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# ABSTRACT HOMOMORPHISMS OF ALGEBRAIC GROUPS: PROBLEMS AND BIBLIOGRAPHY

edited by

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A small conference on abstract homomorphisms of algebraic groups (and related topics) was held at The Pennsylvania State University May 28-June 2, 1979. A collection of problems was compiled at the end of the conference and has been circulating privately for a year. Since it is continuing to attract interest, the three of us who organized the conference are publishing this updated version to make it more generally available. The names attached to the problems indicate which participants suggested them, but they are certainly not exclusive claims of originality. The original formulations were preserved to a large extent; the later additions are placed in brackets. At the end is a bibliography of recent, unfamiliar, or herein-cited publications in the area of abstract homomorphisms.

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Besides the three of us, those attending the conference were E. Connors, K. Dennis, J. Faulkner, J. Ferrar, E. Formanek, J. Freeman, A. Hahn, A. Johnson, B. McDonald, O.T. O'Meara, C. Riehm, L. Vaserstein, and R. Ware. We would like to thank them all for their help in compiling and revising the list of problems. We are grateful to the National Science Foundation and to Mathematics Department of Pennsylvania State University for financial support of the conference.

PROBLEMS.

I. (O.T. O'Meara, L. Vaserstein) Let $R$ and $R'$ be rings (or integral domains), let $V, V'$ be projective modules of rank $\geq 3$ over $R$ and $R'$, respectively, and let

$$\alpha\colon (\mathrm{End}_R V)^* \to (\mathrm{End}_{R'} V')^*$$

be a group isomorphism. Are the rings $R$ and $R'$ Morita-equivalent? Recall: two rings are Morita-equivalent if there are projective modules $M, M'$ such that $\mathrm{End}_R M \cong \mathrm{End}_{R'} M'$. For example

(i) If $R, R'$ are commutative domains and $V, V'$ are free, then $R \cong R'$ (cf. O.T. O'Meara, [1974**a**].

(ii) If $R = \mathrm{Mat}_n R'$, then we have an isomorphism $(\mathrm{Mat}_m R)^* = GL_m R \to (\mathrm{Mat}_{mn} R')^* = GL_{mn} R'$.

(iii) In the example of O'Meara [1977c, pp. 139-141] the rings $\mathcal{O}$ and $\mathcal{O}_2$ are Morita-equivalent by construction.

A similar question is raised for other classical groups. [A. Hahn [1980 + f] has obtained some results on this problem. First, the problem has to be restated to permit for Morita equivalence of $R$ to $R'$ or to the opposite of $R'$. Next, if $R = R_1 \oplus R_2$ and $R' = R_1' \oplus R_1'$ one can get Morita equivalence of $R_1$ to $R_1'$ and of $R_2$ to the opposite of $R_2'$. Hahn has shown that then the answer is positive if $V$ and $V'$ are free of rank $\geq 5$ and $R$ and $R'$ are Ore domains or semi-simple rings, or maximal orders over Dedekind domains. The proof is by reduction to known results of O.T. O'Meara et al. Hahn thinks that the question has a negative answer for general projective modules. ]

II. (A. Hahn, D. James) Study homomorphisms of full subgroups of isotropic (Witt index $\geq 2$) classical groups into other classical groups. Assume perhaps that the image of the subgroup generated by transvections (or Eichler transformations) is full in the target group. In particular, study the homomorphisms of the whole classical group over noncommutative domains with division ring of quotients. Can this be done for more general rings? (See also XVII and XXIII).

III. (L. Vaserstein) Let $G$ be a classical group over a ring $A$ (possibly non-commutative), let $G'$ be a classical group of the same type over a skew field $A'$, and let $\alpha : G(A) \to G(A')$ be a surjective homomorphism. Is it possible that $\mathrm{Ker}(\alpha)$ contains the subgroup generated by elementary matrices?

If the answer is negative it would be possible to prove that $EG(A)$, the group generated by elementary matrices, is characteristic.

