DIFFERENTIAL FORMAL GROUPS OF J. F. RITT

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This paper is an attempt to understand the last four papers of Joseph Fels Ritt.

0.1. Ritt's work on differential formal groups was inspired by a paper of Solomon Bochner (Ann. Math. 47(1946)), in which the latter introduced what are now called formal Lie groups. Ritt published four papers on the subject in short succession—the first was submitted in February of 1949, the last in August of 1950—and also reported on this research at the International Congress of Mathematicians, 1950, in Cambridge, Mass., see Proc. ICM 1950, vol. I, 207-208. Ritt died on January 5, 1951.

From the speed with which the papers were published, and from the variety of questions raised and settled in them, one can judge that Ritt was very enthusiastic about the subject.

0.1.1. In the first paper, Ritt introduces his groups and classifies the "one-dimensional" ones. There are two such groups, each of which can be considered to be defined on the set $\mathfrak{F}$ of "functions" of one variable. The first group has as its operation the addition of functions, and can be construed as a "spread out" additive group $\mathbf{G}_a$, or rather $\mathbf{G}_{a,\mathfrak{F}}$ taken as a group over the ground field. The second group has substitution of functions as its operation. It is called "substitutional" by Ritt, and we will denote it by $\mathbf{G}_s$. This group can be considered to be the formal group associated with the group of diffeomorphisms of the line.

0.1.2. In the second paper, Ritt proves that his notion of group is equivalent to that of a type of Lie algebra. This approach presents a difficulty: the Lie algebra must be given a linearly compact topology. Ritt circumvents this problem by showing that his groups are described by structure constants—that is, by Lie coalgebras, cf. [Nich 2].

0.1.3. In the third paper, Ritt classifies the "two-dimensional" differential formal groups. It follows from his explicit formulas that all two-

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dimensional groups are extensions of one-dimensional ones. The most interesting of the two-dimensional groups have a one-dimensional commutative normal subgroup (which must be $\mathbf{G}_a$), with $\mathbf{G}_s$ being the quotient. The modern approach to the problem of classifying such groups would be

(i) to study one-dimensional differential $\mathbf{G}_s$-modules, and
(ii) to study all possible extensions by $\mathbf{G}_s$ of each one-dimensional $\mathbf{G}_s$-module.

It follows from Ritt's results that each one-dimensional $\mathbf{G}_s$-module is described by an element h from the field k of constants of the differential field. His formulas show (as was pointed out to us by D. Kazhdan) that the action of $\mathbf{G}_s$ on the one-dimensional module M_h indexed by $h \in k$ is the same as the action of the diffeomorphisms of the line on the set of differentials of weight h. The module M_h can be considered to be a highest weight module, with highest weight h.

Ritt's computations imply as well that non-trivial extensions of $\mathbf{G}_s$ by M_h exist only when $h = 0, 1, 2, 5$. It is strange that only these highest weights occur: one would expect that non-trivial extensions exist for $h \in \mathbf{Z}^+$ —i.e., for dominant integral highest weights.

Finally, Ritt computes the group $\operatorname{Ext}(\mathbf{G}_s, M_h)$. The calculation yields that this group is two-dimensional if $h = 0$, and 1-dimensional if $h = 1$, 2, or 5.

0.1.4. The fourth paper studies the questions of subgroups and normal subgroups, and their relative positions. Here Ritt makes his groups into "formal varieties" (following a suggestion of E. Kolchin, as footnoted at the beginning of the second paper), and studies the relations of this notion to the definition of a formal group via Hopf algebras.

0.1.5. As a final historical note, let us note that in 1956 J. Dieudonné began the study of formal groups with a particular emphasis on characteristic p.

0.1.6. In the present paper we study Lie algebras of formal groups. In the special case considered by Ritt, the Lie algebras correspond biunivoquely to formal groups (see sections 3.4, 3.5 below). Our results classify those Lie algebras of formal groups which are "simple" (see Theorems 4.6, 5.2, and 6.3.3 below). Some initial structural results in the "general" case are obtained (see Proposition 4.4.5 and Theorem 4.7.1). Thus we generalize the second paper of Ritt and a small part of his third paper.

0.2. Our work began with the observation that the Lie algebra of the group $\mathbf{G}_s$ is the Witt algebra W_1. This indicates, as do our remarks at the end of 0.1.1, the connection between Ritt's groups and the Lie algebras of Cartan type (or pseudo-groups of transformations). The theory of these was initiated by Sophus Lie himself, further developed by E. Cartan, and then brought into its modern form by V. Guillemin, V. Kac, D. Quillen, I. Singer, and S. Sternberg (cf. [Gui 1, Gui 2, Gui 3, Gui 4, Gui 5, Kac, Sing]). However this theory, at least to our taste, did not feel complete. It handled simple objects readily (cf. [Gui 2]), but was rather awkward when dealing with the non-simple objects (cf. [Gui 1]). We felt that the additional "differential" structure would make the object more tractable, and believe that this has turned out to be the case. However, our proof of the classification theorem follows quite closely that of V. Guillemin [Gui 1, Gui 2].

0.3. Our exposition is quite general, at least in the beginning. The reader wishing to know the motivation behind the different notions presented should consult Appendix II. The basic object is the algebra $K[P]$ of differential operators, where K is a field and P is a Lie ring acting as derivations of K. It turns out that $K[P]$ has a twisted Hopf algebra structure—it is a Hopf K/k-algebra, where k is a subfield of K on which P acts trivially. Our initial constructions involve only this Hopf structure. Given a K/k-Hopf algebra B and B-modules M, N, one can define natural B-module structures on $M \otimes_K N$ and $\mathrm{Hom}_K(M, N)$. Then one can define a B-algebra to be a K-algebra A which is a B-module in such a way that the multiplication $A \otimes_K A \to A$ is a B-module map. The notions of B-coalgebra and B-bialgebra are defined similarly. Thus one can define a formal B-group of Ritt to be a complete topological B-bialgebra A which both has a unique maximal ideal and is finitely generated as a topological B-algebra. Under some additional finiteness conditions on B and A, the Lie algebra of the formal group is linearly compact. This allows the application of the methods and results of V. Guillemin [Gui 1] and R. Blattner [Bla 2]. We use these techniques to classify those K-algebras which admit a simple linearly compact $K[P]$-algebra structure. We then study the set of non-isomorphic $K[P]$-structures on each such K-algebra, as follows. We fix one easily-constructed structure as a reference point. Each additional structure is described by a kind of 1-cocycle—in our case, a flat connection with values in the Lie algebra. As is explained in Appendix II, when the algebra is finite-dimensional the connection represents the fixing of a direct product structure on a principal bundle whose fiber is the adjoint group of our Lie algebra.

0.4. We now give a more detailed account of the contents of each section.

Section 1 introduces the basic objects: Hopf K/k-algebra B, B-algebra, B-coalgebra, B-bialgebra. We describe different constructions involving B-modules, and in 1.3.2 state an important result relating produced and induced modules. In 1.4 we determine criteria for a K-algebra structure on a B-module to give a B-algebra. We develop in 1.5, 1.6 a language for describing B-module (B-algebra) structures on a K-module (K-algebra). Section 1.7 contains a "comparison" statement for two B-structures on a K-module. In 1.8 we introduce split B-structures—which are important since they can often be explicitly constructed. We study in 1.9 the action of the group of K-automorphisms of a B-algebra (B-module) on the set of B-structures—clearly there is no reason to distinguish between structures equal up to K-automorphism. Section 1.10 is technical, and develops notions to be used in sections 4 and 6.

In section 2 we specialize from arbitrary Hopf K/k-algebras to the case of the algebras of differential operators $K[P]$. These algebras resemble universal enveloping algebras, as is reflected, for example, in 2.1.4, 2.1.8. For $K[P]$, the comparison of structures (cf. 1.7, 1.9 above) takes a particularly nice form: the difference between two $K[P]$-algebra structures on a K-algebra M is a flat connection with values in $\mathrm{Der}_K M$ (cf. 2.2.7, 2.2.8). In 2.3 we develop techniques for handling the split structure which, as mentioned above, will be our reference point. (These techniques will be used in sections 6 and 7.) As we will need certain finiteness conditions on our $K[P]$-modules, we restrict our attention to the case in which P has a linearly compact topology. We introduce a natural class of B-modules—the B-modules of finite kind. The dual of such a module is said to be of cofinite kind. The cofinite kind property is easy to check (cf. 2.6). Modules of finite kind satisfy a Noetherian condition (cf. 2.5.5), and behave well under induction (cf. 2.5.6). Most of the results of 2.4, 2.5, 2.6 are quite simple if P is finite-dimensional; we consider the infinite case because such algebras arise naturally, and to include them in our treatment is not unduly taxing.

In section 3 we introduce Ritt's formal groups, and relate his definition to ours. We also show that every linearly compact Lie $K[P]$-algebra of cofinite kind is the Lie algebra of a formal group of Ritt.

Section 4 contains the main results of the paper. Here we investigate the Lie algebras of the formal groups of Ritt, primarily by adapting the techniques of V. Guillemin [Gui 1] and R. Blattner [Bla 2]. The first conceptual result of this section is Proposition 4.4.5, which says that our Lie algebras have a well-defined radical. We then proceed to study those Lie algebras which have trivial radical. The first step is to show that the minimal closed $K[P]$-ideals of such algebras have no closed ideals of their own, and are therefore "simple." Then we study the simple $K[P]$-algebras. We show that for each simple $K[P]$-algebra G there exists an open Lie K/k-subalgebra $\tilde{P}$ of P and an algebraically simple linearly compact $K[\tilde{P}]$-algebra S such that G is isomorphic to $\mathrm{Hom}_{K[\tilde{P}]}(K[P], S)$ as a $K[\tilde{P}]$-algebra (Theorem 4.6). In Theorem 4.7.1 we then describe the quotient of our Lie algebra by its radical, and also describe its quotient by the sum of the minimal ideals of the quotient by the radical.

To complete the picture given by the above-mentioned Theorem 4.6, we must describe the $K[\tilde{P}]$-structures on a simple Lie algebra S. We obtain such a description in section 5 for the case in which S is finite-dimensional. Our result says that every such structure splits after an appropriate field extension.

When S is of Cartan type, split structures are not of cofinite type. We still use them as reference points, however, and describe in section 6 the $K[\tilde{P}]$-structures on S in terms of flat connections. The main difficulty is to express and prove the validity of a condition for a $K[\tilde{P}]$-structure to be of cofinite kind. Each connection defines an open Lie K/k-subspace of $\tilde{P}$ of codimension $n = \Sigma_{i<0} \dim(\mathrm{Der}_K S)_i$. For the case $S = W_n$, we exhibit a bijection between such subspaces and the set of $K[\tilde{P}]$-structures on S. For the cases S_n, H_n, we classify the structures corresponding to a particular such subspace by means of elements of K^*/k^* and $GL(n, k)/Sp(n, K)k^*$ respectively; the case $S = K_n$ is somewhat more complicated (cf. Theorem 6.3.3).

In section 7 we briefly look at the question: Can formal groups of Ritt be considered to be formalizations (completions of the local ring at identity) of some algebraic structures? The answer we get is: "Yes, if their Lie algebras are finite-dimensional" and "No, if their Lie algebras are simple of Cartan type."

Finally, in section 8, we list some open problems; in Appendix I, we present some of the Hopf algebra which are used in this paper; and in Appendix II we describe some of the related differential geometry.

0.5. *Some notations and conventions.* Most of the Hopf algebra notation is explained in Appendix I. The notions of K/k-space, K/k-algebra, K/k-coalgebra etc. are defined in sections 1.1, 1.2. The notation B is used primarily for a Hopf K/k-algebra, and P for a Lie K/k-algebra. The Hopf K/k-algebra $K[P]$ of differential operators is defined in section 2.1. The notions of B-module, B-algebra etc. are defined in section 1.4. Split B-structures are defined in section 1.8. Topological (linearly compact and discrete) objects are first dealt with in section 2.4; cl stands for closure in the appropriate topology.

It would be handy to be acquainted with [Bla 1, Bla 2, Gui 1, Gui 2, Nich 3].

0.6. The main results of this paper were announced in [Weis].

0.7. The authors express their deep gratitude to V. Guillemin, V. Kac, D. Kazdhan, E. Kolchin, B. Kostant, J. O'Sullivan, M. Takeuchi, S. Sternberg, and J. Tits for many useful conversations and explanations, which had a great influence upon this paper.

1. Generalities. In this section, we will establish the basic properties of the structures we will be working with in as much generality as seem worthwhile. We first recall some concepts (see [Nich 3] for more detail). Some of the notions discussed below were already considered by other authors. For example, our sections 1.1.2, 1.1.3, 1.1.4, 1.1.5, 1.2.1 are close to material discussed by M. Sweedler on pp. 88, 104, 134, 108, 123 respectively of his paper (Groups of simple algebras, Publ. IHES, no. 44, 1975). Also M. Takeuchi considered in his papers K/k-algebras and K/k-bialgebras where K may be non-commutative (see, for example, J. algebra 42(1976), p. 327, or J. Math. Soc. Japan 29(1977), p. 460).

1.1. *K/k-structures.* Let K/k be a field extension.

1.1.1. *Definition.* A K/k-space is a $K \otimes_k K$-module.

When we refer to "the" K-structure of a K/k-space V we mean the K-structure obtained from the action of $K \otimes_k 1$. The K-structure obtained from $1 \otimes_k K$ will be called the "right" K-structure. For $v \in V$, $\lambda \in K$ we will write λv for $(\lambda \otimes_k 1)v$, and $v\lambda$ for $(1 \otimes_k \lambda)v$. A K/k-space is a (K, K)-bimodule. A K-module is a K/k-space via the multiplication $K \otimes_k K \to K$.

When V, V' are K/k-spaces, $V \otimes V'$ denotes their tensor product as K-spaces; thus, for $v \in V$, $v' \in V'$, $\lambda \in K$ we have $\lambda v \otimes v' = v \otimes \lambda v'$. We write $V \otimes_r V'$ for the "bimodule" tensor product, in which $v\lambda \otimes_r v' = v \otimes_r \lambda v'$. We consider $V \otimes_r V'$ to be a K/k-space via $(\lambda \otimes_k \lambda')(v \otimes_r v') = \lambda v \otimes_r v'\lambda'$.

When V is a K/k-space, $V^* = \operatorname{Hom}_K(V, K)$ is also a K/k space, with the action given by $\langle(\lambda \otimes_k \lambda')x \mid v\rangle = \langle x \mid (\lambda \otimes_k \lambda')v\rangle = \langle x \mid \lambda v \lambda'\rangle = \lambda\langle x \mid v\lambda'\rangle$ for $x \in V^*$, $v \in V$, λ, $\lambda' \in K$.

1.1.2. *Definition.* A K/k-algebra is a K/k-space A which is also a k-algebra, with $(\lambda a)b = \lambda(ab)$, $(a\lambda)b = a(\lambda b)$, $(ab)\lambda = a(b\lambda)$ for $a, b \in A$, $\lambda \in K$.

