpha}$ and $h_\alpha \in G_0$ (see [3], Chapter II, §2). Since $\dim G_\alpha = \dim G_{-\alpha} = 1$, we obtain that $[g, \sigma(g)] \neq 0$, which contradicts the commutativity of $d\phi(Z)$. Thus $d\phi(Z) = 0$, and the lemma is proved.

We now turn to concrete examples. Suppose that

$$\mathbf{C}_{2,a} = \begin{pmatrix} 2 & -1 \\ a & 2 \end{pmatrix}, \qquad \mathbf{C}_{2,\infty} = \begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}$$

are matrices of characteristic 3 and

$$\mathbf{C}_{3,a} = \begin{pmatrix} 0 & 1 & 0 \\ a & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \mathbf{F}_{4,a} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ a & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \Delta_n = \begin{bmatrix} 1 & 1 & & & 0 \\ 1 & 0 & 1 & & \\ & 1 & \ddots & \ddots & \\ & & \ddots & 0 & 1 \\ 0 & & & 1 & 0 \end{bmatrix}$$

are matrices of characteristic 2.

Table 1

p	Cartan matrix A	$\dim G(A)$	m	p-structure	$\mathcal{G}(A)$	Isomorphisms
3	$\mathbf{C}_{2,a}$ $a \in (\mathbf{k} \cup \infty) \setminus (-1, 0)$	10	3	exists	$\mathbf{A}_1 \times \mathbf{A}_1$	$a = a'$, $a = -a' - 1$
2	$\mathbf{C}_{3,a}$ $a \in \mathbf{k} \setminus (0,1)$	16	4	none	$\mathbf{A}_1 \times \mathbf{A}_1 \times \mathbf{A}_1$	$a = \dfrac{\alpha a' + \beta}{\gamma a' + \delta}$, $\alpha, \beta, \gamma, \delta \in \mathbf{F}_2 \begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix} \neq 0$
2	$\mathbf{F}_{4,a}$ $a \in \mathbf{k} \setminus (0,1)$	34	8	exists	$\mathbf{A}_3 \times \mathbf{A}_1$	$a = a'$, $a = \dfrac{1}{a'}$
2	Δ_n $n = 1, 2, \ldots$	$2n^2 + n$	$2n - 1$	none	$\mathbf{C}_n$	$n = n'$

Proposition 3.6. *All the contragredient Lie algebras enumerated in Table 1 are finite dimensional and simple. Table 1 indicates their dimensions, the greatest number of the gradation m, the existence of a p-structure, the irreducible component of the group of automorphisms $\mathcal{G}(A)$, and also all cases of isomorphisms of these Lie algebras.*

Proof. The proof of finite dimensionality and the computation of the dimension of the Lie algebras from Table 1 are easy to carry out on the basis of the following obvious considerations. If $G(A) = \tilde{G}(A)/J(A)$ is a contragredient Lie algebra, then, since $g \in \tilde{G}_i$, $i > 1$, and $[g, f_i] = 0$ for all $i \in I$, it follows that $g \in J(A)$, and since there exists an $i \in I$ such that $[g, f_i] \in J(A)$, it follows that $g \in J(A)$. For $i < -1$, the same results may be obtained as for $i > 1$ by exchanging e_i and f_i, which can be done because the automorphism σ exists. Applying this reasoning, we find successively the bases of the spaces $G_{\pm 2}, G_{\pm 3}, \cdots, G_{\pm m}$. At the same time, from these computations we find that the contragredient Lie algebras with Cartan matrices $\mathrm{C}_{2,a}$, $\mathrm{C}_{3,a}$, and $\mathrm{F}_{4,a}$ satisfy condition a) of Lemma 3.4. From this lemma it therefore follows that all the differentiations of the Lie algebras $\mathrm{C}_{2,a}$ and $\mathrm{F}_{4,a}$ are inner and $\mathrm{C}_{3,a}$ is an ideal of codimension 1 in the Lie algebra of differentiations. Therefore, in particular, $\mathrm{C}_{2,a}$ and $\mathrm{F}_{4,a}$ have p-structures. As in the proof of Lemma 3.4, this process of reasoning leads to the conclusion that in the Lie algebra Δ_n the differentiations $(\operatorname{ad} g)^2$, together with the inner differentiations, generate the space of all differentiations.

The fact that the Lie algebras of Table 1 are simple follows from their finite dimensionality and from Lemma 3.3.

We turn to the computation of the groups $\mathcal{G}(A)$ for the Lie algebras of Table 1. For this we shall first find the subalgebra $H(A) = d\phi(\operatorname{Lie} \mathcal{G}(A)) \subset D(A)$. The Lie algebra $H(A)$ is a homogeneous subalgebra with respect to the weight decomposition of $D(A)$ under the action of the torus $\mathfrak{T}(A)$, and $D(A) = \bigoplus_{\alpha \in X} \tilde{G}_\alpha$, where, by virtue of the existence of the automorphism σ, the subalgebra $H(A)$ contains, together with $\tilde{G}_\alpha$, $\tilde{G}_{-\alpha}$. For the sake of brevity, we shall call such a subalgebra symmetric. We have, further, that $H(A) = H_0(A) \oplus H_1(A)$ is a direct sum of spaces which are invariant with respect to $\mathfrak{T}(A)$, where $H_0(A) \cap \operatorname{ad}(G(A)) = 0$ and $H_1(A)$ is a symmetric subalgebra in $G(A)$. We shall prove that $D(A) = H_0(A) \oplus \operatorname{ad}(G(A))$. For $\mathrm{C}_{2,a}$, $\mathrm{F}_{4,a}$, and $\mathrm{C}_{3,a}$ this is obvious, since for the first two $H_0(A) = 0$ and for $\mathrm{C}_{3,a}$, $H_0(A) \subset P(A)$. For Δ_n this follows from the fact that if g is a weight vector and $(\operatorname{ad} g)^2 \neq 0$, then, by Lemma 1.2, $E(t(\operatorname{ad} g)^2)$ is a one-parameter group in $\mathcal{G}(A)$.

Thus, it is only left for us to compute the Lie algebra $H_1(A)$. First, note that $H_1(A) \neq G(A)$, since otherwise all the maximal tori of $G(A)$ would be conjugate to the torus G_0 (Proposition 1.6) and the matrix A would be equivalent to the Cartan matrix of a simple Lie algebra of the classical type, which obviously is not so. Further, for all the Lie algebras of Table 1, we shall construct a group of automorphisms $\mathcal{G}'(A)$ for which the Lie algebra $H'_1(A) = d\phi(\operatorname{Lie} \mathcal{G}'(A))$ is a maximal symmetric subalgebra in $H_1(A)$. By the same token, the Lie algebra $H_1(A) = H'_1(A)$ will be computed.