IV. (A. Hahn, L. Vaserstein) Study isomorphisms of full subgroups of orthogonal and unitary groups in the case where the Witt index is 1 or 0. In the case of Witt index 0 assume, for example, the subgroup is full of "plane rotations". When the Witt index is 1, one may have to make additional assumptions (e.g. about the existence of "sufficiently many" semisimple elements normalizing unipotent subgroups) as the following example possibly shows:

Let $k = \mathbb{Q}$ and $H = SO_n(f,\mathbb{Z})$ where $f$ is an integral quadratic form such that $f \otimes \mathbb{Q}$ has Witt index 1. Then the subgroup $\widetilde{H}$ of $H$ generated by elementary matrices is a free product of two commutative groups. This implies that $\widetilde{H}$ has non-standard automorphisms. (However, it is not clear that $\widetilde{H}$ is full.)

V. (B. McDonald) Do there exist rings $R$ such that not all automorphisms of $GL_n(R)$ are standard for some $n \geq 3$?

Perhaps some non-dimensional rings $R$ could yield such examples. Recall that a ring $R$ is non-dimensional if some pair of free modules with bases of different (finite) cardinality are isomorphic.

C. Riehm suggests that in this case one consider an isomorphism $\varphi: Mat_n(R) \xrightarrow{\sim} Mat_m(R)$ and pull an outer automrophism

$\sigma_m$ *of* $GL_m(R)$ *to* $GL_n(R)$ (i.e., consider $\varphi^{-1} \circ \sigma_m \circ \varphi$). This might be a non-standard isomorphism of $GL_n(R)$. For $\sigma_m$ to exist one must assume (says L. Vaserstein) that $R$ has an involution.

However, the ring $R$ of transformations which act non-trivially only on finite-dimensional subspaces, or the ring $R$ of matrices with only a finite number of non-zero elements in each row and column, satisfies the above conditions. Nevertheless, all automorphisms are standard by O.T. O'Meara [1977c]. We have, of course, $R \cong Mat_n(R)$ for $n \geq 1$.

VI. (A. Hahn, C. Riehm) To provide a unified setting for the isomorphism theory of classical groups over (possibly non-commutative) integral domains one should consider pseudo-orthogonal groups over algebras $D$ with involution $\sigma$ such that either $D$ is a division algebra or $D = D_1 \oplus D_2$ and $\sigma$ acts as a permutation of $D_1$ and $D_2$.

VII. (C. Riehm) Let $h: V \times V \to R$ be a non-degenerate $(J,\varepsilon)$-hermitian form where $\varepsilon = \pm 1$, $J$ is an involution on $R$, and $R$ is semi-simple (or say a division algebra, or the direct sum of a division algebra with itself).

Let $\sigma \in U(h)$ so that

$$h(\sigma x, \sigma y) = h(x,y) \quad \forall x,y \text{ in } V.$$

Define $\sigma' = 1 - \sigma$, and we have

$$h(\sigma' x, y) + h(x, \sigma' y) = h(\sigma' x, \sigma' y).$$

This shows that $h(\sigma'x,y)$ depends only on $\sigma'y$ and not just on $y$, so we can define $h_\sigma:\sigma'V \times \sigma'V \to R$ by

$$h_\sigma(\sigma'x,\sigma'y) = h(\sigma'x,y). \qquad (*)$$

Then $h_\sigma$ is a J-sesquilinear form on $\sigma'V$, non-degenerate, and $f = h_\sigma$ satisfies (by *)

$$f(u,v) + \varepsilon f(v,u) = h(u,v) \qquad (**).$$

Conversely, if one has a non-degenerate sesquilinear form $f:U \times U \to R$ on a subspace $U$ satisfying (**), then there is a unique $\sigma \in U(h)$ such that $f = h_\sigma$. So we have a bijection

$$U(h) \leftrightarrow \{(U,f) \text{ with above properties}\}.$$

This is a generalization of the Cayley parametrization of the orthogonal group. This parametrization has many formal properties reflecting the group structure of $U(h)$, and C. Riehm feels it should be used in the context of O'Meara's techniques (note that $\sigma'V$ is the residual space of $\sigma$).