Note that the multiplication of a K/k-algebra A defines a map $A \otimes_r A \to A$. A unit for a K/k-algebra A is a unit 1 for the k-algebra A such that $\lambda 1 = 1\lambda$ for $\lambda \in K$. Thus a K/k-algebra with unit is equipped with a map $K \to A$ of K/k-algebras.

(Note that a K/k-algebra is *not* a $K \otimes_k K$-algebra.)

1.1.3. *Definition.* A K/k-coalgebra is a K/k-space C which is also a K-coalgebra with counit, with $\Delta(c\lambda) = \Sigma_i c_{1i}\lambda \otimes c_{2i} = \Sigma_i c_{1i} \otimes c_{2i}\lambda$ for $c \in C$, $\lambda \in K$.

We will often write $\Delta(c) = \Sigma\, c_1 \otimes c_2$ instead of $\Delta(c) = \Sigma_i c_{1i} \otimes c_{2i}$.

1.1.4. PROPOSITION. *Let C, D be K/k-coalgebras. Then $C \otimes_r D$ is a K/k-coalgebra. If $c \in C$, $d \in D$, $\Delta(c) = \Sigma_i c_{1i} \otimes c_{2i}$, $\Delta(d) = \Sigma_j d_{1j} \otimes d_{2j}$ then $\Delta(c \otimes_r d) = \Sigma_{i,j}(c_{1i} \otimes_r d_{1j}) \otimes (c_{2i} \otimes_r d_{2j})$ and $\epsilon(c \otimes_r d) = \epsilon(c\epsilon(d))$.*

Proof. We first show that our coproduct is well-defined. Given d_1, $d_2 \in D$, define $R_{d_1,d_2}: C \times C \to (C \otimes_r D) \otimes (C \otimes_r D)$ by $R_{d_1,d_2}(c_1, c_2) = (c_1 \otimes_r d_1) \otimes (c_2 \otimes_r d_2)$ for c_1, $c_2 \in C$. Then for $\lambda \in K$, we have $R_{d_1,d_2}(\lambda c_1, c_2) = (\lambda c_1 \otimes_r d_1) \otimes (c_2 \otimes_r d_2) = (c_1 \otimes_r d_1) \otimes (\lambda c_2 \otimes_r d_2) = R_{d_1,d_2}(c_1, \lambda c_2)$. Thus R_{d_1,d_2} passes to $C \otimes C$. Now define, for each $c \in C$, a map $L_{\Delta(c)}: D \times D \to (C \otimes_r D) \otimes (C \otimes_r D)$ by: $L_{\Delta(c)}(d_1, d_2) = R_{d_1,d_2}(\Delta(c))$. For $\lambda \in K$, we have $L_{\Delta(c)}(\lambda d_1, d_2) = R_{\lambda d_1,d_2}(\Delta(c)) = R_{\lambda d_1,d_2}(\Sigma_i c_{1i} \otimes c_{2i}) = \Sigma_i (c_{1i} \otimes_r \lambda d_1) \otimes (c_{2i} \otimes_r d_2) = \Sigma_i (c_{1i}\lambda \otimes_r d_1) \otimes (c_{2i} \otimes_r d_2) = R_{d_1,d_2}(\Delta(c\lambda)) = R_{d_1,d_2}(\Sigma_i c_{1i} \otimes c_{2i}\lambda) = \Sigma_i (c_{1i} \otimes_r d_1) \otimes (c_{2i}\lambda \otimes_r d_2) = \Sigma_i (c_{1i} \otimes_r d_1) \otimes (c_{2i} \otimes_r \lambda d_2) = R_{d_1,\lambda d_2}(\Delta(c)) = L_{\Delta(c)}(d_1, \lambda d_2)$. Thus $L_{\Delta(c)}$ passes to $D \otimes D$.

We may now define $\Delta: C \times D \to (C \otimes_r D) \otimes (C \otimes_r D)$ by $\Delta(c, d) = L_{\Delta(c)}(\Delta(d))$ for $c \in C$, $d \in D$. Then for $\lambda \in K$ we have $\Delta(c\lambda, d) = L_{\Delta(c\lambda)}(\Delta(d)) = \Sigma_{i,j}(c_{1i}\lambda \otimes_r d_{1j}) \otimes (c_{2i} \otimes_r d_{2j}) = \Sigma_{i,j}(c_{1i} \otimes_r \lambda d_{1j}) \otimes (c_{2i} \otimes_r d_{2j}) = L_{\Delta(c)}(\Delta(\lambda d)) = \Delta(c, \lambda d)$. Thus Δ passes to $C \otimes_r D$, and our coproduct is well-defined.

Next we show that the counit is well-defined. We define $\epsilon: C \times D \to K$ by $\epsilon(c, d) = \epsilon(c\epsilon(d))$ for $c \in C$, $d \in D$. Then for $\lambda \in K$ we have $\epsilon(c\lambda, d) = \epsilon(c\lambda\epsilon(d)) = \epsilon(c\epsilon(\lambda d)) = \epsilon(c, \lambda d)$. Thus ϵ passes to $C \otimes_r D$, as required.

The properties of Δ are easy to verify. We show that ϵ is a left counit. For $c \in C$, $d \in D$ we have $(\epsilon \otimes id)\Delta(c \otimes_r d) = \Sigma_{i,j}\, \epsilon(c_{1i} \otimes_r d_{1j})c_{2i} \otimes_r d_{2j} = \Sigma_j \Sigma_i\, \epsilon(c_{1i}\epsilon(d_{1j}))c_{2i} \otimes_r d_{2j} = \Sigma_j \Sigma_i\, \epsilon(c_{1i})c_{2i}\epsilon(d_{1j}) \otimes_r d_{2j}$ (since $\Delta(c\epsilon(d_{1j})) = \Sigma_i\, c_{1i}\epsilon(d_{1j}) \otimes c_{2i} = \Sigma_i\, c_{1i} \otimes c_{2i}\epsilon(d_{1j})) = \Sigma_{i,j}\, \epsilon(c_{1i})c_{2i} \otimes_r \epsilon(d_{1j})d_{2j} = c \otimes_r d$. Similarly, ϵ is a right counit, and we are done.

1.1.5. *Definition.* A K/k-bialgebra is a K/k-coalgebra B, which is also a K/k-algebra with unit in such a way that the maps $B \otimes_r B \to B$ and $K \to B$ are K/k-coalgebra maps.

Remark. Since $K \to B$ is a K/k-coalgebra map, we have $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$. Since $B \otimes_r B \to B$ is a K/k-coalgebra map, we have $\Delta(ab) = \Sigma\, a_1b_1 \otimes a_2b_2$ and $\epsilon(ab) = \epsilon(a\epsilon(b))$ for a, $b \in B$. These four conditions can be used to define the K/k-bialgebra structure. In contrast to the k-bialgebra case, we cannot require $\Delta: B \to B \otimes B$ to be a K/k-algebra map, as $B \otimes B$ is not even a k-algebra.

1.2. *Hopf K/k-algebras.*

1.2.1. *Definition.* A Hopf K/k-algebra is a K/k-bialgebra B equipped with an additive map $E: B \to B \otimes_r B$—the "antiproduct"—satisfying

(E 0) $E(\lambda b) = \lambda E(b) = E(b)\lambda$ for $\lambda \in K$, $b \in B$

(E 1) $\Sigma\, E(b_1)b_2 = b \otimes_r 1$, for $b \in B$

(E 2) $(\mathrm{id} \otimes E)\Delta = (\Delta \otimes \mathrm{id})E$

(E 3) $\mu E(b) = \epsilon(b)1$, where $b \in B$ and $\mu: B \otimes_r B \to B$ is the multiplication map.

For $b \in B$, we will write $E(b) = \Sigma_i\, E_{1i}(b) \otimes_r E_{2i}(b)$. Here E_{1i} and E_{2i} are not actual functions of b; we simply mean that if $E(b) = \Sigma_i\, x_i \otimes_r y_i$ for some x_i, $y_i \in B$, then we will write $E_{1i}(b)$ for x_i, and $E_{2i}(b)$ for y_i. We will write $E(b) = \Sigma\, E_1(b) \otimes_r E_2(b)$ when it seems to be possible to do so without causing additional confusion. Our axioms for E imply [Nich 3] that for all $b \in B$ we have

(E 4) $\Sigma\, E_1(b)\epsilon(E_2(b)) = b$.

1.2.2. Proposition (cf. [Nich 3, Theorem 1, Proposition 6, Proposition 7]). *Let B be a K/k-bialgebra. Suppose that B is pointed as a coalgebra. Then B has an antiproduct iff the grouplike elements of B form a group. In this case, the antiproduct is unique, and satisfies*

(E 5) $E(bc) = \Sigma E_1(b)E_1(c) \otimes_r E_2(c)E_2(b)$ *for* $b, c \in B$,
(E 6) $(E \otimes \mathrm{id})\Delta = (\mathrm{id} \otimes F)E : B \to B \otimes_r B \otimes_r B$,

where F *is the antiproduct of* B^{coop}—*the* K/k-*bialgebra obtained from* B *by reversing its coproduct.*

1.2.3. *Remark.* We will always assume that our antiproducts satisfy (E 5) and (E 6). In particular, we assume that B^{coop} is Hopf if B is.

1.3. *B-modules $M \otimes N$ and* $\mathrm{Hom}(M, N)$. Let B be a K/k-bialgebra, and let M, N be left B-modules. Then $M \otimes_K N$ has a left B-module structure, given by $b(m \otimes n) = \Sigma\, b_1 m \otimes b_2 n$ for $b \in B$, $m \in M$, $n \in N$. If B is a Hopf K/k-algebra, then $\mathrm{Hom}_K(M, N)$ has a left B-module structure, given by $(b\phi)(m) = \Sigma E_1(b)(\phi(E_2(b)m))$ for $b \in B \phi \in \mathrm{Hom}_K(M, N)$, and $m \in M$.

1.3.1. The modules $M \otimes N$ and $\mathrm{Hom}(M, N)$ have an associativity property. Before giving it, we recall some notions from [Hig].

Let $A \subset B$ be rings, and let M be a left A-module. Then $B \otimes_A M$ and $\mathrm{Hom}_A(B, M)$ are left B-modules, via $b(b' \otimes m) = bb' \otimes m$ and $(b\phi)(b') = \phi(b'b)$ for $b, b' \in B$, $m \in M$, $\phi \in \mathrm{Hom}_A(B, M)$. These modules are called (respectively) the B-modules induced and produced from the A-module M, and have the following universal property.

Proposition. *Let A, B, M be as above. Let N be a left B-module.*

(i) *For each left A-module map $\phi : M \to N$, there is a unique B-module map $\theta : B \otimes_A M \to N$ satisfying $\theta(1 \otimes m) = \phi(m)$, $m \in M$.*
(ii) *For each left A-module map $\phi : N \to M$, there is a unique B-module map $\theta : N \to$ $\mathrm{Hom}_A(B, M)$ satisfying $\theta(n)(1) = \phi(n)$, $n \in N$.*

1.3.2. Proposition (cf. [Nich 3, Theorem 2]). *Suppose that $A \subset B$ are Hopf K/k-algebras. Let V be a left A-module, and let W be a left B-module. Then* $\mathrm{Hom}_K(B \otimes_A V, W) \simeq \mathrm{Hom}_A(B, \mathrm{Hom}_K(V, W))$ *as B-modules.*

1.3.3. Suppose that W is linearly compact, and each $b \in B$ acts continuously on W. Then $\mathrm{Hom}_K(B \otimes_A V, W)$ and $\mathrm{Hom}_K(V, W)$ are linearly compact, dual to the discrete spaces $(B \otimes_A V) \otimes W^*$ and $V \otimes W^*$ respectively. It is easy to verify that when $\mathrm{Hom}_A(B, \mathrm{Hom}_K(V, W))$ is given the finite-open topology, the isomorphism of Proposition 1.3.2 is a homeomorphism.

1.4. *B-algebras.* We now move on to new concepts.

1.4.1. *Definition.* Let B be a K/k-bialgebra. A B-algebra (respectively, B-coalgebra, B-bialgebra, B-Hopf algebra) is a B-module M which is also a K-algebra (respectively, K-coalgebra, K-bialgebra, K-Hopf algebra) in such a way that all of its structures maps are B-module homomorphisms.

Note that any B-module can be considered to be a B-algebra with trivial multiplication.

1.4.2. *Warning.* B is *not* a B-algebra. Indeed, the multiplication map $B \times B \to B$ is only k-bilinear, and so does not in general pass to $B \otimes B \to B$; and even when it does, the map $B \otimes B \to B$ is not generally a B-module map.

1.4.3. Let B be a Hopf K/k-algebra, and let M be a B-module. Recall from 1.3 that $\mathrm{End}_K M = \mathrm{Hom}_K(M, M)$ is a B-module, via $(bQ)(m) = \Sigma E_1(b)(Q(E_2(b)m)$ for $b \in B$, $Q \in \mathrm{End}_K M$, and $m \in M$.

PROPOSITION. *$\mathrm{End}_K M$ is a B-algebra.*

Proof. For $b \in B$, we will define $b^\times \in \mathrm{End}_K M$ by $b^\times(m) = bm$ for all $m \in M$. Then for $b \in B$, $Q \in \mathrm{End}_K M$, we have $bQ = \Sigma_i E_{1i}(b)^\times Q E_{2i}(b)^\times$.

Now let $b \in B$, Q_1, $Q_2 \in \mathrm{End}_K M$. We have

$$\sum_j (b_{1j}Q_1)(b_{2j}Q_2) = \sum_{i,j,k} E_{1i}(b_{1j})^\times Q_1 E_{2i}(b_{1j}^\times)^\times E_{1k}(b_{2j})^\times Q_2 E_{2k}(b_{2j})^\times$$

$$= \sum_{i,j,k} E_{1i}(E_{1j}(b)_{1k})^\times Q_1 E_{2i}(E_{1j}(b)_{1k})^\times E_{1j}(b)_{2k}{}^\times$$

$$\cdot\, Q_2 E_{2j}(b)^\times \text{ (by } (E_2))$$

$$= \sum_j E_{1j}(b)^\times Q_1 Q_2 E_{2j}(b)^\times \text{ (by (E 1))}$$

$$= b(Q_1 Q_2), \text{ as required.}$$

1.4.4. Let M be a K-algebra. For each $x \in M$, define $L_x \in \mathrm{End}_K M$ by $L_x(y) = xy$ for all $y \in M$.

PROPOSITION. *Let M be a K-algebra which is also a B-module. Then M is a B-algebra iff $bL_x = L_{bx}$ for all $b \in B$, $x \in M$.*

Proof. Note that our condition is that

$$(bx)y = \sum_i E_{1i}(b)(x(E_{2i}(b)y)) \quad \text{for all} \quad b \in B, x, y \in M.$$

First assume that M is a B-algebra. We have

$$\sum_i E_{1i}(b)(x(E_{2i}(b)y)) = \sum_{i,j} (E_{1i}(b)_{1j}x)(E_{1i}(b)_{2j}(E_{2i}(b)y))$$

(since M is a B-algebra)

$$= \sum_{i,j} (b_{1i}x)(E_{1j}(b_{2i})E_{2j}(b_{2i})y) \text{ (by (E 2))}$$

$$= \sum_i (b_{1i}x)(\epsilon(b_{2i})y) \text{ (by (E 3)}$$

$$= (bx)y \text{ (since } \epsilon \text{ is a counit), as required.}$$

Conversely, assume that

$$(bx)y = \sum_i E_{1i}(b)(x(E_{2i}(b)y)) \quad \text{for} \quad b \in B, x, y \in M.$$

Then

$$\sum_i (b_{1i}x)(b_{2i}y) = \sum_{i,j} E_{1j}(b_{1i})(x(E_{2j}(b_{1i})b_{1i})b_{2i}y)) = b(xy) \text{ (by (E 1))}.$$

Thus, M is a B-algebra.