By Lemma 1.2, the automorphisms for the Lie algabras $C_{2,a}$ will be $E(te_1)$ and $E(t[e_1e_2e_2])$, $t \in k$. The symmetric subalgebra in $C_{2,a}$ containing the elements e_1 and $[e_1e_2e_2]$ is clearly a maximal symmetric subalgebra and is isomorphic to $A_1 \oplus A_1$.

Let

$$A_i(t)e_j = E(te_i)e_j \quad \forall i, j \in I,$$

$$A_i(t)f_j = E(te_i)f_j \quad \forall i \neq j,$$

$$A_i(t)f_i = f_i + th_i + te_i.$$

It can immediately be verified that $A_i(t)$ may be extended to an automorphism of the Lie algebras $C_{3,a}$ and $F_{4,a}$ for $i \neq 2$ and of the Lie algebra Δ_n for $i \neq 1$. It can also immediately be verified that the mapping $E(t[e_1e_2e_3e_2])$, defined on the generators e_i and f_i, $i \in I$, may be extended to an automorphism of the Lie algebras $C_{3,a}$ and $F_{4,a}$, and the mapping $E(t[e_1e_2e_1])$, to an automorphism of Δ_n. In all cases, the symmetric subalgebras in $G(A)$ containing all the above-mentioned elements are maximal symmetric subalgebras isomorphic in the case of $C_{3,a}$ to the Lie algebra $A_1 \oplus A_1 \oplus A_1$, in the case of $F_{4,a}$, to $A_1 \oplus A_3$, and in the case of Δ_n, to D_n.

Therefore the Lie algebra $H(A)$ is computed in the cases $C_{2,a}$, $C_{3,a}$, $F_{4,a}$. Since by Lemma 3.5 the corresponding group $\mathcal{G}(A)$ is reductive, we obtain that it is isomorphic in these cases to those groups which appear in Table 1.

As can now easily be seen, in the case of Δ_n the Lie algebra $H(A)$ is isomorphic to C_n. The reductiveness of the group $\mathcal{G}(A)$ in this case may be proved on the basis of the information obtained concerning the Lie algebra $H(A)$, just as Lemma 3.5. Therefore $\mathcal{G}(A)$ is a group of type C_n.

We now turn to the proof of the fact that the Lie algebras of Table 1 are nonisomorphic. It is clear that the Lie algebras from the different rows of this table are not isomorphic and also that Δ_n and Δ'_n are isomorphic only for $n = n'$. Furthermore, by Lemma 3.2, if the Lie algebras $G(\tilde{A})$ and $G(A)$ from the same row of Table 1 are isomorphic, then there exists an isomorphism Φ which takes any weight subspace in $G(\tilde{A})$ with respect to $\mathcal{T}(\tilde{A})$ into a weight subspace in $G(A)$ with respect to $\mathcal{T}(A)$. Suppose that $\Phi(\tilde{e}_i) \subset G_{\tilde{\alpha}_i} \subset G(A)$. The weights $\tilde{\alpha}_i$, $i \in I$, generate a basis over $\mathbf{Q}$ in the group of characters of the torus $\mathcal{T}(A)$ and $\tilde{\alpha}_i - \tilde{\alpha}_j$ is not a weight for $i \neq j$. Such a system of weights is called a system of simple roots. Suppose that $e_{\pm\tilde{\alpha}_i} \in G_{\pm\tilde{\alpha}_i}$ and $h_{\tilde{\alpha}_i} = [\tilde{e}_{\alpha_i}, \tilde{e}_{-\alpha_i}]$. Clearly the matrix $(\tilde{\alpha}_j(h_{\tilde{\alpha}_i}))$ is equivalent to $\tilde{A}$. The matrix $\tilde{A}$ will be called the Cartan matrix of the system of simple roots $\tilde{\alpha}_i$. Note that if the systems of simple roots are conjugate with respect to the group $\mathcal{G}(A)$, then their Cartan matrices are equivalent.

Therefore $G(A)$ and $G(\tilde{A})$ are isomorphic if and only if $G(A)$ contains a system of simple roots with Cartan matrix $\tilde{A}$. For the Lie algebra $C_{2,a}$ there exists only one system of simple roots which is not conjugate to the standard system α_1, α_2, and this is the system of roots α_2, $-(\alpha_1 + 2\alpha_2)$, corresponding to the Cartan matrix $\mathbf{C}_{2,-a-1}$. For the Lie algebra $\mathbf{C}_{3,a}$ there exist, besides the standard system, three pairwise nonconjugate systems of simple roots with Cartan matrices

$$\mathbf{C}_{3,a+1}, \mathbf{C}_{3,a-1} \text{ and } \begin{bmatrix} 0 & 1 & a \\ 1 & 0 & a+1 \\ a & a+1 & 0 \end{bmatrix}.$$

Finally, for the Lie algebra $\mathbf{F}_{4,a}$ there exist, besides the standard system, four pairwise nonconjugate systems of simple roots with Cartan matrices

$$\mathbf{F}_{4,a-1}, \quad \begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & a+1 & 0 \\ 0 & a+1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \widetilde{\mathbf{F}}_a = \begin{bmatrix} 0 & 1 & a+1 & 0 \\ 1 & 0 & a & 0 \\ a+1 & a & 0 & a \\ 0 & 0 & a & 0 \end{bmatrix}, \quad \widetilde{\mathbf{F}}_{a^{-1}}.$$

The proof of the proposition is complete.

Theorem 3.7. *For characteristics 3 and 2, all the finite-dimensional simple contragredient Lie algebras are exhausted by the simple Lie algebras of the classical type and by the Lie algebras enumerated in Table 1.*

We shall first prove Theorem 3.7 for characteristic 3.

Lemma 3.8. *Suppose that the matrix A satisfies condition* (M) *and let* $(\ ,\ )$ *be an invariant bilinear form on $G(A)$. Then, if α is a weight of $G(A)$ with respect to $\mathfrak{I}(A)$ and $(\alpha, \alpha) \neq 0$. then $2\alpha + 3\beta$ is not a weight for any $\beta \in X$ and* $\dim G_\alpha = 1$.