Furthermore, it can be generalized easily to pseudo-quadratic forms so that all the "classical" groups in characteristic two can be accommodated as well. Can one hope for a "unified" automorphism theory of the classical groups using this setting?

VIII. (B. McDonald) Suppose $R$ is a commutative ring with zero divisors (e.g. take $R = k \oplus k$, $k$ field). Can the automorphism theory à la O'Meara be formulated here? For example, suppose $R$ is a ring, $T$ a multiplicatively closed subset with no zero-divisors, and $S = T^{-1}R$ a ring having all rank one projectives free (see McDonald, [1976c]). Can the theory of residual spaces be formulated for $S$?

IX. (B. McDonald) It is known that if $R$ is commutative and $n \geqq 3$ then for $GL_n(R)$:

(i) The normal subgroups are standard (Golubchik, Suslin).

(ii) If $\frac{1}{2} \in R$ and modulo an argument on idempotents, the automorphisms are standard (McDonald).

On the other hand, both fail at $n = 2$ for some rings $R$. If $R$ is a ring for which normal subgroups are standard does this imply automorphisms are standard? and conversely? (say for $GL_n(R)$) In what way are the solutions of the two problems related? (See also XVII). [W. Waterhouse [1980d] has removed the restrictions on idempotents in (ii). B. McDonald [1980 + j,k] has shown that for $n = 2$ the automorphisms are standard if the ring is commutative and has many units; in [1980c] he described normal subgroups over such rings.]

X. (E. Connors) Characterize fields $k$ such that $k$ has a quadratic separable extension $K$ with the group $S$ of elements of norm 1 in $K$ isomorphic to $k^*$ (clearly, $k$ is infinite).

XI. (E. Connors, A. Johnson) Find Aut $\Omega_3(V)$ and Aut $\Omega_4(V)$ when $V$ is anisotropic (any characteristic). If $V$ is isotropic then $P\Omega(V)$ is isomorphic to $PSL_3(k)$, $PSL_3(k) \times PSL_3(k)$ or $PSL_3(K)$ where $K$ is a quadratic separable extension of $k$. Thus in the case of isotropic $V$ the answer is known.

XII. (A. Hahn, L. Vaserstein, B. Weisfeiler) Let $H$ be a full subgroup of $GL_n(D)$, $D$ a skew field (resp. other classical groups). Does there exist a ring $A$ with ideal $I$ such that $H \supset E_n(A,I)$ and $AA^{-1} = A^{-1}A = D$ (resp., the corresponding object in other classical groups). The answer is positive if $D = \mathbb{Q}$ $H \subseteq SL_n(\mathbb{Z})$.

More generally, let $G$ be a split algebraic semisimple group defined over $k$. Suppose that $H$ is a dense subgroup of $G(k)$ generated by its intersections with root subgroups. Does there exist a ring $A$, an ideal $I$ and a structure of an A-scheme on $G$ such that $H \supset EG(A,I)$, the subgroup generated by root subgroups with entries in $I$?

If $k$ is finite then the "irreducible" subgroups generated by their intersections with root subgroups are described by W. Kantor [1979b] and A. Wagner [1974d]. Wagner has also determined the subgroups of $PGL(V)$ (over finite fields) generated by "reflections" (= homologies = dilatations).

XIII. (L. Vaserstein) Let $E_n(R)$ denote the subgroup of $GL_n(R)$ generated by $E_{ij}(a)$ (elementary transvections) where $a \in R$. If $R$ is commutative, it was shown by Suslin that $E_n(R)$ is normal in $GL_n(R)$ (cf. B. McDonald [1978d, appendix]). Show $E_n(R)$ is normal in $GL_n(R)$ when $R$ is a noncommutative ring.

XIV. (B. Weisfeiler) Let $D$ be a finite-dimensional central division algebra over a field $k$, $D^1$ the set of elements of reduced norm 1 in $D$, and $\tilde{D}$ the subring of $D$ generated by $D^1$. Do normal subgroups of $D^1$ correspond to ideals of $\tilde{D}$ (congruence subgroup problem)?