1.4.5. Corollary. *A Lie K-algebra M which is also a B-module is a B-algebra iff* $b(\operatorname{ad} x) = \operatorname{ad}(bx)$ *for all* $b \in B$, $x \in M$.

1.4.6. *Remark.* Suppose that a B-algebra M has a unit 1. Then for $b \in B$, we have $b1 = (b1)1 = \Sigma_i E_{1i}(b)(E_{2i}(b)1)$ (by 1.4.4) $= \epsilon(b)1$ (by (E 3)).

1.4.7. Proposition. *Let M be a B-algebra. For* $g \in \operatorname{Aut}_K M$, $p \in B$ *primitive, we have* $g^{-1}(pg) \in \operatorname{Der}_K M$.

(*Here* $pg \in \operatorname{End}_K M$ *is defined by the B-module structure on* $\operatorname{End}_K M$, *and* $g^{-1}(pg)$ *is the product of* g^{-1} *and* pg *in* $\operatorname{End}_k M$. *The map* $(dg)(p) = g^{-1}(pg)$ *is called the* logarithmic derivative *of g (in the direction p).*)

Proof. For $a, b \in M$, we have $(pg)(ab) = p(g(ab)) - g(p(ab)) = p(g(a)g(b)) - g(p(a)b + a(p(b))) = (p(g(a)))g(b) + g(a)(p(g(b)) -$

$g(p(a))g(b) - g(a)g(p(b)) - (pg)(a)g(b) + g(a)(pg)(b)$. Thus $g^{-1}(pg) \in \mathrm{Der}_K M$, as required.

1.5. *Universal K/k-algebras and K/k-coalgebras.*

1.5.1. PROPOSITION (cf. [Nich 1]). *Let V be a K/k-space. Then there exists an associative K/k-algebra $T_{K/k}(V)$ and a map $i: V \to T_{K/k}(V)$ of K/k-spaces such that: if A is an associative K/k-algebra with* 1 *and $f: V \to A$ is a map of K/k-spaces, then there exists a unique K/k-algebra homomorphism $F: T_{K/k}(V) \to A$ with $F \circ i = f$.*

Remark. $T_{K/k}(V)$ is analogous to the usual tensor algebra, but is formed using $\otimes_r$.

We now sketch the more complicated construction of the universal K/k-coalgebra on a K/k-space.

1.5.2. PROPOSITION. *Let C be a K/k-coalgebra. Then $C^* = \mathrm{Hom}_K(C, K)$ is a $K \otimes_k K$-algebra with unit.*

Proof. Let $\lambda, \lambda' \in K$, a, $b \in C^*$, $c \in C$. Then $\langle(\lambda \otimes_k \lambda')ab \mid c\rangle = \langle ab \mid \lambda c \lambda'\rangle = \langle a \otimes b \mid \Delta(\lambda c \lambda')\rangle = \langle a \otimes b \mid \Sigma_i \lambda c_{1i} \lambda' \otimes c_{2i}\rangle = \langle \lambda a \lambda' \otimes b \mid \Sigma_i c_{1i} \otimes c_{2i}\rangle = \langle(\lambda a \lambda')b \mid c\rangle$. Thus, $(\lambda \otimes_k \lambda')ab = ((\lambda \otimes_k \lambda')a)b$. Similarly, $(\lambda \otimes_k \lambda')ab = a((\lambda \otimes_k \lambda')b)$.

1.5.3. In the next few subsections, we will use the symbol A° to denote the "dual coalgebra" of an algebra A, as in [Swe, Ch. 6]. The symbol will have a different meaning later in the paper.

PROPOSITION. *Let A be a $K \otimes_k K$-algebra with unit. Then A° is a K/k-coalgebra.*

Proof. Since A is a K-algebra with unit, A° is a K-coalgebra with counit. Recall [Swe, Proposition 6.0.3] that $f \in A^*$ lies in A° iff there exists f_{1i}, $f_{2i} \in A^*$ so that $f(ab) = \Sigma_i f_{1i}(a) f_{2i}(b)$ for all a, $b \in A$; in this case, $\Delta(f) = \Sigma_i f_{1i} \otimes f_{2i}$. If $f \in A^\circ$ and $\lambda \in K$ then for a, $b \in A$ we have $(f\lambda)(ab) = f((ab)\lambda) = f((a\lambda)b) = \Sigma_i f_{1i}(a\lambda) f_{2i}(b) = \Sigma_i (f_{1i}\lambda)(a) f_{2i}(b)$. Thus $f\lambda \in A^\circ$, and $\Delta(f\lambda) = \Sigma_i f_{1i}\lambda \otimes f_{2i}$. Similarly, $\Delta(f\lambda) = \Sigma_i f_{1i} \otimes f_{2i}\lambda$.

1.5.4. PROPOSITION (cf. [Swe, Theorem 6.0.5]). *The functor $(\ )^\circ$: $K \otimes_k K$-algebras $\to$ K/k-coalgebras is an adjoint to the functor $(\ )^*$: K/k-coalgebras $\to$ $K \otimes_k K$-algebras.*

1.5.5. PROPOSITION. *Let V be a $K \otimes_k K$-module. There exists an associative $K \otimes_k K$-algebra $T_{K \otimes_k K}(V)$ and a map $i: V \to T_{K \otimes_k K}(V)$ of $K \otimes_k K$-modules such that if A is an associative $K \otimes_k K$-algebra with* 1 *and $f: V \to A$ is a map of $K \otimes_k K$-modules, then there exists a unique $K \otimes_k K$-algebra map $F: T_{K\otimes_k K}(V) \to A$ so that $F \circ i = f$.*

Proof. This is standard.

Exactly as in [Swe, section 6.4] we now deduce the following.

1.5.6. Proposition. *Let V be a K/k-space. Then there exists a K/k-coalgebra C and a map $\pi: C \to V$ of K/k-spaces so that for every K/k-coalgebra D and every map $f: D \to V$ of K/k-spaces, there exists a unique K/k-coalgebra map $F: D \to C$ with $\pi \circ F = f$.*

The pair (C, π) is called a cofree K/k-coalgebra on V.

1.6. *The universal K/k-bialgebra of a K-algebra.* Recall [Swe, 7.0] that a measuring of a K-coalgebra C on a K-algebra A is a K-linear map $\psi: C \to \mathrm{End}_K A$ such that $\psi(c)(ab) = \Sigma\, \psi(c_1)(a)\psi(c_2)(b)$, $\psi(c)(1) = \epsilon(c)1$ for $c \in C$, a, $b \in A$.

1.6.1. *Definition.* Let C be a K/k-coalgebra, and A a K-algebra. A K/k-measuring of C on A is a map $\psi: C \to \mathrm{End}_k A$ of K/k-spaces, such that $\psi(c)(ab) = \Sigma_i\, \psi(c_{1i})(a)\psi(c_{2i})(b)$ for all $c \in C$, a, $b \in B$. When A has a unit, we require in addition that $\psi(c)(1) = \epsilon(c)1$ for all $c \in C$.

Note that for $\lambda \in K$, $c \in C$, $a \in A$, we have $\psi(\lambda c)(a) = (\lambda\psi(c))(a) = \lambda(\psi(c)(a))$, and $\psi(c\lambda)(a) = (\psi(c)\lambda)(a) = \psi(c)(\lambda a)$. In particular, when A has a unit 1 we have $\psi(c)(\lambda) = \psi(c)(\lambda 1) = \psi(c\lambda)1 = \epsilon(c\lambda)1$ for $\lambda \in K$.

1.6.2. *Remark.* Let C be a K/k-coalgebra. Define $\psi: C \to \mathrm{End}_k K$ by: $\psi(c)(a) = \epsilon(ca)$ for $c \in C$, $a \in K$. Then for a, $b \in K$ we have $\Sigma\, \psi(c_1)(a) \cdot \psi(c_2)(b) = \Sigma\, \epsilon(c_1 a)\epsilon(c_2 b) = \epsilon(\Sigma\, \epsilon(c_1 a)c_2 b) = \epsilon(cab)$, since $\Delta(cab) = \Sigma\, c_1 a \otimes c_2 b$. Thus C measures K.

1.6.3. Proposition. (cf. [Swe, Theorem 7.0.4]). *Let A be a K-algebra. There is a K/k-coalgebra $\tilde{A}$ and a K/k-measuring $\theta: \tilde{A} \to \mathrm{End}_k A$ with the following universal property: if $\psi: C \to \mathrm{End}_k A$ is a K/k-measuring on A, then there exists a unique K/k-coalgebra map $F: C \to \tilde{A}$ with $\psi = \theta \circ F$.*

Proof. Let (E, π) be a cofree K/k-coalgebra on $\mathrm{End}_k A$. Let $\tilde{A}$ be the sum of all subcoalgebras of E on which π is a K/k-measuring of A, and let θ be the restriction of π to $\tilde{A}$. Then, as in [Swe, Theorem 7.0.4], $(\tilde{A}, \theta)$ has the required property.

1.6.4. Proposition. *$\tilde{A}$ has a unique K/k-algebra structure for which $\tilde{A}$ is a K/k-algebra and $\theta: \tilde{A} \to \mathrm{End}_k A$ is a K/k-algebra map.*

Proof. Define $\psi:\tilde{A} \otimes_r \tilde{A} \to \mathrm{End}_k A$ by $\psi(x \otimes_r y)(a) = \theta(x)(\theta(y)(a))$ for $x, y \in \tilde{A}$, $a \in A$. ψ is a K/k-measuring of $\tilde{A}$ on A, and thus gives rise to a K/k-coalgebra map $\tilde{A} \otimes_r \tilde{A} \to \tilde{A}$. The K-module action of K on A is a K/k-measuring and thus defines a K/k-coalgebra map $K \to \tilde{A}$. It is easy to use the universal property of $\tilde{A}$ to show that these K/k-coalgebra maps have the correct properties.

We will call $\tilde{A}$ the universal K/k-bialgebra of the K-algebra A.

1.6.5. PROPOSITION. *Let C be a K/k-coalgebra, A a K-algebra, and $\psi: C \to \mathrm{End}_k A$ a measure. For $g \in \mathrm{Aut}_K A$, define $g^{-1}\psi g: C \to \mathrm{End}_k A$ by: $(g^{-1}\psi g)(c)(a) = g^{-1}(\psi(c)(g(a)))$ for $c \in C$, $a \in A$. Then $g^{-1}\psi g$ is a measure.*

Proof. It follows from the fact that g is k-linear that $g^{-1}\psi g$ is a map of K/k-spaces. For $a, b \in A$, $c \in C$ we have $(g^{-1}\psi g)(c)(ab) = g^{-1}(\psi(c)(g(ab))) = g^{-1}(\psi(c)(g(a)g(b))) = g^{-1}(\Sigma\, \psi(c_1)(g(a))\psi(c_2)(g(b))) = \Sigma g^{-1}(\psi(c_1)(g(a)))g^{-1}(\psi(c_2)(g(b))) = \Sigma(g^{-1}\psi g)(c_1)(a)(g^{-1}\psi g)(c_2)(b)$. Thus $g^{-1}\psi g$ is a measure.

1.6.6. Let B be a K/k-bialgebra.

PROPOSITION. *Let M be a K-algebra. Then a B-algebra structure on M is a homomorphism $F: B \to \tilde{M}$ of K/k-bialgebras.*

Proof. Suppose that M is a B-algebra. Define $f: B \to \mathrm{End}_k M$ by: $f(b)(m) = bm$. Then f measures M. Therefore there is a unique K/k-coalgebra map $F: B \to \tilde{M}$ with $\theta \circ F = f$.

For $a, b \in B$, we have $\theta(F(a)F(b)) = \theta(F(a))\theta(F(b)) = f(a)f(b) = f(ab) = \theta(F(ab))$. The universal property of $\tilde{M}$ (applied using $B \otimes_r B$) then gives that F is a K/k-algebra map. Thus F is a K/k-bialgebra map. The converse is clear.

1.6.7. COROLLARY. *Let M be a K-algebra. Then a B-algebra structure on M is a homomorphism $f: B \to \mathrm{End}_k M$ of K/k-algebras which measures M.*

Proof. We take $f = \theta \circ F$ in the above.

1.6.8 PROPOSITION. *Let B be a K/k-bialgebra, M a K-algebra, and $\psi: B \to \mathrm{End}_k M$ a B-algebra structure on M. Then for each $g \in \mathrm{Aut}_K M$, $g^{-1}\psi g$ is a B-algebra structure on M.*

Proof. By 1.6.5 and 1.6.7 it suffices to check that $g^{-1}\psi g: B \to$

$\mathrm{End}_k M$ satisfies $(g^{-1}\psi g)(ab) = (g^{-1}\psi g)(a)(g^{-1}\psi g)(b)$ for all $a, b \in B$, and this is immediate.

1.7. Note that a B-module structure on a K-space M is given by a homomorphism $\phi: B \to \mathrm{End}_k M$ of K/k-algebras. When B is a Hopf K/k-algebra, we can compare two B-module structures as follows.

PROPOSITION. *Let B be a Hopf K/k-algebra. If $\phi_1, \phi_2: B \to \mathrm{End}_k M$ are B-module structures, then $\Sigma_i\, \phi_1(E_{1i}(b))\phi_2(E_{2i}(b)) \in \mathrm{End}_K M$, $b \in B$.*

Proof. Let $\lambda \in K$, $b \in B$. By (E 0), we have $\Sigma\, \lambda E_1(b) \otimes_r E_2(b) = E(\lambda b) = \Sigma E_1(b) \otimes_r E_2(b)\lambda$. Thus for $m \in M$ we have $\Sigma\, \phi_1(E_1(b))(\phi_2(E_2(b))(\lambda m)) = \Sigma\, \phi_1(E_1(b))\phi_2(E_2(b)\lambda)(m)) = \Sigma\, \phi_1(\lambda E_1(b))(\phi_2(E_2(b))(m) = \lambda\Sigma\, \phi_1(E_1(b))(\phi_2(E_2(b))(m))$.

1.8. *Split B-structures.* We first show how to construct a B-algebra from a k-algebra.

1.8.1. PROPOSITION. *Let M be a B-algebra, and let N be a k-algebra. Then $M \otimes_k N$ is a B-algebra, via $b(m \otimes_k n) = bm \otimes_k n$ for $b \in B$, $m \in M$, $n \in N$.*

Proof. It is clear that $M \otimes_k N$ is a B-module, isomorphic to a direct sum of $\dim_k N$ copies of M. We must verify that the given action is a measuring. We have $b((m \otimes_k n)(m' \otimes_k n')) = b(mm' \otimes_k nn') = b(mm') \otimes_k nn' = \Sigma_i (b_{1i}m)(b_{2i}m') \otimes_k nn' = \Sigma_i (b_{1i}m \otimes_k n)(b_{2i}m' \otimes_k n')$, as required.