Proof. Exactly as in Proposition 24 of [3], it can be proved that if A is a matrix of order ≥ 3 in characteristic 3 all the elements of which are equal to 2, then $\dim G(A) = \infty$.

Since $G_{(\alpha)}$ and $G_{(-\alpha)}$ are dual with respect to the form $(\ ,\ )$, we have that $\dim G_{(\alpha)} \leq 2$. If $\dim G_\alpha = 2$, then $[G_\alpha, G_\alpha] \neq 0$, and consequently $G_{2\alpha} \neq 0$; therefore $\dim G_{(\alpha)} > 2$. Thus $\dim G_\alpha = 1$. We shall now prove that $G_{2\alpha+3\beta} = 0$.

If this is not so, then, reasoning as in Lemma 19 of [3], we find that the contragredient Lie algebra with Cartan matrix $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$ is finite dimensional, which is not so, as shown in §7 of [3]. The lemma is proved.

Lemma 3.9. *If A is a matrix of order 2 over a field of characteristic 3 and $\dim G(A) < \infty$, then either A has a zero row, or A is equivalent to the matrix E, or A is equivalent to the matrix $\mathbf{C}_{2,a}$, where $a \in \mathbf{k} \cup \infty$.*

Proof. We must show that if $A = \begin{pmatrix} 2 & b \\ a & 2 \end{pmatrix}$, then either $a = -1$ or $b = -1$. If $A = \begin{pmatrix} 2 & a \\ -1 & 0 \end{pmatrix}$, then $a = -1$, and the case $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is impossible. If $a \neq 0$ and $b \neq 0$, then in all three cases the conditions of Lemma 3.8 are satisfied. It is easy to show that α_1 and $2\alpha_1 + 3\alpha_2$ are roots and $(\alpha_1, \alpha_1) \neq 0$, if, in the first case, a and b are different from -1 and, in the second, $a \neq -1$ which, by Lemma 3.8, implies that these Lie algebras are infinite dimensional. The third case drops out, since then $\alpha_1 + \alpha_2$ and $2(\alpha_1 + \alpha_2)$ are roots and $(\alpha_1 + \alpha_2, \alpha_1 + \alpha_2) \neq 0$. Therefore it remains to show that the cases $A = \begin{pmatrix} 2 & 0 \\ a & 2 \end{pmatrix}$, $a \neq 0$, and $A = \begin{pmatrix} 2 & 0 \\ -1 & 0 \end{pmatrix}$ are impossible. In these cases, in the system of roots $(\alpha_1 + 2\alpha_2, \alpha_2 + 2\alpha_1)$ the Cartan matrix is equal to $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$, and consequently the corresponding contragredient Lie algebras are infinite dimensional.

Lemma 3.10. *If A is a matrix of order 3 over a field of characteristic 3 satisfying condition* (m) *of Lemma* 3.3, *then it is equivalent to the Cartan matrix of the simple Lie algebra of the classical type* $\mathbf{A}_3$, $\mathbf{B}_3$, *or* $\mathbf{C}_3$.

Proof. Note that every principal submatrix of order 2 in the matrix A must be one of those enumerated in Lemma 3.9. The Cartan matrices of the systems of roots

$$(\alpha_{i_1}, \alpha_{i_2} + \alpha_{i_3}),\ (\alpha_{i_1} + \alpha_{i_2}, \alpha_{i_3} + \alpha_{i_3}) \text{ and } (\alpha_{i_1} + 2\alpha_{i_2}, \alpha_{i_3}),$$

where (i_1, i_2, i_3) is some permutation, also must be matrices enumerated in Lemma 3.9. By direct computation it can be verified that these conditions and condition (m) are satisfied only by the matrices $\mathbf{A}_3$, $\mathbf{B}_3$, $\mathbf{C}_3$ and also by $\tilde{\mathbf{A}}_2$, $\tilde{\mathbf{B}}_2$, $\tilde{\mathbf{C}}_2$ from Table 1 of [3]. But, as has been shown in §7 of [3], the last three Lie algebras are infinite dimensional.

Lemma 3.11. *If A is a matrix over a field of characteristic $p \geq 3$ satisfying condition* (m) *of Lemma* 3.3 *and* $\dim G(A) < \infty$, *then for any set of distinct numbers* $i_1, i_2, \cdots, i_r \in I$, $r > 2$, *the following equation is true*:

$$a_{i_1 i_2} a_{i_2 i_3} \ldots a_{i_{r-1} i_r} a_{i_r i_1} = 0.$$

Proof. We shall prove the lemma by induction on r. For $r = 3$, the lemma is true in accordance with Lemma 3.10. For $r > 3$, consider the system of roots

$$(\alpha_{i_1}, \alpha_{i_2}, \ldots, \alpha_{i_{r-2}}, \alpha_{i_{r-1}} + \alpha_{i_r}).$$

Then, as can easily be seen, if $a_{i_1 i_2} \cdots a_{i_r i_1} \neq 0$, then for the Cartan matrix of this system of roots

$$\tilde{a}_{12} \cdot \tilde{a}_{23} \ldots \tilde{a}_{r-1, 1} \neq 0.$$

The lemma is proved.

Now it is easy to complete the proof of Theorem 3.7 for $p = 3$. Since $G(A)$ is simple, by Lemma 3.3 A satisfies condition (m). For matrices of order 2 and 3, Theorem 2 follows from Lemmas 3.9 and 3.10. Now let A be a matrix of order > 3 over a field of characteristic 3 satisfying condition (m) and suppose that $\dim G(A) < \infty$. From Lemma 3.11, since $a_{ij} = 0$, it clearly follows that $a_{ji} = 0$ for any $i, j \in I$. We shall prove that after the transformation to the equivalent matrix, we have $a_{ii} = 2$, and if $(a_{ij}, a_{ji}) \neq (0, 0)$, then $(a_{ij}, a_{ji}) = (-1, 1)$ or $(-1, -2)$. By condition (m), there exists a k such that $a_{jk} \neq 0$. Taking the principal submatrix of the matrix A corresponding to the indices i, j, k of Lemma 3.9, we obtain the required result.

Suppose that S is a Dynkin scheme of the matrix A. By condition (m), S is connected. By Lemma 3.10, every connected subscheme consisting of three points is one of the Dynkin schemes A_3, B_3, or C_3. By Lemma 3.11, S does not contain any cycles. In accordance with the results of [3], §7, S does not contain any schemes from Tables 1–3 of [3]. As can easily be seen, a Dynkin scheme which has the properties enumerated above may only be a Dynkin scheme of a Lie algebra of the classical type. Theorem 3.7 is proved for $p = 3$.