[The above question is known to have positive answer over $\mathfrak{p}$-adic fields (C. Riehm [1972a]) and over number fields for quaternions (G. Margulis [1980b]).]

XV. (K. Dennis, W. van der Kallen) Consider $SL_n(\mathbb{Z})$, $n \geqq 3$. Then every $M \in SL_n(\mathbb{Z})$ is a product of elementary matrices. Let $v(M)$ be the minimal number of elementary matrices needed to express $M$.

Is $v(M)$ bounded when $M$ varies over $SL_n(\mathbb{Z})$? The bound may depend on $n$. This is proved modulo the Riemann conjectures by van der Kallen.

It is also known that $SL_n(\mathbb{Z}) = [SL_n(\mathbb{Z}), SL_n(\mathbb{Z})]$. Is every matrix in $SL_n(\mathbb{Z})$ a product of a bounded number of commutators?

C. Riehm suggests the method of J. Williamson's form on the residual space (see VII) may help with the first question.

[The problem was solved positively by D. Carter and G. Keller [1980+b,c].]

XVI. (H. Bass) Consider $\mathbb{Z}[x]$; then $\dim \mathbb{Z}[x] = 2$. Since the quotient fields of $\mathbb{Z}[x]$ are finite it follows that every maximal ideal of $\mathbb{Z}[x]$ contains a cyclotomic polynomial. Let $\varphi_m(x) = X^m - 1$. Let $S$ be the multiplicative system generated by $X$ and all $\varphi_m(x)$ and set $R = S^{-1}\mathbb{Z}[x]$. Because of the above remark, $\dim R = 1$.

Questions: (i) is $SL_n(R) = E_n(R)$?

(ii) is $R$ Euclidean?

(iii) is $R$ of stable rang 1?

[Solved by H.W. Lenstra [1980+i]. The answers are negative. For more background on this problem see [1980+a,d,e].]

XVII. (B. Weisfeiler) A proposed general statement for homomorphisms.

Let $k,k'$ be infinite fields. Let $G$ be an absolutely simple algebraic group defined over $k$, or a projective classical group over a skew field with center $k$; likewise $G'$. Let $H$ be a subgroup of $G(k)$ and $\alpha: H \to G'(k')$ a group homomorphism. Then there exist

(a) a subring $A \subseteq k$,

(b) a structure of a group scheme over $A$ on $G$ (if $G$ is algebraic), or a lattice $M$ in the underlying space (if $G$ is classical) such that $G(A) \supseteq H$,

(c) a $k'$-algebra $K$ (commutative of finite dimension),

(d) a ring homomorphism $\varphi: A \to K$,

(e) a special $k'$-isogeny $\beta: {}^{\varphi}G_A \to G'$ (if $G$ is algebraic), or a $k'$-linear map from ${}^{\varphi}M$ to the underlying space of $G'$,

(f) a homomorphism $\gamma: H \to Z_G{}'(\alpha(H))$, such that

$$\alpha(h) = \gamma(h) \cdot \rho(\varphi(h)) \quad \text{for} \quad h \in H.$$

This statement is false for arbitrary $H,\alpha,G'$. For example, it is false for $SL_2(\mathbb{F}_2[t])$ (I. Reiner, [1957a]). Also we can take $H$ be free (by J. Tits [1972a]) and then we can map $H$ anywhere and in any fashion.

But it is proved for the following cases:

(i) $\alpha$ an isomorphism, $\alpha(H)$ and $H$ full, with some additional conditions on $G$ (J. Dieudonné, O.T. O'Meara et al)

(ii) $G$ algebraic isotropic (cf. XVIII), $H \supseteq G^+$, $G'$ algebraic and $\alpha(H)$ dense (A. Borel, J. Tits [1973a]);

(iii) $k = k' = \mathbb{R}$, $H \supseteq G(\mathbb{R})^\circ$ (J. Tits [1974c]);

[(iv) $\alpha$ an epimorphism, $G$ and $H$ certain subgroups of projective orthogonal groups of equal dimension (A. Dress [1964a]. F. Knüppel [1977b]);]

(v) $k$ real closed (B. Weisfeiler [1979f]);

[(vi) $\alpha$ an "irreducible" homomorphism, $H = G$ and $G'$ projective orthogonal groups and $\dim G \geq \dim G' = 3$ (D. James [1980+g,h])].