1.8.2. COROLLARY. *Let M° be a k-algebra. Then $K \otimes_k M^\circ$ is a B-algebra, via $b(\lambda \otimes_k m) = \epsilon(b\lambda) \otimes_k m$ for $b \in B$, $\lambda \in K$, $m \in M$.*

1.8.3. *Definition.* We say that $M = K \otimes_k M^\circ$ is a *split* B-algebra (or a split B-module if $M^\circ M^\circ = 0$). A representation of a B-algebra M in the form $M = K \otimes_k M^\circ$ with the above action of B is called a *splitting*. When $M = K \otimes_k M^\circ$ is a splitting of M, the image of $b \in B$ under the module structure map $B \to \mathrm{End}_k M$ will be denoted b°.

1.8.4. PROPOSITION. *Suppose that K is algebraically closed of characteristic zero. Let L be a Lie K-algebra which is either finite-dimensional and simple or else linearly compact and of Cartan type. Then L can be given a split B-algebra structure.*

Proof. Since K is algebraically closed, L is split, and thus defined over $\mathbf{Q}$. That is, $L = K \otimes_k L^\circ$ for some $\mathbf{Q}$-algebra L°.

1.9. *Higher logarithmic derivatives.*

1.9.1. PROPOSITION. *Let B be a K/k-bialgebra. Assume that B is irreducible and cocommutative as a coalgebra. Then $E: B \to B \otimes_r B$ is a coalgebra map.*

Proof. By induction, using the coradical filtration (Appendix I). Clearly E is a coalgebra map on $B_0 = K1$. Assume that E is a coalgebra map on B_{n-1}, and take $b \in P_n(B) = B_n \cap \ker \epsilon$. Write $\Delta(b) = b \otimes 1 + 1 \otimes b + \Sigma_i b_i \otimes b_i'$, where $b_i, b_i' \in P_{n-1}(B)$. Then by (E 1), $E(b) = b \otimes_r 1 - 1 \otimes_r b - \Sigma_i E(b_i)b_i'$. Thus, $\Delta(E(b)) = \Delta(b \otimes_r 1) - \Delta(1 \otimes_r b) - \Sigma_i \Delta(E(b_i))\Delta(b_i) = \Delta(b \otimes_r 1) - \Delta(1 \otimes_r b) - \Sigma_i((E \otimes E)\Delta(b_i'))\Delta(b_i')$ by induction. Now $\Sigma_i b_i \otimes b_i' = \Delta(b) - b \otimes 1 - 1 \otimes b = \Sigma b_1 \otimes b_2 - b \otimes 1 - 1 \otimes b$. Thus, $\Sigma_i((E \otimes E)\Delta(b_i))\Delta(b_i') = \Sigma((E \otimes E)(b_1 \otimes b_2))(b_3 \otimes b_4) - ((E \otimes E)\Delta(b))\Delta(1) - ((E \otimes E)\Delta(1))\Delta(b) = \Sigma E(b_1)b_3 \otimes E(b_2)b_4 - (E \otimes E)\Delta(b) - \Delta(1 \otimes_r b) = \Sigma E(b_1)b_2 \otimes E(b_3)b_4 - (E \otimes E)\Delta(b) - \Delta(1 \otimes_r b)$ (since B is cocommutative) $= \Sigma(b_1 \otimes_r 1) \otimes (b_2 \otimes_r 1) - (E \otimes E)\Delta(b) - \Delta(1 \otimes_r b)$ by (E 1) $= \Delta(b \otimes_r 1) - (E \otimes E)\Delta(b) - \Delta(1 \otimes_r b)$. Thus $\Delta(E(b)) = (E \otimes E)\Delta(b)$, as required.

1.9.2. *Definition.* Let M be a K-algebra. Define inductively subspaces $\mathrm{Der}_K^n M$ of $\mathrm{End}_K M$ by: $\mathrm{Der}_K^0 M = 0$, $\mathrm{Der}_K^1 M = \mathrm{Der}_K M$, $\mathrm{Der}_K^n M = \{f \in \mathrm{End}_k M$: there exist $f_i \in \mathrm{Der}_K^{n_i} M$, $f_i' \in \mathrm{Der}_K^{n-n_i} M$ such that for all $m, m' \in M$, $f(mm') = f(m)m' + mf(m') + \Sigma_i f_i(m)f_i'(m')\}$.

1.9.3. PROPOSITION. *Let B be an irreducible cocommutative K/k-bialgebra. Let M be a B-algebra, and let $g \in \mathrm{Aut}_K M$. Define $dg: B \to \mathrm{End}_K M$ by: $(dg)(b) = g^{-1}(bg)$ for $b \in B$. (The product of g^{-1} with bg is taken within the B-algebra $\mathrm{End}_K M$.) Then*

(i) *dg is a K-measure on M.*
(ii) $(dg)(P_n(B)) \subseteq \mathrm{Der}_K^{nM}$.

Proof. Let $B \in B$, $m, m' \in M$. Then

$$(dg)(b)(mm') = (g^{-1}(bg))(mm')$$

$$= g^{-1}(\sum_i E_{1i}(b)(g(E_{2i}(b)(mm'))))$$

$$= g^{-1}(\sum_{i,j} E_{1i}(b)(g(E_{2i}(b)_{1j}(m)E_{2i}(b)_{2j}(m'))))$$

$$= g^{-1}(\sum_{i,j} E_{1i}(b)(g(E_{2i}(b)_{1j}(m))g(E_{2i}(b)_{2j}(m')))$$

$$= g^{-1}(\sum_{i,j,k} E_{1i}(b)_{1k}(g(E_{2i}(b)_{1j}(m)))E_{1i}(b)_{2k}$$

$$(g(E_{2i}(b)_{2j}(m'))))$$

$$= g^{-1}(\sum_{i,j,k} E_{1j}(b_{1i})(g(E_{2j}(b_{1i})(m)))E_{1k}(b_{2i})$$

$$(g(E_{2k}(b_{2i})(m')))) \quad \text{by} \quad 1.9.2$$

$$= \sum_{i,j,k} g^{-1}(E_{1j}(b_{1i})(g(E_{2j}(b_{1i})(m))))$$

$$g^{-1}(E_{1k}(b_{2i})(g(E_{2k}(b_{2i})(m'))))$$

$$= \sum_i (dg)(b_{1i})(m)(dg)(b_{2i})(m'),$$

as required. If M has a unit, then $(dg)(b)(1) = g^{-1}(\Sigma E_1(b)(g(E_2(b)(1))) = g^{-1}(\Sigma E_1(b)g(\epsilon(E_2(b))1)) = g^{-1}(\Sigma E_1(b)\epsilon(E_2(b))1) = g^{-1}(b1)$ by (E 4) $= g^{-1}(\epsilon(b)1) = \epsilon(b)1$. Thus, dg is a K-measure.

If $b \in P_n(b)$, then $\Delta(b) = b \otimes 1 + 1 \otimes b + \Sigma_i\, b_i \otimes b_i'$, where $b_i \in P_{n_i}(B)$, $b_i' \in P_{n-n_i}(B)$. Thus (ii) follows immediately from (i) by induction.

1.9.4. Corollary. *Let B be a K/k-algebra, M a K-algebra, $g \in \mathrm{Aut}_K M$. Define $dg: P(B) \to \mathrm{End}_K M$ by: $(dg)(p) = g^{-1}(pg)$. Then $(dg)(P(B)) \subseteq \mathrm{Der}_K M$.*

Proof. Drop down to the subbialgebra generated by K and $P(B)$, and apply 1.9.3. (Or simply repeat the calculation for this easy case.)

The function $(dg)(p)$ is called the *logarithmic derivative* of g (in the direction p.)

1.9.5. The following related result will be used later.

Proposition. *Let B be an irreducible commutative K/k-bialgebra. Let M be a B-algebra. Then $\mathrm{Der}_K M$ is a B-algebra.*

Proof. Recall (1.4.3) that $\mathrm{End}_K M$ is a B-algebra. It is easy to see that if a map $(x, y) \to xy$ defines a B-algebra structure on a B-module, then so does $(x, y) \to xy - yx$. Thus, it suffices to show that $\mathrm{Der}_K M$ is a B-submodule of $\mathrm{End}_K M$.

Accordingly, let $b \in B$, $g \in \mathrm{Der}_K M$, $x, y \in M$. We have

$$\begin{aligned}
(bg)(xy) &= \sum_i E_{1i}(b)(g(E_{2i}(b)(xy))) \\
&= \sum_{i,j} E_{1i}(b)(g((E_{2i}(b)_{1j}(x))(E_{2i}(b)_{2j}(y)))) \\
&= \sum_{i,j} E_{1i}(b)(g(E_{2i}(b)_{1j}(x))E_{2i}(b)_{2j}(y)) \\
&\quad + \sum_{i,j} E_{1i}(b)(E_{2i}(b)_{1j}(x)g(E_{2i}(b)_{2j}(y)) \\
&= \sum_{i,j,k} E_{1i}(b)_{1k}(g(E_{2i}(b)_{1j}(x)))E_{1i}(b)_{2k}(E_{2i}(b)_{2j}(y)) \\
&\quad + \sum_{i,j,k} E_{1i}(b)_{1k}(E_{2i}(b)_{1j}(x))E_{1i}(b)_{2k}(g(E_{2i}(b)_{2j}(y))) \\
&= \sum_{i,j,k} E_{1j}(b_{1i})(g(E_{2j}(b_{1i})(x)))E_{1k}(b_{2i})(E_{2k}(b_{2i})(y)) \\
&\quad + \sum_{i,j,k} E_{1j}(b_{1i})(E_{2j}(b_{1i})(x))E_{1k}(b_{2i})(g(E_{2k}(b_{2i})(y))) \quad \text{by} \quad 1.9.1 \\
&= \sum_{i,j} E_{1j}(b_{1i})(g(E_{2j}(b_{1i})(x)))\epsilon(b_{2i})y \\
&\quad + \sum_{i,j,k} E_{1j}(b_{1i})xE_{1k}(b_{2i})(g(E_{2k}(b_{2i})(y))) \quad \text{by} \quad (\mathrm{E}\ 3) \\
&= \sum_j E_{1j}(b)(g(E_{2j}(b)(x)))y + \sum_k xE_{1k}(b)(g(E_{2k}(b)(y))) \\
&= (bg)(x)y + x(bg)(y).
\end{aligned}$$

Thus, $bg \in \mathrm{Der}_K M$, as required.

1.10. *Convolution algebras.*

1.10.1. *Definition.* Let C be a K-coalgebra, and A a K-algebra. The *convolution algebra* structure on $\mathrm{Hom}_K(C, A)$ is given by $(fg)(c) = \Sigma_i f(c_{1i})g(c_{2i})$, $f, g \in \mathrm{Hom}_K(C, A)$, $c \in C$.

1.10.2. *Remark.* We will always assume that C is associative. Then $\mathrm{Hom}_K(C, A)$ is associative (Lie) (abelian) if A is.

1.10.3. *Remark.* If C is a (K, T)-bimodule, then $\mathrm{Hom}_K(C, A)$ is a left T-module via $(tf)(c) = f(ct)$, $f \in \mathrm{Hom}_K(C, A)$, $t \in T$, $c \in C$.

1.10.4. PROPOSITION. *Let B be a K/k-bialgebra, and let R be a K-algebra. Then* $\mathrm{Hom}_K(B, R)$ *is a B-algebra.*

Proof. We have that $\mathrm{Hom}_K(B, R)$ is a B-module by 1.10.3. For $f, g \in \mathrm{Hom}_K(B, R)$, $b, b' \in B$ we have

$$\begin{aligned}(b(fg))(b') &= (fg)(b'b) \\ &= \sum_{i,j} f(b'_{1i}b_{1j})g(b'_{2i}b_{2j}) \\ &= \sum_{i,j} (b_{1j}f)(b'_{1i})(b_{2j}g)(b'_{2i}) \\ &= (\sum_j (b_{1j}f)(b_{2j}g))(b').\end{aligned}$$

Thus $b(fg) = \Sigma(b_1f)(b_2g)$, and $\mathrm{Hom}_K(B, R)$ is a B-algebra.

1.10.5. PROPOSITION. *Let $A \subset B$ be K/k-bialgebras, and let R be an A-algebra. Then* $\mathrm{Hom}_A(B, R)$ *is a B-subalgebra of* $\mathrm{Hom}_K(B, R)$.

Proof. We first check that $\mathrm{Hom}_A(B, R)$ is closed under products. Let $f, g \in \mathrm{Hom}_A(B, R)$, $a \in A$, $b \in B$. Then

$$\begin{aligned}(fg)(ab) &= \sum_{i,j} f(a_{1i}b_{1j})g(a_{2i}b_{2j}) \\ &= \sum_{i,j} (a_{1i}f(b_{1j}))(a_{2i}g(b_{2j})) \\ &= a(\sum_j f(b_{1j})g(b_{2j})) \text{ (since } R \text{ is an } A\text{-algebra)} \\ &= a((fg)(b)).\end{aligned}$$

Since $\mathrm{Hom}_A(B, R)$ is clearly a B-submodule of $\mathrm{Hom}_K(B, R)$, the proof is complete.

1.10.6. PROPOSITION. *Let B, A, R be as in* 1.10.5. *Let G be a B-algebra, and let $\phi: G \to R$ be an A-algebra map. Then the map $\theta: G \to$* $\mathrm{Hom}_A(B, R)$ *given by $\theta(x)(y) = \phi(yx)$, $x \in G$, $y \in B$ is a B-algebra map.*

Proof. We know from 1.3.1 that θ is a B-module map. We check that θ is preserves products. Let $x, x' \in G$, $y \in B$. Then $(\theta(x)\theta(x'))(y) = \Sigma_i \theta(x)(y_{1i})\theta(x')(y_{2i}x')) = \Sigma_i \phi(y_{1i}x)\phi(y_{2i}x') = \phi(\Sigma_i(y_{1i}x)(y_{2i}x')) = \phi(y(xx'))$ (since G is a B-algebra) $= \theta(xx')(y)$. This completes the proof.

1.10.7. PROPOSITION. *Let B be an irreducible cocommutative K/k-bialgebra. Let C be a B-coalgebra, and let A be a B-algebra. Then the convolution algebra* $\mathrm{Hom}_k(C, A)$ *is, with the B-module structure of* 1.3, *a B-algebra.*

Proof. Let $b \in B$, f, $g \in \mathrm{Hom}_K(C, A)$, $c \in C$. Then

$$
\begin{aligned}
(b(fg))(c) &= \Sigma E_1(b)((fg)(E_2(b)(c))) \\
&= \Sigma E_1(b)(f(E_2(b)_1(c_1)g(E_2(b)_2(c_2))) \\
&= \Sigma E_1(b)_1(f(E_2(b)_1(c_1)))E_1(b)_2(g(E_2(b)_2(c_2))) \\
&= \Sigma E_1(b_1)(f(E_2(b_1))(c_1)))E_1(b_2)(g(E_2(b_2)(c_2))) \quad \text{by} \quad 1.9.1 \\
&= \Sigma(b_1(f))(c_1)(b_2(g))(c_2) \\
&= \Sigma(b_1(f)b_2(g))(c)
\end{aligned}
$$

Thus $b(gf) = \Sigma\, b_1(f)b_2(g)$, as required.