The proof of Theorem 3.7 for characteristic 2 requires much more involved computations. We give only an outline of the proof.

Lemma 3.12. *If A is a matrix of order 2 over a field of characteristic 2 and $\dim G(A) < \infty$, then either A contains a zero row, or A is equivalent to the matrix E, or A is equivalent to the matrix A_2 or Δ_2.*

Lemma 3.13. *If A is a matrix of order 3 over a field of characteristic 2 having property* (m) *and* $\dim G(A) < \infty$, *then A is equivalent either to A_3 or to Δ_3 or to $\mathrm{C}_{3,a}$, $a \in \mathbf{k}$, or to the matrix*

$$\begin{pmatrix} 0 & a+1 & a \\ a+1 & 0 & 1 \\ a & 1 & 0 \end{pmatrix}$$

for some $a \in \mathbf{k}$.

Lemma 3.14. *If A is a matrix of order 4 over a field of characteristic 2 having property* (m) *and* $\dim G(A) < \infty$, *then A is equivalent either to one of the matrices A_4, D_4, Δ_4, $\mathrm{F}_{4,a}$, $a \in \mathbf{k}$, or to one of the matrices*

$$\begin{bmatrix} 0 & a & 0 & 0 \\ a & 0 & a+1 & 0 \\ 0 & a+1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & a & a+1 & 0 \\ a & 0 & 1 & 0 \\ a+1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

for some $a \in \mathbf{k}$.

Lemma 3.15. *If A is a matrix of order 5 in characteristic 2 satisfying condition* (m) *of Lemma 3.5 and* $\dim G(A) < \infty$, *then $G(A)$ is a simple Lie algebra of the classical type.*

These lemmas are proved on the basis of the same considerations that were employed in Lemmas 3.9 and 3.10. Furthermore, just as in Lemma 3.11, it may be shown that for any set of distinct numbers $i_1, i_2, \cdots, i_r \in I$, $r \geq 4$, it is true that

$$a_{i_1 i_2} \ldots a_{i_{r-1} i_r} a_{i_r i_1} = 0.$$

Theorem 3.7 may now be proved for characteristic 2 just as for characteristic 3, on the basis of Lemmas 3.12–3.15, taking into account the condition that contragredient Lie algebras with the Cartan matrices given in Lemmas 3.13 and 3.14 are isomorphic to the Lie algebras $C_{3,a}$ and $F_{4,a}$, respectively (see the proof of Proposition 3.6).

Remark. It is not difficult to prove Lemma 3.5 for $p > 3$ not using the fact that $G(A)$ satisfies condition a). Therefore Theorem 2.1 and Lemma 3.11 give a new and simple proof of the fact that every simple finite-dimensional contragredient Lie algebra of characteristic $p > 5$ is a Lie algebra of the classical type.

Remark. In A. I. Kostrikin's article [16], a family of Lie 3-algebras $L(\epsilon)$ has been constructed. It is easy to show that $L(\epsilon) = C_{2, 2\epsilon/(1+\epsilon)}$.

§4. Applications to the classification of simple Lie algebras and group schemes

In this section, $\mathbf{k}$ is an algebraically closed field of characteristic $p > 5$.

A. **Definition.** Suppose that L is a Lie algebra without a center. Let $L_0 = \operatorname{ad} L \cap d\phi(\operatorname{Lie} \mathcal{G}(L))$. The algebra L will be called *primitive* if L_0 is a maximal $\mathcal{G}(L)$-invariant subalgebra in L and L_0 does not contain nonzero ideals of L.

Note that L is primitive if the largest reduced subscheme in the scheme of its automorphisms is a maximal subscheme. All the known examples of primitive Lie algebras may be divided into two classes. The first class consists of the simple Lie algebras of the classical type; for these, $L = L_0$. The second class is contained among the graded Lie algebras $G = \bigoplus_{i \in Z} G_i$ of Cartan type w_n, s_n, h_n, cs_n, ch_n and k_n (for their definitions see [4]), and also among the filtered Lie algebras with which they are associated; for these $L_0 \subseteq \bigoplus_{i \geq 0} G_i$. For $L \neq L_0$, following the process in [1], we shall construct in L a noncondensing $\mathcal{G}(L)$-filtration. Suppose that L_{-1} is a minimal $\mathcal{G}(L)$-invariant subspace in L containing L_0 and distinct from L_0. Since L_0 is a maximal $\mathcal{G}(L)$-invariant subalgebra in L, we have $L = L_{-1}^d$ for some d. Let

$$L_{-k} = L_{-1}^k, \quad L_k = \{l \in L_{k-1} : [lL_{-1}] \subset L_{k-1}\}, \quad k \geqslant 1.$$

Then, as is easy to see, $L = L_{-d} \supset \cdots \supset L_{-1} \supset L_0 \supset L_1 \supset \cdots$ is a filtered Lie algebra, where all the subspaces L_i are $\mathcal{G}(L)$-invariant. The constructed filtration will be called a standard filtration of the primitive Lie algebra L.

Theorem 4.1. *Let L be a primitive Lie algebra over $\mathbf{k}$, $L_0 = \operatorname{ad} L \cap d\phi(\operatorname{Lie} \mathcal{G}(L))$. Then either $L = L_0$, and then L is a simple Lie algebra of the classical type, or $L \neq L_0$, and then the graded Lie algebra G, associated with the standard filtration in L,* *is isomorphic to one of the algebras of the Cartan type* w_n, s_n, h_n, cs_n, ch_n *or* k_n. *If* G *is a Lie algebra of Cartan type* w_n, cs_n, ch_n, *or* k_n, *then* $L \simeq G$.

Note at once that if $L = L_0$, then, by virtue of Corollary 2.10, L is a Lie algebra of the classical type. Therefore, in what follows, we shall assume that $L \neq L_0$. Suppose that

$$L = L_{-d} \supset \ldots \supset L_{-1} \supset L_0 \supset L_1 \supset \ldots$$

is a standard filtration in L and let $G = \bigoplus_{i\in \mathbf{Z}} G_i$ be the associated graded Lie algebra. The following properties of G are obvious:

1°. If $x \in G_i$, $i \geq 0$, and $[xG_{-1}] = 0$, then $x = 0$.

2°. $G_{-1}^i = G_{-i}$, $i \geq 0$.