In the cases (i), (ii), (iii) $\alpha$ is assumed, explicitly or implicitly, to be a monomorphism.

The statement (if and when proved) implies that the normal subgroups $N$ of $H$ such that $H/N$ can be embedded into a simple algebraic group, or in a classical group, are congruence subgroups. On the other hand it would follow if a description of the normal subgroups of $H$ were discovered and proved to be very nice.

XVIII. (B. Weisfeiler) Let $G$ be an isotropic absolutely almost simple algebraic group over a field $k$ of characteristic zero. Let $G'$ be an algebraic group over a field $k'$. Let $\alpha : G(k) \to G'(k')$ be a group homomorphism. It seems that the method of J. Tits [1974c] goes through and gives the same answer. The reason is that there are subgroups $F_i \simeq \mathbb{G}_a \times \mathbb{G}_m$, $i = 1,\dots,\dim G$, defined over $k$ in $G$ such that the subgroups $G_i = [F_i, F_i] \simeq \mathbb{G}_a$ generate $G$.

It seems also that the same would apply with weakened assumptions on the characteristic of $k$ if one can find in $G$ subgroups $F_i = \mathbb{G}_\alpha \times \mathbb{G}_m$, $i = 1,\ldots,\dim G$, such that (i) $\mathbb{G}_m$ acts on $\mathbb{G}_a$ via a character non-divisible by characteristic, (ii) $F_i$ is defined over $k$, (iii) the groups $G_i = [F_i, F_i]$ generate $G$ separably.

XIX. (B. Weisfeiler) It seems that the proof of A. Borel and J. Tits [1973a] applies in the following situation: $G$ is a classical isotropic group over a skew field $D$ with center $k$ or $G$ is a "twisted" isotropic $B_2, G_2, F_4$ (infinite groups of Suzuki and Ree type). The assumptions on $G'$ should, however, be the same as in Borel-Tits (in order to be able to use Zariski closure).

XX. (B. Weisfeiler) Study homomorphisms between subgroups of exceptional anisotropic groups. Of exceptional interest would be to study groups related to Jordan division algebras.

It seems that the groups of type $D_4$ should be studied first. For that one needs a description of them. The geometry should be the geometry on the set $X$ of two-dimensional tori $T$ such that $Z_G(T)$ is of type $A_2$. Two such tori $T_1, T_2$ are incident if there exists $T_3 \in X$ such that $Z_G D(\langle T_1, T_2\rangle)^\circ = T_3$.

This would be the regular part of the geometry but it would be interesting to complete it by adding unipotent groups. One should understand this geometry and get some kind of F.T.P.G. for it.

For further applications one also needs an association theorem: If $\tilde{T},\overline{T}$ are two maximal tori split over a cubic extension then there exists a sequence of $m = 4$ or $5$ such tori $\tilde{T} = T_o, T_1, \ldots, T_m = \overline{T}$ such that $T_i \cap T_{i+1} \in X$.

It is better to do the whole construction without assuming that we are given a cubic field extension. Assume only that we are given a cubic separable algebra over $k$.

For $F_4$ one needs to study the geometry of subalgebras in the exceptional Jordan algebra. As <u>points</u> one should take 3-dimensional separable "associative" subalgebras, and as <u>lines</u> the 9-dimensional simple "associative" subalgebras. The relation of $F_4$ to $D_4$ in this geometry is similar to the relation of $G_2$ to $A_2$, and this latter was studied in Weisfeiler [1980+n].

For $E_6$ we have a rank 3 geometry. But by the time we are through with $D_4$ and $F_4$ there will be no difficulty in handling $E_6$.