2. The algebra of differential operators. We now take a more restricted viewpoint.

2.1. *Differential operators.* A Lie K/k-algebra is a K-vector space P which is also a Lie k-algebra, equipped with a map $\partial: P \to \mathrm{Der}_k K$ of K-vector spaces and Lie k-algebras such that

$$[x, \lambda y] = \partial(x)(\lambda)y + \lambda[x, y] \quad \text{for} \quad \lambda \in K, x, y \in P.$$

Note that a Lie K/k-algebra is not a K/k-algebra, nor even a K/k-space.

2.1.2. *Definition.* Let P be a Lie K/k-algebra. We define $K[P]$ to be the associative k-algebra generated by K and P, subject to the relations (for $\lambda \in K$, $p, p_1, p_2 \in P$) $\lambda p = \gamma(\lambda, p)$, where $\gamma : K \times P \to P$ is the vector space structure map, $p_1 = \lambda p + \partial(p)(\lambda)1$, $p_1 p_2 - p_2 p_1 = [p_1, p_2]$.

2.1.3. Note that $K[P]$ is a K/k-algebra. The natural map $i : P \to K[P]$ has a universal property, as follows.

PROPOSITION. *Let A be an associative K/k-algebra with unit. Then for each K-linear map $\phi : P \to A$ satisfying*

$$\phi([x, y]) = [\phi(x), \phi(y)], \qquad \phi(x)\lambda = \lambda\phi(x) + \partial(x)\lambda 1 \quad \text{for} \quad x, y \in P,$$

$\lambda \in K$, there is a unique K/k-algebra map $\Phi : K[P] \to A$ with $\Phi \circ i = \phi$.

This follows immediately from the definition.

2.1.4. PROPOSITION (Poincaré-Birkhoff-Witt). *Let P be a Lie K/k-algebra. Let $\{p_i\}_{i \in I}$ be a K-basis of P. Totally order the index set I. Then $\{p_{i_1}^{e_i} \cdot \cdots \cdot p_{i_n}^{e_n} : i_1 \leq \cdots \leq i_n, 0 \leq e_1, \ldots, e_n\}$ is a K-basis of $K[P]$.*

The proof is the same as the usual proof of the Poincaré-Birkhoff-Witt theorem.

2.1.5. COROLLARY. *Let A be a K/k-algebra. There is a $1 - 1$ correspondence between K/k-algebra maps $K[P] \to A$ and K-linear maps $\phi : P \to A$ with $\phi([x, y]) = [\phi(x), \phi(y)]$ and $\phi(x)\lambda = \lambda\phi(x) + \partial(x)\lambda 1$ for $x, y \in P$, $\lambda \in K$.*

Proof. This follows from 2.1.3 and the fact that $i : P \to K[P]$ is injective.

2.1.6. PROPOSITION (cf. [Nich 3, Example 2, Theorem 1, Proposition 6, Proposition 7]). *$K[P]$ is a Hopf K/k-algebra, with Δ, E given on the generators $p \in P$ by: $\Delta(p) = p \otimes 1 + 1 \otimes p$, $E(p) = p \otimes_r 1 - 1 \otimes_r p$.*

2.1.7. If B is any K/k-bialgebra, the set of primitive elements $P = \{p \in B : \Delta(p) = p \otimes 1 + 1 \otimes p\}$ is closed under commutation and under left multiplication by elements of K; thus, P is a K-vector space and a Lie k-algebra. By 1.6.2, the map $\partial : P \to \mathrm{End}_k K$, $\partial(p)(\lambda) = \epsilon(p\lambda)$ for $p \in P$, $\lambda \in K$, takes values in $\mathrm{Der}_k K$. Since $p\lambda = \Sigma\, \epsilon(p_1\lambda)p_2 = \lambda p + \epsilon(p\lambda)1$, it follows easily that P is a Lie K/k-algebra.

Note that the subalgebra of B generated by K and P is a K/k-bialgebra.

2.1.8. When characteristic $k = 0$, we have the following characterization of the algebra of differential operators.

PROPOSITION (cf. [Nich 3, Theorem 4], [Wint, B.3]). *Let B be a K/k-bialgebra which is cocommutative and irreducible as a coalgebra. Assume that characteristic $k = 0$. Then B is a Hopf K/k-algebra, and is isomorphic as a Hopf K/k-algebra to the algebra of differential operators defined by its Lie K/k-algebra of primitives P.*

2.2. *Comparison of structures.* We will now assume that characteristic $k = 0$, and that $B = K[P]$, where P is the Lie K/k-algebra of primitives of B. We wish to compare the possible B-algebra structures on a K-algebra M.

2.2.1. PROPOSITION. *Let $\phi_1, \phi_2 : B \to \mathrm{End}_k M$ be two B-algebra structures on M. Then $(\phi_1 - \phi_2)(P) \subseteq \mathrm{Der}_K M$.*

Proof. Combine 1.6.6 with 1.7.

2.2.2. PROPOSITION.

(i) *B-module structures on M are given by K-linear maps $\phi : P \to \mathrm{End}_k M$ satisfying $\phi([x, y]) = [\phi(x), \phi(y)]$, $\phi(x)\lambda = \lambda\phi(x) + \epsilon(x\lambda)I$ for $x, y \in P$, $\lambda \in K$.*

(ii) *Such a ϕ defines a B-algebra structure on M iff $\phi(P) \subseteq \mathrm{Der}_k M$.*

Proof. As $\mathrm{End}_k M$ is a K/k-algebra, (i) follows from 2.1.5. Since P generates B, (ii) now follows from 1.6.6.

2.2.3. *Definition.* Let L be a Lie B-algebra. A differential m-form with values in L is a K-m-linear map $\omega : P^m \to L$ (where $P^m = P \times P^{m-1}$) such that, for each permutation σ on m symbols, $\omega(p_1, \ldots, p_m) = (\det \sigma)\omega(p_{\sigma(1)}, \ldots, p_{\sigma(m)})$ for all $p_1, \ldots, p_m \in P$.

2.2.4. We denote by $\Omega^m(P, L)$ the space of m-forms with values in L. We define $d : \Omega^m(P, L) \to \Omega^{m+1}(P, L)$ by

$$(d\omega)(p_0, \ldots, p_m) = \frac{1}{m+1} \sum_{i=0}^{m} (-1)^i p_i(\omega(p_0, \ldots, \hat{p}_i, \ldots, p_m))$$

$$+ \frac{1}{m+1} \sum_{0 \le i < j \le m}^{m} (-1)^{i+j}$$

$$\cdot\, \omega([p_i, p_j], p_0, \ldots, \hat{p}_i, \ldots, \hat{p}_j, \ldots, p_m)).$$

For $m = 1$, we have

$$(d\omega)(p,q) = \frac{1}{2}[p(\omega(q)) - q(\omega(p)) - \omega([p,q])].$$

2.2.5. PROPOSITION. *Let $\phi: P \to \mathrm{Der}_k M$ define a B-algebra structure on the K-algebra M.*

(i) *If $\tilde{\phi}$ is another B-algebra structure on M, then*

$$\omega = \tilde{\phi} - \phi \in \Omega^1(P, \mathrm{Der}_K M)$$

(ii) *If $\omega \in \Omega^1(P, \mathrm{Der}_K M)$ then $\phi + \omega: P \to \mathrm{Der}_k M$ is a B-algebra structure on M iff $d\omega = -\frac{1}{2}[\omega, \omega]$ (where $[\omega, \omega](p, q) = [\omega(p), \omega(q)]$ for $p, q \in P$).*

Proof. Part (i) follows from 2.2.1.

We now consider $\tilde{\phi} = \phi + \omega$ for $\omega \in \Omega^1(P, \mathrm{Der}_K M)$. Then $\tilde{\phi}$ is K-linear. For $p \in P$, $m \in M$, $\lambda \in K$ we have $(\tilde{\phi}(p)\lambda)(m) = \tilde{\phi}(p)(\lambda m) = \phi(p)(\lambda m) + \omega(p)(\lambda m) = \lambda\phi(p)(m) + \epsilon(p\lambda)(m) + \lambda\omega(p)(m) = \lambda\tilde{\phi}(p)(m) + \epsilon(p\lambda)(m)$. Thus $\tilde{\phi}(p)\lambda = \lambda\tilde{\phi}(p) + \epsilon(p\lambda)I$. Thus by 2.2.2 $\tilde{\phi}$ is a B-algebra structure iff $\tilde{\phi}([p, q]) = [\tilde{\phi}(p), \tilde{\phi}(q)]$ for $p, q \in P$. Now $\tilde{\phi}([p, q]) = \phi([p, q]) + \omega([p, q]) = [\phi(p), \phi(q)] + \omega([p, q])$. We have $[\tilde{\phi}(p), \tilde{\phi}(q)] = [\phi(p) + \omega(p), \phi(q) + \omega(q)] = [\phi(p), \phi(q)] + [\phi(p), \omega(q)] + [\omega(p), \phi(q)] + [\omega(p), \omega(q)]$. Thus we require $\omega([p, q]) - [\phi(p), \omega(q)] - [\omega(p), \phi(q)] = [\omega(p), \omega(q)]$. Now for any $T \in \mathrm{End}_K M$, $p \in P$ we have $[\phi(p), T] = p(T)$, by definition of the B-module action that $\mathrm{End}_K M$ inherits from M. Thus our condition is $\omega([p, q]) - p(\omega(q)) + q(\omega(p)) = [\omega(p), \omega(q)]$, or $-2d\omega = [\omega, \omega]$, as required.

2.2.6. PROPOSITION. *Let M be a K-algebra. The action of $g \in \mathrm{Aut}_K M$ on a B-algebra structure $\phi: P \to \mathrm{Der}_k M$ is given by $g^{-1}\phi g = \phi + dg$, where $dg \in \Omega^1(P, \mathrm{Der}_K M)$ is given by $(dg)(p) = g^{-1}(pg)$ (cf. 1.4.7).*

Proof. For $p \in P$, $m \in M$ we have $(dg)(p)(m) = (g^{-1}(pg))(m) = g^{-1}(\phi(p)(g(m)) - g(\phi(p)(m)) = g^{-1}(\phi(p)(g(m)) - \phi(p)(m) = (g^{-1}\phi g - \phi)(p)(m)$. Thus $g^{-1}\phi g - \phi = dg$, as required.

2.2.7. *Remark.* Let us fix a B-algebra structure $\phi: P \to \mathrm{Der}_k M$. By 2.2.5, all other B-algebra structures on M are of the form $\phi + \omega$, where $\omega \in \Omega^1(P, \mathrm{Der}_K M)$ and $d\omega = -\frac{1}{2}[\omega, \omega]$. Now $g^{-1}(\phi + \omega)g = g^{-1}\phi g + g^{-1}\omega g = \phi + g^{-1}\omega g + dg$. Thus, g acts on ω by $g^{-1}(\omega) = g^{-1}\omega g + dg$.

Thus the orbit of ω under $\mathrm{Aut}_K M$ is a connection with values in $\mathrm{Der}_K M$; the condition $d\omega = -\frac{1}{2}[\omega, \omega]$ means that this connection is flat. Therefore 2.2.6 can be restated as follows.

2.2.8. *Let one B-algebra structure on M be given. Then the set of all B-algebra structures on M can be identified with the set of flat connections with values in* $\mathrm{Der}_K M$.

2.2.9. *Remark.* The $K[P]$-module structures on M were classified in [Jac] for the case $\dim_K P = 1$, M finite-dimensional.

2.3. *Split B-structures.*

2.3.1. LEMMA. *Let M be a B-module. Let* $h: K[[X]] \to \mathrm{End}_K M$ *be a K-algebra homomorphism. Write* $x = h(X)$, $\exp x = h(\exp X)$. *If characteristic* $k = p > 0$ *assume that* $x^p = 0$.) *Set* $g = \exp x \in \mathrm{Aut}_K M$. *Then*

$$dg(q) = -\sum_{i>0} (-1)^i \frac{(\mathrm{ad}\, x)^{i-1}}{i!}(qx) \quad \text{for} \quad q \in P.$$

Proof. We will use the notation of Proposition 1.4.3: for $b \in B$, we define $b^\times \in \mathrm{End}_k M$ by $b^\times(m) = bm$, $m \in M$.

Note that in the algebra of formal power series in two non-commuting indeterminates X and Y we have $(\exp X)^{-1} Y (\exp X) = \exp(\mathrm{ad}(-X))(Y)$. Thus for $g = \exp x$, $q \in P$ we have

$$g^{-1} q^\times g = \exp(\mathrm{ad}(-x))(q^\times) = \sum_{i \geq 0} \frac{(-1)^i}{i!} (\mathrm{ad}\, x)^i (q^\times).$$

Now $(\mathrm{ad}\, x)(q^\times) = [x, q^\times] = -qx$. Thus

$$g^{-1} q^\times g = q^\times - \sum_{i>0} \frac{(-1)^i}{i!} (\mathrm{ad}\, x)^{i-1}(qx).$$

So

$$(dg)(q) = g^{-1}(qg) = g^{-1}(q^\times g - g q^\times) = g^{-1} q^\times g - q^\times$$

$$= -\sum_{i>0} \frac{(-1)^i}{i!} (\mathrm{ad}\, x)^{i-1}(qx),$$

as required.

2.3.2. PROPOSITION. *Let* $M = K \otimes_r M^\circ$ *be a split B-module.*

(i) *Let y be an element of $End_k M^\circ \subseteq \mathrm{End}_K M$ which is the value at Y of a K-algebra homomorphism $K[[Y]] \to \mathrm{End}_K M$. (If characteristic $k = p > 0$, assume $y^p = 0$.) Let $x = \lambda y \in \mathrm{End}_K M$. Then $d(\exp x) = (d\lambda)x \in \Omega^1(P, \mathrm{End}_K M)$ where for $\lambda \in K^*$, $d\lambda \in \Omega^1(P, K)$ is defined by $(d\lambda)(q) = \lambda^{-1}(q(\lambda))$.*

(ii) *If $m_1, m_2, \ldots$ is a k-basis of M° and $gm_i = \lambda_i m_i$ for $g \in \mathrm{End}_K M$, $\lambda_i \in K^*$, then $(dg)(q)(m_i) = (d\lambda_i)(q)(m_i)$ for $q \in P$.*

Proof. The second assertion follows directly from the definition. For the first assertion, we use the lemma. Note that for $q \in P$, we have $qx = q(\lambda y) = q(\lambda)y$; thus, $(\mathrm{ad}\, x)(qx) = 0$, and the lemma gives

$$(d \exp x)(q) = q(\lambda)y = \lambda^{-1}q(\lambda)x = ((d\lambda)x)(q),$$

as required.