Let $\mathfrak{N}$ be an unipotent radical of the group $\mathcal{G}(L)$. By Corollary 2.10, $L_1' = d\phi(\mathrm{Lie}\,\mathfrak{N}) \cap L_0 \neq 0$. Since the unipotent radical lies in the kernel of the irreducible representation, L_1' lies in the kernel of the representation of L_0 on L_{-1}/L_0. From this, by the definition of a standard filtration, it follows that $L_1' \subset L_1$. In particular, we obtain

3°. $G_1 \neq 0$.

Since the spaces L_i are $\mathcal{G}(L)$-invariant, $\mathcal{G}(L)$ acts by automorphisms on the Lie algebra G, preserving the gradation. Since $[L_1', L_i] \subset [L_1, L_i] \subset L_{i+1}$, $\mathfrak{N}$ acts trivially on G. Letting $\mathcal{H} = \mathcal{G}(L)/\mathfrak{N}$, we obtain the following properties.

4°. There exists a reductive subgroup $\mathcal{H}$ in Aut G such that $\mathcal{H}G_i \subset G_i$.

5°. The $\mathcal{H}$-module G_{-1} is exact and simple.

6°. $\mathrm{ad}(G_0) \subset d\phi(\mathrm{Lie}\,\mathcal{H})$.

To prove Theorem 4.1, we shall need several lemmas. The fundamental lemma is the following slightly modified version of Theorem 3 of [4].

Lemma 4.2. *Let* $G = \bigoplus_{i\in\mathbf{Z}} G_i$ *be a finite-dimensional graded Lie algebra satisfying conditions* 1°–6°. *Then if the* G_0*-module* G_1 *is exact, there exists a homogeneous ideal* $I \subset \bigoplus_{i<-1} G_i$ *such that* G/I *is either a graded Lie algebra of Cartan type* w_n, s_n, h_n, cs_n, ch_n, *or* k_n, *or a simple Lie algebra of the classical type with one of the standard gradations.*

Proof. Let I be a maximal homogeneous ideal lying in $\bigoplus_{i\leq -1} G_i$. Let $\overline{G} = G/I = \bigoplus \overline{G}_i$. Since the ideal I is obviously $\mathcal{H}$-invariant, $\mathcal{H}$ acts by automorphisms on $\overline{G}$, where all the properties 1°–6° are preserved for $\overline{G}$. By virtue of 4°, $[\mathrm{Lie}\,\mathcal{H}, \overline{G}_i] \subset \overline{G}_i$. Letting $G_0' = \mathrm{Lie}\,\mathcal{H}$ and $G_i' = G_i$ for $i \neq 0$, in the space $G' = \bigoplus_{i\in\mathbf{Z}} G_i'$ we obtain in the natural manner, by virtue of 6°, the structure of a graded Lie algebra containing $\overline{G}$ as a homogeneous ideal. Clearly the Lie algebra G' also satisfies all the conditions 1°–6°, where equality holds in 6°.

By virtue of 1° and because G_1' is an exact G_0'-module, the graded Lie algebra G' is transitive (using the terminology of [4]). By 6°, G_0' is a Lie algebra of the classical type.(1) In order to make use of Theorem 3 of [4], it remains for us to prove that the G'_0-module G'_{-1} is simple and is a p-representation.

Let $\Lambda = \Sigma k_i\lambda_i$ ($k_i \in \mathbf{Z}$; the λ_i are fundamental weights) be the highest weight of the $\mathcal{H}$-module G'_{-1} (see condition 5°) and let $\mathrm{M} = \Sigma m_i\lambda_i$ be the lowest weight of any simple submodule G''_1 of the $\mathcal{H}$-module G'_1. Let $\overline{\Lambda} = \Sigma \overline{k}_i\lambda_i$ and $\overline{\mathrm{M}} = \Sigma\overline{m}_i\lambda_i$, where $\overline{k}_i$ and $\overline{m}_i$ are images of k_i and m_i under the homomorphism $\mathbf{Z} \to \mathbf{F}_p$. From the results of [15] it follows that the G'_0-module G'_{-1} (analogously G''_1) is isomorphic to the tensor product of the simple G'_0-module with the highest weight $\overline{\Lambda}$ and the trivial module, where G'_{-1} is simple if and only if $0 \le k_i < p$ for all i.

Let V_Λ be the space of all the highest weight vectors of the G'_0-module G'_{-1}. By 1°, for any $x \in V_\Lambda$ and $y \in V_\mathrm{M}$, we have $[x, y] = B(x, y)\, e_{-\alpha}$, where $\alpha = -(\Lambda + \mathrm{M})$ is a root of the group $\mathcal{H}$, and where the bilinear form $B(x, y)$ brings about a nonsingular coupling of the space V_M with some subspace V'_Λ of the space V_Λ.

We shall show that $\dim V_\mathrm{M} = 1$. If this is not so, then we shall take two linearly independent vectors $y_1, y_2 \in V_\mathrm{M}$ and their dual vectors $x_1, x_2 \in V'_\Lambda$ with respect to the bilinear form $B(x, y)$. We shall consider two cases. First, suppose that $\alpha \neq 0$; for definiteness take $\alpha > 0$. From the proof of Lemma 4.1 of [4] it follows that $\overline{\Lambda} = \lambda_1$ and $\overline{\mathrm{M}} = -\lambda_1 - \theta$, where $\alpha = \theta$ is the highest root of G'_0. Therefore, in particular, $\overline{\Lambda}(h_\alpha) \neq 0$. Let $e_i = [y_i, e_\alpha]$ and $f_i = x_i$, $i = 1, 2$. By Lemma 2.1 of [4] these elements generate an infinite-dimensional algebra, which is impossible. If $\alpha = 0$, then from Lemma 2.4 of [4] it follows that $[e_{-\alpha}x_i] \neq 0$ and the elements $e_i = y_i$ and $f_i = x_i$, $i = 1, 2$, once again generate an infinite-dimensional algebra. Thus $\dim V_\mathrm{M} = 1$, and therefore $0 \le m_i < p$ for all i. If $\alpha = 0$, then from this it follows that $0 \le k_i < p$ and, therefore, $\dim V_\Lambda = 1$. If $\alpha \neq 0$, then $\mathrm{M} = -\lambda_1 - \theta$, $\Lambda = -\mathrm{M} - \theta = \lambda_1$ and again $\dim V_\Lambda = 1$. From this it follows that the G'_0-module G'_{-1} is simple and is a p-representation.

Thus the Lie algebra G' satisfies all the conditions of Theorem 3 from [4], and consequently G', and also $\overline{G}$, are Lie algebras of one of the types enumerated in the formulation of the lemma.