XXI. (B. Weisfeiler) Study the geometries which arise in the homomorphism problem:

(a) The geometry of subalgebras of an algebra. Let $A$ be an algebra over $k$ (which we consider as a functor), $k$ infinite. Let $S_i(A)$ be the set of subalgebras of dimention $i$. Then

$S_i(A)$ is a subset in the Grassmanian $G_{\dim A,i}$. Take its closure $\overline{S}_i(A)$. The subspaces belonging to different irreducible components of $\overline{S}_i(A)$ (probably, they should be cleverly chosen), with incidence relation given by inclusion, give a geometry.

(b) The geometry of subtori of an algebraic semi-simple group $G$. Here again we consider a completion of $G$, say $\overline{G}$, and then we complete the set of subtori in the Chow scheme of $\overline{G}$. Then the irreducible components ofthe completion with inclusion relation give a geometry.

The geometry in (a) has been studied by J. Faulkner [1973b] and by B. Weisfeiler [1980+n], and in a different context by M. Gerstenhaber and R.M. May [1980+p].

In the first case we get a known geometry, a Tits geometry. The second is probably a new example. The geometry in (b) is not known; the only relevant case is D. James and B. Weisfeiler [1980a]. It is easy to understand a similar picture for orthogonal groups. But even for groups of type $A_n$ the full geometry of tori is not known.

XXII. (J. Faulkner, B. Weisfeiler) Study Tits building with homomorphisms (à la W. Klingenberg [1956a]).

One of the possible formulations is this: Let $B$ be a subset of a Tits building $\mathcal{B}$ with distinguished subsets inherited from $\mathcal{B}$. Let $\alpha: B \to \mathcal{B}'$ be an inclusion preserving mapping into another Tits building $\mathcal{B}'$. There should be some conditions of completeness on $B$ (that $B$ "sufficiently" represents $\mathcal{B}$). Then show that there is a local ring $A \subseteqq k$ (where $k$ is associated to $\mathcal{B}$) and a structure of a group (scheme) over $A$ on the group $G$ associated to $\mathcal{B}$ so that $\alpha$ is induced by a composition of a homomorphism $\varphi:A \to k'$ and a homomorphism of algebraic or classical groups $\beta:{}^{\varphi}G_A \to G'$.

Another possible formulation should be an axiomatic characterization of such geometries. We are given some objects with names corresponding to vertices of a Dynkin diagram, and an equivalence relation which induces a homomorphism into a "standard" geometry. Show that the homomorphism is induced as above using a homomorphism of coordinates.

This was done by Klingenberg (loc. cit.) for mappings to projective spaces over skew-fields and appears to have now been done for Moufang projective planes (J. Faulkner, J. Ferrar).

XXIII. (D. James, B. Weisfeiler) Let $M$ be a free A-module, where $A$ is a commutative ring with identity, and $P(M)$ the associated projective space. Thus points (lines) in $P(M)$ are free rank one (two) direct summands of $M$. Similarly take $B,N$ and $P(N)$. A projectivity $\alpha:P(M) \to P(N)$ is a mapping sending points to points and such that for any points $L,L_1,L_2$ in $P(M)$ with $L \subset L_1 + L_2 =$ line, it follows that $\alpha L \subset \alpha L_1 + \alpha L_2$. In D. James and B. Weisfeiler[1980a] a F.T.P.G. is established in this situation when $A$ is a semilocal ring (further assumptions are needed to avoid collapsing of dimension, etc). One obtains a homomorphism $\varphi:A \to B$ and a $\varphi$-semilinear map $\beta:M \to N$ which induce the projectivity $\alpha$. With modifications the proof also works when A is a principal ideal ring and may even go through if A has stable range $\leq 2$ (where, as suggested by C. Riehm, we replace free by projective). What can be established for more general rings, for example, polynomial rings $A = k[X_1,\ldots,X_m]$ with $m \geq 2$? Also, what can be established if we assume $\alpha(\text{line}) \subseteq \text{line}$ (and not $\alpha L \subset \alpha L_1 + \alpha L_2$, as above)? A theorem of this type, with $A = k$, a field, was used in F. Knüppel [1977b], B. Weisfeiler [1979f], D. James [1980+g,h].

[F. Veldkamp [1980+m] has studied this problem too. The paper of W. Stephenson [1969a] is also interesting in this context.]

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