2.4. *Linearly compact Lie K/k-algebras.*

2.4.1. *Definition* (cf. [Guil, Definition 2.1]). A linearly compact Lie K/k-algebra is a Lie K/k-algebra P whose underlying K-vector space structure is linearly compact, and for which the structure maps $[\ ,\ ]: P \times P \to P$ and $\partial: P \to \mathrm{Der}_k K$ are continuous. (Here K and $\mathrm{Der}_k K$ are discrete. A topological vector space is linearly compact iff it is isomorphic as a topological vector space to a product of copies of K.)

2.4.2. *Remark.* Note that $\partial^{-1}(0)$ is an open ideal of P which acts trivially on K. Thus, every open subspace (subalgebra) (ideal) $\tilde{P}$ of P contains an open subspace (subalgebra) (ideal) which acts trivially on K—namely, $\tilde{P} \cap \partial^{-1}(0)$.

2.4.3. *Definition.* Let P be a Lie K/k-algebra which is either linearly compact or discrete. Let M be a $K[P]$-algebra. We say that M is a linearly compact (discrete) $K[P]$-algebra if M is a linearly compact (discrete) topological vector space, in such a way that the module action $P \times M \to M$ is continuous.

2.4.4. *Remark.* If M is a linearly compact (discrete) $K[P]$-algebra, then each $b \in B = K[P]$ acts continuously on M. If P is discrete and each $p \in P$ acts continuously on M, then M is a linearly compact (discrete) $K[P]$-algebra.

2.4.5. PROPOSITION. *Let P be a linearly compact Lie K/k-algebra, and let M be a $K[P]$-algebra which is linearly compact as a topological* *vector space. Then M is a linearly compact $K[P]$-algebra iff the following conditions hold.*

(i) *For each $p \in P$, the map $M \to M$ sending m to pm for $m \in M$ is continuous.*
(ii) *For each open subspace U of M, the set $P_U = \{p \in P : pM \subseteq U\}$ is open in P.*

Proof. We must show that the structure map $\psi : P \times M \to M$ is continuous iff (i) and (ii) hold.

First assume that ψ is continuous. Then (i) holds. Let U be an open subspace of M. Then there are open subspaces P_0 of P, M_0 of M with $\psi(P_0, M_0) \subseteq U$. Write $M = M_0 + Km_1 + \cdots + Km_r$. There are open subspaces $P_1, \ldots, P_r$ of P with $\psi(P_i, m_i) \subseteq U$. Then $P_0 \cap P_1 \cap \cdots \cap P_r \cap \bar{\partial}^1(0)$ is an open subspace contained in P_U, so P_U is open.

Conversely, assume that (i) and (ii) hold. We will show that ψ is continuous at each $(p, m) \in P \times M$. Let U be an open subspace of M. Let M_0 be an open subspace of M with $\psi(p, M_0) \subseteq U$. Then $\psi(p + P_U, m + M_0) \subseteq \psi(p, m) + \psi(p, M_0) + \psi(P_U, M) \subseteq \psi(p, m) + U$. Thus ψ is continuous.

2.4.6. Corollary. *Let M be a linearly compact $K[P]$-algebra. Let N be an open subspace of M. Then the stabilizer of N in P is open.*

Proof. The stabilizer of N contains P_N.

2.4.7. Corollary. *Let $\phi : P \to \mathrm{Der}_k M$ define a linearly compact B-algebra structure on the linearly compact K-algebra M. Let $\tilde{\phi} = \phi + \omega$ be another B-algebra structure on M, where $\omega : P \to \mathrm{Der}_k M$ satisfies $d\omega = -\frac{1}{2}[\omega, \omega]$ (cf. 2.2.5). Then $\tilde{\phi}$ is a linearly compact B-algebra structure iff the following conditions hold.*

(i) *For each $p \in P$, $\omega(p) : M \to M$ is continuous.*
(ii) *For each open subspace U of M, the subspace $P_U = \{p \in P : \omega(p)(M) \subseteq U\}$ is open.*

Proof. First, let us note that $\tilde{\phi}$ is a linearly compact B-algebra structure on M iff the K-bilinear map $\psi : P \times M \to M$ given by $\psi(p, m) = \omega(p)(m)$ for $p \in P$, $m \in M$ is continuous. It is clear that if ψ is continuous, then conditions (i) and (ii) hold.

Next, assume that (i) and (ii) hold. Let U be an open subspace of M. Let $p \in P$, $m \in M$. Since $\omega(p)$ is continuous, we can find an open subspace

M_0 of M with $\psi(p, M_0) \subseteq U$. Thus $\psi(p + P_U, m + M_0) \subseteq \psi(p, m) + \psi(p, M_0) + \psi(P_U, M) \subseteq \psi(p, m) + U$. This shows that ψ is continuous.

2.4.8. PROPOSITION. *Let P be a linearly compact Lie K/k-algebra. Let M be a $K[P]$-module. Then M is a discrete $K[P]$-module iff M^* is a linearly compact $K[P]$-module.*

Proof. Let us first assume that M is a discrete $K[P]$-module. We will verify (i) and (ii) of 2.4.5 for M^*.

Fix $p \in P$, $m \in M$. If $x \in M^*$ vanishes on m and on pm, then px vanishes on m. Thus the map $x \to px$ is continuous.

Let U be an open subspace of M^*. Say $U = ann_M(Km_1 + \cdots + Km_r)$, $m_i \in M$. Then $ann_P(m_1) \cap \cdots \cap ann_P(m_r) \cap \partial^{-1}(0)$ is open and contained in P_U. So (ii) holds.

Next, assume that M^* is a linearly compact $K[P]$-module. To show that M is a discrete $K[P]$-module, we must show that for each $m \in M$, $ann_P m$ is open in P. Now $U = ann_{M^*}(m)$ is open in M^*. If $p \in P_U$, $x \in M^*$, then $px \in U$, so $0 = (px)(m) = p(x(m)) - x(pm)$. Thus $P_U \cap \partial^{-1}(0) \subseteq ann_P m$, so $ann_P m$ is open.

2.5. *Modules of finite kind.* Let P be a linearly compact Lie K/k-algebra.

2.5.1. *Definition.* A B-module M is of finite kind if there exist $x_1, \ldots, x_r \in M$ and an open Lie K/k-subalgebra $\tilde{P}$ of P, such that $M = Bx_1 + \cdots + Bx_r$ and $0 = \tilde{P}x_i$, all i. We say that M is of cofinite kind if M is a linearly compact B-module, and the continuous dual M^* of M is a B-module of finite kind.

2.5.2. Note that for each $x \in M$, $ann_P x = \{p \in P : px = 0\}$ is a Lie K/k-subalgebra of P.

PROPOSITION. *Let $x \in M$. If $ann_P x$ is open, then $ann_P px$ and $ann_P \lambda x$ are open for all $p \in P$, $\lambda \in K$.*

Proof. Write $\tilde{P} = ann_P x$. Then $ann_P px$ and $ann_P \lambda x$ contain the open subspaces $(\text{ad } p)^{-1}(\tilde{P}) \cap \tilde{P}$ and $\partial^{-1}(0) \cap \tilde{P}$, respectively.

2.5.3. COROLLARY. *Let M be a B-module of finite kind. Then every finite subset of M is annihilated by some open Lie K/k-subalgebra of P.*

Proof. Note that if P_1 annihilates y_1 and P_2 annihilates y_2, then $P_1 \cap P_2$ annihilates y_1, y_2 and thus $y_1 + y_2$. Since B is generated by P and K, our result thus follows by repeated application of 2.5.2.

2.5.4. *Remark.* The above corollary shows that a finitely-generated B-module M is of finite kind iff M is a discrete B-module (2.4.3).

2.5.5. THEOREM. *Let M be a B-module of finite kind. Then every submodule and every quotient module of M is of finite kind.*

Proof. The assertion for quotient modules is clear. Then by 2.5.4, it suffices to show that every submodule N of M is finitely generated.

We have $M = Bx_1 + \cdots + Bx_r$, $\tilde{P}x_i = 0$. Now the coradical filtration $\{B_n\}$ of B is given by $B_{-1} = 0$, $B_0 = K$, $B_n = K + P + P^2 + \cdots + P^{n-1}$. Let $X = Kx_1 + \cdots + Kx_r$. Define $M_n = B_nX$. Then $B_iM_j \subseteq M_{i+j}$. Thus $grM = \oplus_{i\geq 0} M_i/M_{i-1}$ is a module over $grB = \oplus_{i\geq 0} B_i/B_{i-1}$. Replacing $\tilde{P}$ by $\partial^{-1}(0) \cap \tilde{P}$, we may assume that $\tilde{P}(K) = 0$. Since the actions of $p_1, p_2 \in P$ on grM commute, it follows that $\tilde{P}$ annihilates grM. Note also that the action of $p \in P$ on grM commutes with the action of $\lambda \in K$. It follows that the grB-action factors through the Noetherian symmetric algebra $S(P/\tilde{P})$. Thus, every grB-submodule of grM is finitely generated.

We apply this result to grN, where N is graded via $N_n = N \cap M_n$. Let $y_1, \ldots, y_s$ be elements of N whose images $\bar{y}_1, \ldots, \bar{y}_s$ in grN generate. We will show by induction on n that $N_n \subseteq By_1 + \cdots + By_s$. This is clear for $n = 0$. Suppose that $N_n \subset By_1 + \cdots + By_s$ for $n < t$, and let $y \in N_t$. Then $\bar{y} = \bar{b}_1\bar{y}_1 + \cdots + \bar{b}_s\bar{y}_s$, some $b_1, \ldots, b_s \in B$. But this means that $y - b_1y_1 - \cdots - b_sy_s \in N_{t-1}$, and we are done by induction.

2.5.6. PROPOSITION. *Let $\tilde{P}$ be an open Lie K/k-subalgebra of P, and let M be a $K[\tilde{P}]$-module. Then M is a $K[\tilde{P}]$-module of finite kind iff $K[P] \otimes_{K[\tilde{P}]} M$ is a $K[P]$-module of finite kind.*

Proof. One direction is clear. Suppose that $M' = K[P] \otimes_{K[\tilde{P}]} M$ is a $K[P]$-module of finite kind. We may take $K[P]$-module generators of M' of the form $1 \otimes m_1, \ldots, 1 \otimes m_r$, where $m_i \in M$. Since 1 is a member of a basis of $K[P]$ as a free right $K[\tilde{P}]$-module (by 2.1.4), we have $M = K[\tilde{P}]m_1 + \cdots + K[\tilde{P}]m_r$. Again by 2.1.4, if W is an open Lie K/k-subalgebra of p with $W(1 \otimes m_i) = 0$ all i, then $(W \cap \tilde{P})m_i = 0$ all i. Thus M is a $K[\tilde{P}]$-module of finite kind, and the proof is complete.

2.6. We will work mainly with B-modules which are duals of modules of finite kind.

PROPOSITION. *Let M be a B-module of finite kind. Then there exists an open subspace N of M^*, such that for all $0 \neq f \in M^*$, Bf is not contained in N.*

Proof. Take a finite dimensional K-subspace X of M with $BX = M$. Let $N = X^{\perp} \subset M^*$. Suppose $f \in M^*$, and $Bf \subseteq N$. Then for $x \in X$, we have $f(x) = 0$. For $p \in P$, we have $f(px) = p(f(x)) - (pf)(x) = 0$. Since P generates B, it follows that f vanishes on $BX = M$, i.e. $f = 0$.

2.6.1. We establish a dual version of Proposition 2.6.

PROPOSITION. *Let M be a discrete B-module. Assume that M^* has an open subspace N, such that Bf is not contained in N for any $0 \neq f \in M^*$. Then M is a B-module of finite kind.*

Proof. We have that the stabilizer $\tilde{P}$ of N in P is an open Lie K/k-subalgebra of P. Let V be the annihilator of N in M. Since N is open, V is finite-dimensional. By 2.5.4, it suffices to show that $M = K[P]V$.

Let $f \in M^*$. By assumption, there exists $b \in B$ such that bf is not contained in N. Since N is open, this means that bf does not vanish on V. Thus f does not vanish on $K[P]V$. Thus $K[P]V = M$.

3. The formal groups of Ritt. In this section, we assume that B is a cocommutative Hopf K/k-algebra.

3.1. *Definition.* Let A be a commutative associative B-algebra with unique maximal ideal J. Assume that A has a complete linear topology. We say that A is a formal B-group if it is equipped with continuous B-algebra homomorphisms

$$\Delta: A \to A \hat{\otimes} A \text{ (comultiplication)}$$

$$\epsilon: A \to K \text{ (counity)}$$

such that $(\Delta \hat{\otimes} id)\Delta = (id \hat{\otimes} \Delta)\Delta$, and $(id \hat{\otimes} \epsilon)\Delta \;\; id = (\epsilon \hat{\otimes} id)\Delta$.

3.1.1. *Remark.* Since ϵ is surjective and J is the unique maximal ideal of A, we have $J = \ker \epsilon$. Since ϵ is a B-module map, we have in particular that J is B-stable.

3.2. We define the Lie coalgebra of the formal group (A, Δ).

For $x \in J$, we have $\Delta(x) = x \otimes 1 + 1 \otimes x \bmod J \hat{\otimes} J$. Thus for x, $y \in J$, we have $\Delta(xy) = xy \otimes 1 + x \otimes y + y \otimes x + 1 \otimes xy \bmod J \hat{\otimes} J^2 + J^2 \hat{\otimes} J$. Now let $\tau: A \hat{\otimes} A \to A \hat{\otimes} A$ be the map which interchanges the factors. Since B is cocommutative, τ is a B-module map. We have $(id - \tau)\Delta(J) \subseteq J \hat{\otimes} J$, and $(id - \tau)\Delta(J^2) \subseteq J \hat{\otimes} J^2 + J^2 \hat{\otimes} J$. By continuity, $(id - \tau)\Delta(clJ^2) \subseteq cl(J \hat{\otimes} J^2 + J^2 \hat{\otimes} J) = J \otimes clJ^2 + clJ^2 \hat{\otimes} J$. Thus $(id - \tau)\Delta$ induces a B-module map $\nu: J/clJ^2 \to J/clJ^2 \hat{\otimes} J/clJ^2$. We will write $M = J/clJ^2$.

The map $\Delta': A \to A \hat{\otimes} A$ given by $\Delta'(a) = \Delta(a) - 1 \otimes a - a \otimes 1$ is coassociative, and sends J into $J \hat{\otimes} J$. Clearly $(id - \tau)\Delta = (id - \tau)\Delta'$. Write $\nu' = (id - \tau)\Delta': J \to J \hat{\otimes} J$, and let $q: J \to M$ be the quotient map. Then $\nu q = (q \otimes q)\nu'$.