At the same time, we have proved that the G_0-module G_{-1} is simple, from which it follows that $I \subset \bigoplus_{i<-1} G_i$. The proof of the lemma is completed.

Lemma 4.3. *Let* $G = \bigoplus_{i\in\mathbf{Z}} G_i$ *be a finite-dimensional graded Lie algebra for which conditions 1°–6° are satisfied. If the* G_0*-module* G_1 *is not exact, then* $G_1 = 0$ *for* $i > 1$.

Proof. Assume the contrary; that is, suppose that $G_2 \neq 0$. Let E_{M_2} be the lowest weighting vector of any simple submodule of the G_0-module G_2. Let $G_0 = G_0^{(1)} \oplus G_0^{(2)}$, where $G_0^{(2)}$ is the kernel of the G_0-module G_1. There exists a highest weighting vector F_Λ of the G_0-module G_{-1} for which $[F_\Lambda E_{\mathrm{M}_2}] = x_\lambda$ is a nonzero weighting vector of G_1.

(1) To avoid misunderstanding, we point out that the terminology here differs from that in [4]: in [4], a Lie algebra of the classical type is a Lie algebra of a simple (and not reductive) group.

There exists a sequence of root vectors $e_{i_1}, \cdots, e_{i_r}$ corresponding to the simple roots of the Lie algebra $G_0^{(1)}$ for which the vector $E_{M_1} = [x_\lambda e_{i_1} \cdots e_{i_r}]$ is the lowest weighting vector of the G_0-module G_1. Letting $x_\mu = [E_{M_2} e_{i_1} \cdots e_{i_r}]$, we obviously have $[F_\Lambda x_\mu] = E_{M_1}$. Note that $[x_\mu f_j] = 0$ if $f_j = e_{-\alpha_j}$, where α_j is a simple root of $G_0^{(2)}$. As in Lemma 4.1 of [4], we obtain that $[F_\Lambda E_{M_1}] = e_\theta$, where θ is the highest root of $G_0^{(2)}$. Let $f_1 = e_{-\alpha_1}$, where α_1 is a simple root of $G_0^{(2)}$ for which $\theta - \alpha_1$ is a root. We have $[[f_1 F_\Lambda F_\Lambda] x_\mu] = [[f_1 F_\Lambda] E_{M_1}] = [f_1 e_\theta] = 0$. Therefore, since clearly $[f_1 F_\Lambda F_\Lambda]$ is the highest weighting vector of G_2, we have $[[f_1 F_\Lambda F_\Lambda], E_{M_2}] \neq 0$. As can easily be seen, in view of Lemma 2.4 and Theorem 2 of [3] this contradicts the finite-dimensionality of G. The lemma is proved.

Lemma 4.4. *Let $G = \bigoplus_{i \in \mathbf{Z}} G_i$ be the Lie algebra of Cartan type $\mathbf{w}_n$, $\mathbf{cs}_n$, $\mathbf{ch}_n$, $\mathbf{k}_n$ or a Lie algebra of the classical type with one of the standard gradations; then $I = 0$ and $G \simeq L$.*

Proof. Let $\mathcal{T}$ be a maximal torus in $\mathcal{G}(L)$. It is clear that the $\mathcal{T}$-modules G and L are isomorphic. In all the cases enumerated in the lemma, with the exception of $\mathbf{A}_{lp-1}$, there exists a one-dimensional subtorus $\mathcal{T}_0$ in $\mathcal{T}$ acting trivially on G_0. Then $\mathcal{T}_0$ acts as a scalar on G_i, and here the character of $\mathcal{T}_0$ on G_k is equal to $k\lambda$, where λ is the character of $\mathcal{T}_0$ on G_1. These characters of the torus $\mathcal{T}_0$ are also realized on L. Let $\mathfrak{G}_k = \{l \in L : t(l) = \lambda^k(t) l \ \forall\, t \in \mathcal{T}_0\}$. In the case of $\mathbf{A}_{lp-1}$, denote by $\mathfrak{G}_k$ the sum of the weighting spaces in L corresponding to those same weights as the weighting spaces from G_k. In this way we transform L into a graded Lie algebra which is clearly isomorphic to G. Since $I + L_0$ is a proper subalgebra of L containing L_0, it follows, since L_0 is maximal, that $I = 0$. The lemma is proved.

Lemma 4.5. *Let $G = \bigoplus_{i \in \mathbf{Z}} G_i$ be the Lie algebra of Theorem 4.1. If $\overline{G} = G/I$ is a Lie algebra of Cartan type $\mathbf{s}_n$ or $\mathbf{h}_n$, then $I = 0$.*

Proof. In all the cases enumerated in the lemma, $\overline{G}_{-2} = 0$. Therefore by Lemma 4.2 we have $I = \bigoplus_{i \leq -2} G_i$. In particular, $[G_{-2}, G_1] = 0$. This means that $[L_{-2} L_1] \subset L_0$. Assume that $L_{-2} \neq 0$. Consider the $\mathcal{G}(L)$-module L_{-2}/L_0. Let $\omega : L_{-2} \to L_{-2}/L_0$ be the natural projection. Then $\omega(L_{-1})$ is a simple $\mathcal{H}$-submodule in L_{-2}/L_0 isomorphic to the $\mathcal{H}$-module G_{-1}. Since $L_{-2} = [L_{-1} L_{-1}]$, all the remaining factors of L_{-2}/L_0 are contained in the $\mathcal{H}$-module $G_{-1} \wedge G_{-1}$ (the surface square). But the $\mathcal{H}$-module G_{-1} is a simpler module of the group $\mathbf{A}_n$ or $\mathbf{C}_n$, and therefore $G_{-1} \wedge G_{-1}$ does not contain a submodule isomorphic to G_{-1}. From this it follows that the $\mathcal{G}(L)$-module L_{-2}/L_0 may be decomposed into a direct sum of the module $\omega(L_{-1})$ and some module M. Take in M the simple $\mathcal{G}(L)$-submodule M_1. It is an $\mathcal{H}$-module which is not isomorphic to the $\mathcal{H}$-module G_{-1}. Let $L'_{-1} = \omega^{-1}(M_1)$.

Since $[L_{-2} L_1] \subset L_0$, we have $[L'_{-1} L_1] \subset L_0$. Therefore, if we construct a standard filtration over L'_{-1}, then for the graded Lie algebra $G' = \bigoplus G'_i$ associated with it, the G_0-modules G_i and G'_i for $i \geq 1$ will be isomorphic, and G_{-1} and G'_{-1} will be nonisomorphic, where $G'_{-1} \subset G_{-1} \wedge G_{-1}$. This obviously contradicts Lemma 4.2.