It is clear that $(id + \tau)\nu = 0$ (coanticommutativity). Now let $\sigma: J \hat{\otimes} J \hat{\otimes} J \to J \hat{\otimes} J \hat{\otimes} J$ permute the factors cyclicly. Writing

$$(\Delta' \hat{\otimes} id)\Delta'(x) = (id \hat{\otimes} \Delta')\Delta'(x) = \sum_i x_{1i} \hat{\otimes} x_{2i} \hat{\otimes} x_{3i} \quad \text{for} \quad x \in J.$$

we have

$$(id \hat{\otimes} \nu')\nu'(x) = \sum_i x_{1i} \hat{\otimes} x_{2i} \hat{\otimes} x_{3i} - \sum_i x_{1i} \hat{\otimes} x_{3i} \hat{\otimes} x_{2i}$$

$$- \sum_i x_{3i} \hat{\otimes} x_{1i} \hat{\otimes} x_{2i} + \sum_i x_{3i} \hat{\otimes} x_{2i} \hat{\otimes} x_{1i}.$$

Thus $(id + \sigma + \sigma^2)(id \hat{\otimes} \nu')\nu' = 0$ (co-Jacobi identity). Since $\nu q = (q \otimes q)\nu'$, ν also satisfies the co-Jacobi identity.

Definition. (M, ν) is the Lie B-coalgebra of the formal B-group (A, Δ).

3.2.1. *Definition.* The topological dual M^* of M, equipped with the map $\nu^*: M^* \hat{\otimes} M^* \to M^*$, is called the *Lie B-algebra* of the formal B-group (A, Δ).

3.2.2. *Remark.* The Lie algebra of a formal B-group may be zero.

3.2.3. *Remark.* In contrast to the situation of [Nich 2], the Lie coalgebra of a formal B-group need not be the union of its finite-dimensional sub coalgebras.

3.3. Let P be a linearly compact Lie K/k-algebra, and let $B = K[P]$.

3.3.1. *Definition.* A formal B-group (A, Δ) is called a *Ritt group* if A is generated as a topological algebra by a B-module of finite kind.

3.3.2. *Remark.* The Lie coalgebra of a Ritt group is a B-module of finite kind.

3.4. Every linearly compact Lie $K[P]$-algebra G of cofinite kind is the Lie algebra of a formal group of Ritt. Indeed, let M be the continuous dual of G. Let A be the completion of the symmetric algebra $S(M)$ with respect to the $M \cdot S(M)$-adic topology. The elements of A can be represented in the form $a = \Sigma_{i=0}^{\infty} a_i$, with $a_i \in S(M)_i$; A has unique maximal ideal $J = \{a \in A : a_0 = 0\}$. Note that $clJ^2 = \{a \in A : a_0 = a_1 = 0\}$. We will use the Lie structure of G to define the coproduct.

Since the Lie bracket $[\ ,\ ]: G \times G \to G$ is continuous, it defines a linear map $[\ ]: G \hat{\otimes} G \to G$. The continuous dual $\nu: M \to M \otimes M$ of $[\ ]$ is a Lie coalgebra structure map for M. Then $\Delta: M \to A \hat{\otimes} A$ is defined from ν using the Campbell-Hausdorff formula as in [Nich 2, section 5]; if $x \in M$,

$$\nu(x) = \sum_i x_i \otimes x_i', \qquad (id \otimes \nu)\nu(x) = \sum_{i,j} x_i \otimes x_{ij}' \otimes x_{ij}'',$$

then

$$\Delta(x) = x \hat{\otimes} 1 + 1 \hat{\otimes} x + \frac{1}{2} \sum_i x_i \hat{\otimes} x_i'$$

$$+ \frac{1}{12}\Big(\sum_{i,j} x_i x_{ij}' \hat{\otimes} x_{ij}'' + \sum_{i,j} x_{ij}'' \hat{\otimes} x_i x_{ij}'\Big)$$

$$+ \text{ terms of higher degree.}$$

We extend Δ to all of A by continuity. The counit $\epsilon: A \to K$ is defined by $\epsilon(1) = 1$, $\epsilon(M) = 0$, ϵ continuous. Since A is generated as a topological algebra by M, we see that (A, Δ) is a formal group of Ritt whose Lie algebra is G.

3.5. Let us discuss the relationship between our definition and that of Ritt.

Ritt considered the case in which P is one-dimensional over K, $B = K[P]$, M is a free B-module, and A is the completed symmetric algebra over M as in 3.4 above. Our generalization permits more general P and more general M. By including arbitrary B-modules of finite kind, we can allow M to be finite-dimensional. When P is finite dimensional, the condition that the Lie coalgebra of a Ritt group be free as a B-module excludes the Lie algebras of Cartan type H_n, S_n, K_n. Thus our assumptions include additional important examples.

Instead of Lie coalgebras, Ritt considered their structure constants. They appear in his construction as follows. Since M is free and $P = Kp$ is one-dimensional, we can take elements $m^1. \ldots, m^q$ of M so that $\{p^i m^j\}$ is a K-basis of M. The formal group structure is given by $p^i m^j \to p^i m^j \otimes 1 + 1 \otimes p^i m^j + \Sigma_{k,l,r,s}\, a^{ij}_{klrs} p^k m^l \otimes p^r m^s$ + terms of higher degree. Set $c^{ij}_{klrs} = a^{ij}_{klrs} - a^{ij}_{rskl}$. Then the c^{ij}_{klrs} are the structure constants for $\nu: M \to M \otimes M$—that is, $\nu(p^i m^j) = \Sigma_{k,l,r,s}\, c^{ij}_{klrs} p^k m^l \otimes p^r m^s$. The relation (52) from [Ritt 2]:

$$c^{ij+1}_{klrs} = \partial(p)(c^{ij}_{klrs}) + c^{ij}_{kl-1rs} + c^{ij}_{klrs-1}$$

means simply that ν is a B-homomorphism.

The construction of the Lie algebra with basis $\{e_{ij}\}$ via $[e_{kl}, e_{rs}] = \Sigma_{i,j}\, c^{ij}_{klrs} e_{ij}$ leads to some difficulties. In the first place, the action of P is difficult to describe. Second, the expression $\Sigma_{i,j}\, c^{ij}_{klrs} e_{ij}$ may be infinite—as happens, for example, for some two-dimensional groups from [Ritt 3]. This reflects the fact that the Lie algebra in question is topological.

4. Guillemin's structure results. Let P be a linearly compact Lie K/k-algebra (2.4.1), and write $B = K[P]$. We wish to study the structure of linearly compact Lie B-algebras G (2.4.3). It will be convenient to extend some of the results of [Gui 1, section 6] to B-algebras. We will change notation from P to L to avoid confusion later on.

4.1. Let L be a linearly compact Lie K/k-algebra. Let G be a linearly compact Lie $K[L]$-algebra.

4.1.1. Lemma (cf. [Gui 1, Lemma 6.2]). *Let S be an open subspace of G. Then $N_L(S) = \{x \in L : xS \subseteq S\}$ is open in L.*

Proof. Since the action $L \times G \to G$ is continuous, we can find open subspaces U_0 of L, V of G with $U_0 V \subseteq S$. Write $G = V + Kx_1 + \cdots + Kx_r$. Again by continuity, there are open subspaces U_i of L with $U_i x_i \subseteq S$. Let $U = U_0 \cap U_1 \cap \cdots \cap U_r \cap \partial^{-1}(0)$. Then U is an open subspace of L, and $UG \subseteq S$. Thus $US \subseteq S$, and so $N_L(S) \supseteq U$ is open.

4.1.2. Lemma (cf. [Gui 1, Proposition 6.2]). *Let A be a subalgebra of G. Then $N_L(A) \subseteq N_L(D_G A)$, where $D_G A = \{x \in A : [x, G] \subseteq A\}$.*

Proof. Let $p \in N_L(A)$, $x \in D_G A$. Since $x \in A$, we have $px \in A$. For $y \in G$, we have $p[x, y] = [px, y] + [x, py]$ since G is a $K[L]$-algebra. We deduce $[px, G] \subseteq A$, as required.

4.1.3. Proposition (cf. [Gui 1, Proposition 6.2]). *Let H be a closed maximal ideal of G. Then $N_L H$ is open.*

Proof. Follow [Gui 1].

4.2. We summarize some straightforward extensions of [Gui 1, section 6] to the situation of 4.1.

4.2.1. Let I be a closed $K[L]$-ideal of G. Let J be a closed maximal ideal of I [Gui 1, Proposition 6.1]. Filter I as in [Gui 1, section 6.3]: $I^0 = I$, $I^1 = J$, $I^{k+1} = D_L(I^k) = \{x \in I^k : [L, x] \subseteq I^k\}$. By induction, each I^k is an ideal of I. The normalizer N of J in L is open, by 4.1.3. Form $gr(I) = \oplus\Sigma_{k=0}^{\infty} I^k/I^{k+1}$, a graded Lie algebra. Since $[N, I^k] \subseteq I^k$[Gui 1, 6.3], the bracket map $L \times I \to I$ induces a map $L/N \times gr(I) \to gr(I)$; in fact, $W = L/N$ acts as commuting derivations of $gr(I)$ of degree -1. Thus we can define an algebra map $\alpha: gr(I) \to \mathrm{Hom}_K(S(W), gr(I))$ via $\alpha(a)(y) = ya$ for $y \in S(W)$, $a \in gr(I)$. (Here $ya = (ady)(a)$ is the extension to the symmetric algebra $S(W)$ of the action of W on $gr(I)$. $\mathrm{Hom}_K(S(W), gr(I))$ is a convolution algebra (1.10), with the coalgebra structure on $S(W)$ being defined by the algebra map $\Delta: S(W) \to S(W) \otimes S(W)$, $\Delta(w) = w \otimes 1 + 1 \otimes w$ for $w \in W$.) The projection onto degree zero $\pi: gr(I) \to I/J$ is an algebra map, and thus induces an algebra map $\psi: gr(I) \to \mathrm{Hom}_K(S(W), I/J)$, given for $a \in gr(I)_r$, $b \in S(W)_s$ by $\psi(a)(b) = \pi(\alpha(a)b) = \delta_{rs}ba$.

4.2.2. PROPOSITION. *The map of graded algebras $\psi: gr(I) \to \mathrm{Hom}_K(S(W), I/J)$ is injective. When characteristic $K = 0$, we have $\mathrm{Hom}_K(S(W), I/J) \simeq S(W^*) \otimes I/J$ as algebras, and thus an injective algebra map $\psi: gr(I) \to S(W^*) \otimes I/J$.*

Proof. To show that ψ is injective, follow [Gui 1, Lemma 6.3].

There is a natural inclusion $W^* = \mathrm{Hom}_K(W, K) \to \mathrm{Hom}_K(S(W), K)$, giving rise to an algebra map from $S(W^*)$ to the convolution algebra $\mathrm{Hom}_K(S(W), K)$. When characteristic $K = 0$, this algebra map is an isomorphism of graded algebras; thus, $\mathrm{Hom}_K(S(W), I/J) = \mathrm{Hom}_K(S(W), K) \otimes I/J \simeq S(W^*) \otimes I/J$.

4.3. We will apply 4.2 with L being the cross product of P and G.

4.3.1. *Definition.* The *cross product* of P and G is the linearly compact Lie K/k-algebra L with underlying linearly compact K-vector space $P \oplus G$, with Lie k-algebra structure $[(p_1, g_1), (p_2, g_2)] = ([p_1, p_2], p_1g_2 - p_2g_1 + [g_1, g_2])$, and with action on K given by $\partial((p, g))(\lambda) = \partial(p)(\lambda)$.

4.3.2. *Remark.* Note that a $K[L]$-ideal of G is the same thing as an ideal of L which is contained in G, or as a B-ideal of G.

4.3.3. Note that G is a linearly compact Lie $K[L]$-algebra. As in 4.2, let I be a closed $K[P]$-ideal of G, and let J be a closed maximal ideal of I.

4.3.4. We write $I^\infty = \bigcap_r D_L^r(J)$ for the largest $K[P]$-ideal of I contained in J (cf. [Gui 1, Proposition 2.4]). As in [Gui 1, section 6.4] we have the following.

PROPOSITION. *Assume that I/J is non-abelian. Let A be a proper closed ideal of I. If $I^\infty \subseteq A$, then $A \subseteq J$.*

4.3.5. COROLLARY. *Assume that I/J is non-abelian. Then there are no closed $K[P]$-ideals strictly contained between I and I^∞.*

4.3.6. COROLLARY. *If the closure of $[I, I]$ is I, then I^∞ is a maximal proper closed B-ideal of G in I.*

Proof. Since $[I, I] \subset J$ is ruled out, 4.3.5 applies.

4.4. From now on, we will assume that K is algebraically closed of characteristic 0, and that G is a B-module of cofinite kind (2.5.1).

4.4.1. Observe that for any subspaces X, Y of G, we have $cl[X, Y] = cl[clX, clY]$ where cl denotes the closure operator.

4.4.2. *Definition.* The *derived series* of G is given by $\mathfrak{D}^0 G = G$, $\mathfrak{D}^{i+1}(G) = cl[\mathfrak{D}^i G, \mathfrak{D}^i G]$.

4.4.3. PROPOSITION.

(i) *Every strictly decreasing sequence of closed $K[P]$-subspaces of G is finite.*
(ii) *Each $\mathfrak{D}^i G$ is a closed $K[P]$-ideal.*

Proof. The first assertion follows from the fact that G^* is of finite kind. For (ii), we need check only that $[\mathfrak{D}^i G, \mathfrak{D}^i G]$ is $K[P]$-invariant, and this follows readily by induction.

4.4.4. *Definition.* A B-subalgebra M of G is solvable if M is closed, and $\mathfrak{D}^n M = 0$ for some n.

4.4.5. PROPOSITION. *The Lie B-algebra G has a largest solvable B-ideal.*

Proof. Let R_1, R_2 be solvable B-ideals of G. The B-ideal $R_1 + R_2$ is closed [Gui 1, Proposition 1.2, Corollary 2], and is solvable since its image in $(R_1 + R_2)/R_2 \simeq R_1/R_1 \cap R_2$ is solvable. To complete the proof, we will show that if $0 = I_0 \subset I_1 \subset \ldots$ is an increasing sequence of solvable B-ideals, then $cl(\bigcup_n I_n)$ is solvable.

Write $R = \bigcup_n I_n$, $S = clR$. Set $R^1 = [R, R]$, $R^{t+1} = [R^t, R^t]$. Then $\mathfrak{D}^t S = clR^t$ by induction. Suppose that for some t we have $\mathfrak{D}^t(S) = \mathfrak{D}^{t+1}(S) \neq 0$. Let J be a maximal proper closed B-ideal of G in $\mathfrak{D}^t(S)$ (4.3.6).

If $I_n \cap \mathfrak{D}^t S \subseteq J$ for all n, then $R \cap \mathfrak{D}^t S \subseteq J$. But this gives $R^t \subseteq J$, and thus $\mathfrak{D}^t S = clR^t \subseteq J$, which is impossible.

Thus we can find n so that $I_{n-1} \cap \mathfrak{D}^t S \subseteq J$, but $I_n \cap \mathfrak{D}^t S \not\subseteq J$. Since $(I_n \cap \mathfrak{D}^t S) + J$ is a closed B-ideal of $\mathfrak{D}^t S$, we have $(I_n \cap \mathfrak{D}^t S) + J = \mathfrak{D}^t S$ by the maximality of J. Thus there is a B-algebra surjection

$$\frac{I_n \cap \mathfrak{D}^t S}{I_{n-1} \cap \mathfrak{D}^t S} \to \frac{\mathfrak{D}^t S}{J}.$$

Thus $(I_n \cap \mathfrak{D}^t S)/(I_{n-1} \cap \mathfrak{D}^t S)$ is not solvable. But there is an injection

$$\frac{I_n \cap \mathfrak{D}^t S}{I_{n-1} \cap \mathfrak{D}^t S} \to \frac{I_n}{I_{n-1}}, \quad \text{and} \quad \frac{I_n}{I_{n-1}}$$

is solvable. This contradiction establishes that $\mathfrak{D}^t S = \mathfrak{D}^{t+1} S \neq 0$ is impossible. Thus (4.4.3) S is solvable.