Proof of Theorem 4.1. *We may assume that $L \neq L_0$ or, equivalently, that L is not* a Lie algebra of the classical type. Let $L = L_{-d} \supset \cdots \supset L_{-1} \supset L_0 \supset \cdots$ be a primitive Lie algebra with a standard filtration. As has been shown, the associated graded Lie algebra *G satisfies conditions 1°–6°.*

In order to apply Lemma 4.2, we must still prove that G_1 is an exact G_0-module. Assume the contrary. Then the $\mathcal{H}$-module G_1 is not exact. Since $L_2 = 0$ by Lemma 4.3, the $\mathcal{H}$-module L_1 is not exact either. But this contradicts Corollary 2.10.

Thus by Lemma 4.2 there exists a homogeneous ideal $I \subset \bigoplus_{i<-1} G_i$ for which $\overline{G} = G/I$ is a Lie algebra of the Cartan or classical type.

By Lemmas 4.4 and 4.5 we have $I = 0$, i.e. $\overline{G} = G$. By Lemma 4.4, the cases for which G is a Lie algebra of the classical type drop out. By the same lemma, $G \simeq L$ for the Lie algebras of types $\mathbf{w}_n$, $\mathbf{cs}_n$, $\mathbf{ch}_n$, and $\mathbf{k}_n$. The theorem is proved.

B. In the remaining part of this article we shall study the group schemes over the field $\mathbf{k}$.

Let us recall some properties of affine group schemes [7]. In what follows, the adjective "affine" is omitted (but understood). If $\mathcal{G}$ is a group scheme, then $\mathcal{G}$ contains a (unique) largest reduced subgroup $\mathcal{G}_{\mathrm{red}}$. The group schemes $\mathcal{G}$ and $\mathcal{G}_{\mathrm{red}}$ are either both reducible or both irreducible. To every group scheme we make correspond its Lie algebra Lie $\mathcal{G}$, which is a p-algebra. Conversely, if G is any Lie p-algebra, then there exists a unique group scheme $\mathcal{G}_G$ such that Lie $\mathcal{G}_G = G$ and $\mathcal{G}_G$ is annihilated by the Frobenius endomorphism. If Φ is a Frobenius endomorphism of the group scheme $\mathcal{G}$ with $\mathcal{G}_\Phi$ its kernel, then Lie $\mathcal{G}_\Phi =$ Lie $\mathcal{G}$. If $\mathcal{H}$ is a normal divisor of the group scheme $\mathcal{G}$, then Lie $\mathcal{H}$ is an ideal in Lie $\mathcal{G}$. Conversely, if H is a $\mathcal{G}$-invariant ideal in Lie $\mathcal{G}$ (in the sense of a p-algebra), then the subgroup $\mathcal{G}_H$ of the group $\mathcal{G}_\Phi$ is the kernel of some suitable purely nonseparable isogeny (which may be defined as the composition of the homomorphism Φ and some group homomorphism which may be annihilated by Φ). By the definition of a purely nonseparable isogeny, its kernel is contained in the kernel of some degree of the Frobenius endomorphism. In particular, $(\mathrm{Ker}\,\omega)_{\mathrm{red}} = 1$.

Below we employ the terms "group" and "group scheme" in one and the same sense.

Definition. The group scheme $\mathcal{G}$ will be called *simple* if every normal divisor in $\mathcal{G}$ lying in $\mathcal{G}_{\mathrm{red}}$ is the kernel of a purely nonseparable isogeny.

Applying the same reasoning as in the proof of Lemma 3.4, we can show that the algebra of differentiations of the Lie p-algebra L of the Cartan type is a Lie p-algebra of the same Cartan type. In particular, from this it follows that these algebras of differentiations do not contain ideals lying in $d\phi(\mathrm{Lie}\ \mathcal{G}(L))$. Therefore every normal divisor in Aut L lying in $\mathcal{G}(L)$ is the kernel of a purely nonseparable isogeny. Thus the scheme of automorphisms of every Lie p-algebra of the Cartan type is a simple group scheme.

Lemma 4.6. *Let $\mathcal{G}$ be a group scheme, let $\mathcal{H} = \mathcal{G}_{\mathrm{red}}$, let G and H be Lie algebras of the groups $\mathcal{G}$ and $\mathcal{H}$, and let H' be a subalgebra in G containing H and invariant with respect to $\mathcal{H}$. Then $\mathcal{G}$ contains a subgroup $\mathcal{H}'$ whose Lie algebra is H' and which* *contains* $\mathcal{H}$.

Proof. Let $\mathcal{H}' = \mathcal{G}_{H'} \subset \mathcal{G}_{\Phi}$. The subgroup of $\mathcal{G}$ generated by $\mathcal{H}'$ and $\mathcal{H}$ is the required group.

The following theorem is the fundamental result of this section.

Theorem 4.7. *Let $\mathcal{G}$ be a simple group scheme without a center, and let $\mathcal{H} = \mathcal{G}_{red}$, $L = \mathrm{Lie}\,\mathcal{G}$, and $L_0 = \mathrm{Lie}\,\mathcal{H}$. Assume that $\mathcal{H}$ is a maximal group subscheme in $\mathcal{G}$ and that $p > 5$. Then the following possibilities exist: either L is a Lie algebra of the classical type, or L is a primitive Lie p-algebra and the graded Lie algebra G associated with its standard filtration is isomorphic to one of the Lie p-algebras of Cartan type w_n, s_n, h_n, cs_n, ch_n, or k_n.*

Before proving the theorem, we shall state some of its corollaries.

Corollary 4.8. *Suppose that $\mathcal{G}$ and $\mathcal{H}$ are the same as in Theorem 4.7, and let $\mathcal{N}$ be a unipotent radical in $\mathcal{H}$. If $\mathcal{G}$ is not a smooth group, then $\mathcal{H}/\mathcal{N}$ is isomorphic to one of the groups $GL(n)$, $SL(n)$, $Sp(n)$, or $CSp(n)$.*

Corollary 4.9. *In the notation of Theorem 4.7, if G is a Lie algebra of Cartan type w_n or k_n, $n \not\equiv -3 \pmod p$, then $\mathcal{G}$ is a scheme of automorphisms of the Lie p-algebra $\mathbf{W}_n$ or $\mathbf{K}_n$, respectively.*

Proof. Since G is a Lie p-algebra, it is isomorphic to $\mathbf{W}_n$ or $\mathbf{K}_n$. From Theorem 4.1 it follows that then L is also a Lie algebra $\mathbf{W}_n$ or $\mathbf{K}_n$ ($n \not\equiv -3 \pmod p$). But all the differentiations of these Lie algebras are inner ([10] and [8]). Therefore Lie (Aut L) = L, i.e. $\mathcal{G}$ = Aut L, as required.

Proof of Theorem 4.7. If C is the center of the Lie algebra L, then $\mathcal{G}_C$ is a central subgroup in $\mathcal{G}$. Therefore $C = 0$. Furthermore, by Lemma 4.6, L_0 is a maximal $\mathcal{H}$-invariant subalgebra in L. If ad $L \subset \mathrm{Lie}\,\mathcal{G}(L)$, then, by Corollary 2.10 (a), L is a Lie algebra of the classical type. Therefore we shall assume that $L \neq L_0$ and that L is not a Lie algebra of the classical type. Note that then, by Corollary 2.10, $Z_{L_6}(N) \subset N$, where $N = \mathrm{Lie}\,\mathcal{N}' \cap L_0$ and $\mathcal{N}'$ is a unipotent radical of $\mathcal{G}(L)$.

It remains for us to show that L_0 does not contain ideals of L. Suppose that I is the largest such ideal, $I \neq 0$. Then, by virtue of what has been said above, $I \cap N \neq 0$. Let Z be the center of the Lie algebra $I \cap N$. Clearly $Z \neq 0$. Let $g \in Z$ and $x \in L$. Then $(\mathrm{ad}\, g)\, x \in I \subset L_0$, $(\mathrm{ad}\, G)^2\, x \in I \subset N$, and $(\mathrm{ad}\, g)^3\, x = 0$. Thus $(\mathrm{ad}\, g)^3 = 0$ for any $g \in Z$. Therefore, by Lemma 1.2, for $p > 3$ and $g \in Z$ we have $E(g) \in \mathcal{G}$. Denote by $\mathcal{M}_1$ the subgroup in $\mathcal{G}$ consisting of the automorphisms of $E(g)$, where $g \in Z$. Let $\mathcal{M}$ be the subgroup in $\mathcal{G}$ generated by the subgroups $\mathcal{G}_1$ and $\mathcal{M}_1$. We shall prove that $\mathcal{M}$ is a normal divisor in $\mathcal{G}$. As a matter of fact, the group $\mathcal{M}$ is clearly invariant with respect to $\mathcal{H}$ and $\mathcal{G}_{\Phi}$, which, because of the maximality of $\mathcal{H}$, generate $\mathcal{G}$. Since $\mathcal{M}_{red} \supset \mathcal{M}_1 \neq 0$, we have reached a contradiction to the conditions on $\mathcal{G}$. The theorem is proved.

The proof of Theorem 4.7 may be modified to prove the following assertion.

Proposition 4.10. *Let $\mathfrak{G}$ be a group scheme without a center which does not have any reduced normal divisors, and let $p > 5$. Suppose that $\mathfrak{H}$ is a subgroup of $\mathfrak{G}$ satisfying the conditions of Theorem 4.7. Then the conclusions of Theorem 4.7 are true for $\mathfrak{G}$.*

Proof. We shall show that the ideal I (see the proof of Theorem 4.7) may be chosen so that $I = \mathrm{Lie}\,\mathfrak{I}$, where $\mathfrak{I}$ is a reduced normal divisor in $\mathfrak{H}$. Then clearly $\mathfrak{I}$ is a normal divisor in $\mathfrak{G}_\Phi \cdot \mathfrak{H}$, from which it follows that $\mathfrak{I}$ is a normal divisor in $\mathfrak{G}$.

Lemma 4.11. *L does not have nil ideals.*

Proof. If R is a nil ideal in L, then $[R, R]$ is also a nil ideal, and consequently we may assume that $[R, R] = 0$. Then the group $\{E(g): g \in R\} \subset \mathfrak{G}$ is the required normal divisor (Lemma 1.2).

Lemma 4.12. *Let $L_1 = \{x \in L_0: \mathrm{ad}\, x|_{L/L_0} = 0\}$. Then $L_1 \subset N$.*

Proof. Let $\mathfrak{H}_1$ be the kernel of the representation of the group $\mathfrak{H}$ on L/L_0. Suppose that $\widetilde{\mathfrak{H}}$ is the normal divisor in $\mathfrak{H}$ generated by all the tori from $\mathfrak{H}_1$. If $\mathfrak{T}$ is a torus from $\widetilde{\mathfrak{H}}$, then $\mathfrak{T}$ acts trivially on $L/\mathrm{Lie}\,\widetilde{\mathfrak{H}}$ ($\mathfrak{T}$ acts trivially on L/L_0, since $\mathfrak{T} \subset \mathfrak{H}_1$; $\mathfrak{T}$ acts trivially on $L_0/\mathrm{Lie}\,\widetilde{\mathfrak{H}}$, since $\widetilde{\mathfrak{H}}$ is a normal divisor in $\mathfrak{H}$). Therefore $\widetilde{\mathfrak{H}}$ acts trivially on $L/\mathrm{Lie}\,\widetilde{\mathfrak{H}}$, and consequently $\mathrm{Lie}\,\widetilde{\mathfrak{H}}$ is an ideal in L. Thus, if $\widetilde{\mathfrak{H}} \neq \{1\}$, we reach a contradiction. This means that $\widetilde{\mathfrak{H}} = \{1\}$, i.e. $\mathfrak{H}_1$ does not contain any tori.

Let I be a maximal ideal of the Lie algebra L lying in L_0. The Lie algebra L/I clearly satisfies all the conditions of Theorem 4.1. From what has been proved above it follows that the factor groups by the unipotent radicals of the groups $\mathfrak{G}(L)$ and $\mathfrak{G}(L/I)$ are isomorphic. The $\mathfrak{G}(L)$- and the $\mathfrak{G}(L/I)$-modules L/L_0 clearly have isomorphic composition series. From Theorem 4.1 it now follows that $L_1 \subset N$. The lemma is proved.

Since $\mathrm{ad}\, I|_{L/L_0} = 0$, by Lemma 4.12 we have that $I \subset N$. Therefore I is a nil ideal in L, which is impossible in view of Lemma 4.11. Proposition 4.10 is proved.

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