4.4.6. *Definition.* The largest solvable B-ideal of G is called the *radical* of G.

4.5. We now assume that G has zero radical.

4.5.1. PROPOSITION. *Let I be a minimal closed B-ideal of G. Then I has no proper closed B-ideals of itself.*

Proof. Let $A \neq I$ be a closed B-ideal of I. Then $cl[A, A]$ is another. If $A = cl[A, A]$, then the inclusion $[G, [A, A]] \subseteq [[G, A], A] \subseteq [I, A] \subseteq A$ shows that A is an ideal of G, and thus (by the minimality of I) that $A = 0$. So if $A \neq 0$, $cl[A, A]$ is properly contained in A. By 4.4.3, we can then find a non-zero A with $[A, A] = 0$.

We now invoke the machinery of 4.2. Filtering A via $A \cap I^n$, we obtain in $gr(I)$ an ideal $H \simeq gr(A)$, with $[H, H] = 0$. Now $gr(I) \hookrightarrow S(W^*) \otimes I/J$. If $H_r \neq 0$, then by [Gui 1, Proposition 5.1] $H_r = U \otimes I/J$ for some subspace U of $S(W^*)_r$. But then $[H, H] \supseteq U^2 \otimes I/J \neq 0$. Thus the case $A \neq 0$ is impossible, and we are done.

4.6. We wish to determine the structure of the minimal closed B-ideals of G. By Proposition 4.5.1, we may assume that G itself has no proper closed B-ideals.

THEOREM. *Let K be algebraically closed of characteristic* 0. *Let G be a linearly compact Lie $K[P]$-algebra of cofinite kind. Assume that G has no proper closed $K[P]$-ideals.*

(i) *If G has no proper closed ideals, then G is simple, and G is K-isomorphic to one of the complete Lie algebras of Cartan type W_n, S_n, H_n, K_n, or to a finite-dimensional simple Lie algebra.*
(ii) *If G is not simple, then there exists an open Lie K/k-subalgebra $\tilde{P}$ of P and a K-simple Lie $K[\tilde{P}]$-algebra S of cofinite kind such that $G \simeq \mathrm{Hom}_{K[\tilde{P}]}(K[P], S)$ as a $K[P]$-algebra. (Here $\mathrm{Hom}_{K[\tilde{P}]}(K[P], S)$ has the algebra structure defined in* 1.10.5). *In particular, G is isomorphic as a Lie K-algebra to $\hat{S}(P/\tilde{P}) \hat{\otimes} S$, where $\hat{S}(P/\tilde{P})$ is the completed symmetric algebra over the K-space $P/\tilde{P}$, and $[a \otimes s, a' \otimes s'] = aa' \otimes [s, s']$ for $s, s' \in S$, a, $a' \in \hat{S}(P/\tilde{P})$.*
(iii) *The Lie $K[P]$-algebra structure on G is completely determined by the Lie $K[\tilde{P}]$-algebra structure on the simple Lie algebra S of* (ii). *In particular, there is a one-to-one correspondence between Lie $K[P]$-structures on G and pairs $(\tilde{P}, \varphi)$, where $\tilde{P}$ is an open Lie K/k-subalgebra of P and $\varphi: \tilde{P} \to \mathrm{End}_k S$ is a Lie $K[\tilde{P}]$-algebra structure of cofinite kind on a simple K-Lie algebra S.*

The proof of assertion (i) is given in [Gui 1, Proposition 4.3]. Assertion (iii) follows directly from (ii). We give the proof of (ii) below, in a number of steps.

4.6.1. Let H be a proper closed maximal ideal of G [Gui 1, Proposition 6.1].

PROPOSITION. *The Lie algebra G/H is non-abelian.*

Proof. If $[G, G] \subseteq H$, then $cl[G, G]$ would be a proper closed $K[P]$-ideal of G, and thus zero. But G is not solvable, hence G/H is non-abelian.

4.6.2. We now invoke 4.2 with L replaced by P, $I = G$, and J replaced by H. We write $\tilde{P}$ for N, the normalizer of J in L. By Proposition 4.2.2, there is an injective map of graded algebras $\psi: gr(G) \to \mathrm{Hom}_K(S(W), G/H))$, where $W = P/\tilde{P}$.

PROPOSITION. *The map ψ is an isomorphism of graded K-algebras.*

Proof. Note first that $\psi(gr(G)_0) = \mathrm{Hom}_K(K, G/H) \simeq G/H$. Thus $\psi(gr(G)_1)$ is a G/H-submodule of $\mathrm{Hom}_K(W, G/H)$.

Since G/H is simple non-abelian (4.5.1) and K is algebraically closed, we have by [Gui 1, Proposition 5.1] (applied to $\mathrm{Hom}_K(W, K) \otimes G/H \simeq \mathrm{Hom}_K(W, G/H)$) that any proper G/H-submodule of $\mathrm{Hom}_K(W, G/H)$ annihilates some $0 \neq w \in W$. But if $w = \bar{l}$, $l \in L$, and $\psi(\bar{H})(w) = 0$, then $[l, H] \subseteq H$, so $l \in N$, $\bar{l} = 0$. Thus $\psi(gr(G)_1) = \mathrm{Hom}_K(W, G/H)$.

We have $[G/H, G/H] = G/H$. To show that ψ is surjective, and thus complete the proof of the proposition, we will prove the following: if R is an algebra over a field of characteristic zero, and $RR = R$, then for every vector-space W the graded algebra $T = \mathrm{Hom}_K(S(W), R)$ is generated by its components in degrees 0 and 1. We assume inductively that the subalgebra A generated by T_0 and T_1 contains $\Sigma_{i=0}^{n-1} T_i$. Let $\{w_i\}$ be a basis of W, and let $\{e_i\}$ be non-negative integers with $\Sigma e_i = n$. To show that $T_n \subset A$, it suffices to show that A_n contains an element h with $h(\Pi w_i^{e_i}) = r$, h vanishing on the other basis elements of $S(W)_n$, where $r \in R$ is arbitrary. Without loss of generality we may assume $e_1 \geq 1$. Now $r = \Sigma_{j=1}^m r_j r_j'$ for some $r_j, r_j' \in R$ by assumption.

By assumption, there exist $h_j \in A_1$, $h_j' \in A_{n-1}$ with $h_j(w_j) = r_j$, $h_j'(w_1^{e_1-1} \Pi_{i>1} w_i^{e_i}) = r_j'$, and h_j, h_j' vanishing on the other basis elements. Then $h = 1/e_1 \Sigma h_j h_j'$ has the required property, and our proof is complete.

4.6.3. Now we will use the map ψ to establish that $G \simeq \mathrm{Hom}_{K[\tilde{P}]}(K[P], G/H)$. The quotient map $\varphi: G \to G/H$ of $K[\tilde{P}]$-algebras gives rise (1.10.6) to a map $\theta: G \to \mathrm{Hom}_{K[\tilde{P}]}(K[P], G/H)$ of $K[P]$-algebras, given by $\theta(x)(y) = \varphi(yx)$ for $x \in G$, $y \in K[P]$. The major part of Theorem 4.6(ii) is contained in the following.

THEOREM. *The map $\theta: G \to \mathrm{Hom}_{K[\tilde{P}]}(K[P], G/H)$ is an isomorphism of $K[P]$-algebras, and a homeomorphism.*

Proof. We give $M = \mathrm{Hom}_{K[\tilde{P}]}(K[P], G/H)$ the finite-open topology. For fixed $y \in K[P]$, the map $G \to G/H$ sending x to $\theta(x)(y) = \zeta(yx)$ is continuous, since P acts continuously on G. Thus θ is continuous. Thus $\ker\theta$ is a closed $K[P]$-ideal of G. Now $\ker \theta \neq G$, since $\theta(x)(1) = \varphi(x) \neq 0$ for $x \in G$, $x \notin H$. Since G has no proper closed $K[P]$-ideals, we must have $\ker\theta = 0$. Thus θ is injective.

We will pass to filtered modules to show that θ is surjective.

Let us first observe that if $x \in G$, $p_1, \ldots, p_r \in P$ and $\theta(x)(p_1, \ldots, p_r) \neq 0$, then $x \in G^{r+1}$. Since θ is injective, we have $\bigcap_r G^r = 0$. Since each

G^r is closed, this means that G is complete with respect to the filtration $\{G^r\}$—that is, given $\{x_r\}$, $x_r \in G^r$, there exists $x \in G$, $x - \Sigma_{i=1}^r x_r \in G^{r+1}$ all r (see [Gui 1, section 2]).

Now define a filtration on M by: $M^0 = M$, $M^{r+1} = \{T \in M: T(K[P]_r) = 0\}$, where $K[P]_r = K + P + P^2 + \cdots + P^r$ is the rth term of the coradical filtration on $K[P]$. It is easy to see that each M^r is closed, that $\cap M^r = 0$, and that $\theta: G \to M$ is filtration-preserving.

Define a map $\eta: gr(M) \to \mathrm{Hom}_K(S(W), G/H)$ as follows. For $T \in M^r$, $w_1, \ldots, w_r \in W$, $w_i = \tilde{p}_i$, $p_i \in P$, set $\eta(\tilde{T})(w_1 \cdots w_r) = T(p_1 \cdots p_r)$. It is easy to check that η is well-defined and injective, and that $\eta \circ gr(\theta) = \psi$. Since ψ is an isomorphism, $gr(\theta)$ must be surjective. Since G is complete, this gives that θ is surjective, and thus that θ is an isomorphism.

By 1.3.3 $M \simeq \mathrm{Hom}_K(K[P] \otimes_{K[\tilde{P}]} (G/H)^*, K)$ is linearly compact. Since θ is a continuous bijection between linearly compact spaces, it is a homeomorphism. This completes the proof of the Theorem above.

4.6.4. To complete the proof of Theorem 4.6(ii), we must show that $S = G/H$ is $K[\tilde{P}]$-module of cofinite kind. Since the continuous dual G^* is (as above) the $K[P]$-module $K[P] \otimes_{K[\tilde{P}]} S^*$, this follows from 2.5.6.

4.7. Let us derive some consequences of Theorem 4.6. We assume that G is a linearly compact Lie B-algebra of cofinite kind, and that G has zero radical.

4.7.1. THEOREM.

(i) *The sum H of the minimal closed B-ideals of G is direct. Each minimal closed B-ideal H_i is isomorphic to $G_i \hat{\otimes} \hat{S}(V_i)$, for some simple Lie K-algebra G_i and finite-dimensional K-space V_i.*

(ii) *The adjoint action of G on H determines imbeddings of B-algebras*

$$G \to \mathrm{Der} H = \bigoplus_i ((\mathrm{Der} G_i) \hat{\otimes} \hat{S}(V_i) + Id_{G_i} \otimes \mathrm{Der} \hat{S}(V_i))$$

and

$$G/H \to \mathrm{Der}\, H/Inn H = \bigoplus_i ((\mathrm{Der} G_i/Inn G_i) \hat{\otimes} \hat{S}(V_i) + Id_{G_i} \otimes \mathrm{Der}\, \hat{S}(V_i))$$

Remark. Recall that $\mathrm{Der} G_i = Inn G_i$ if G_i is finite-dimensional or of Cartan type W_n, K_n, and $\dim_K(\mathrm{Der} G_i/Inn G_i) = 1$ if G_i is of Cartan type S_n, H_n (with exterior derivation acting as scalar multiplication on the natural homogeneous components of these algebras).

Proof. We first show that the sum of the minimal closed B-ideals is direct. Suppose that $I, I_1, I_2, \ldots, I_r$ are minimal closed B-ideals, and that $I \subseteq I_1 + I_2 + \cdots + I_r$. Since G has zero radical, $[I, I] \neq 0$; thus, $[I, I_i] \neq 0$ for some i. But then $cl[I, I_i]$ is a non zero closed B-ideal contained in both I and I_i, forcing $I = I_i$. This shows that the sum of any finite number of minimal closed B-ideals is direct. If $\{I_i\}_{i \geq 1}$ is an infinite family of distinct minimal closed B-ideals, then define $J_r = \Sigma_{i \geq r} I_i$. Since $[I_{r-1}, J_r] = 0$, we have $I_{r-1} \cap clJ_r = 0$. Then $\{clJ_r\}$ is an infinite decreasing sequence of closed B-ideals. Since G is of cofinite type, this cannot happen. Thus, $H = I_1 \oplus \cdots \oplus I_r$, where $I_1, \ldots, I_r$ are the minimal closed B-ideals. The remaining part of assertion (i) follows directly from Theorem 4.6.

The adjoint action of G on H gives us a map $G \to \mathrm{Der}H$ of Lie algebras, which is verified to be a B-algebra map. The kernel of this map is a closed B-ideal of G; if it were non-zero, it would contain a minimal closed B-ideal I of G. But $I \subseteq H$, so this would give $[I, I] \subseteq [I, H] = 0$, $I \subseteq \mathrm{rad}G = 0$. Thus the map is injective. Clearly the composite $G \to \mathrm{Der}H \to \mathrm{Der}H/InnH$ has kernel H.

Let S be a simple linearly compact Lie K-algebra, and let V be a finite-dimensional K-space. The decomposition $\mathrm{Der}(S \hat{\otimes} \hat{S}(V)) = (\mathrm{Der}S) \hat{\otimes} \hat{S}(V) + Id_S \otimes \mathrm{Der}\,\hat{S}(V)$ is explained in [Gui 1, section 5.3]. Here $\mathrm{Der}S \hat{\otimes} \hat{S}(V)$ is identified with the set of maps $\alpha: S \to S \hat{\otimes} \hat{S}(V)$ satisfying $\alpha[x, y] = [x, \alpha(y)] + [\alpha(x), y]$, $x, y \in S$. If we choose a basis $\{v_i\}$ for V and write $\alpha(x) = \Sigma\, \alpha_e(x) v^e$, we find that $\alpha \in \mathrm{Der}S \hat{\otimes} \hat{S}(V)$ iff $\alpha_e \in \mathrm{Der}S$, all e. We have that $Inn(S \hat{\otimes} \hat{S}(V)) \subseteq \mathrm{Der}S \hat{\otimes} \hat{S}(V)$, with $\alpha \in Inn(S \hat{\otimes} \hat{S}(V))$ iff $\alpha_e \in InnS$, all e. Thus the remaining assertions of Theorem 4.7.1(ii) are readily verified.

4.8. *Remark.* R. Block [Blo] has another, more conceptual proof of V. Guillemin's result from [Gui 1]. We decided that it would be easier to extend Guillemin's method to our setting, but it may well be that Block's approach will also extend.

5. $K[P]$-structures on Lie algebras of the form $S \otimes K[[x_1, \ldots, x_n]]$, S simple finite-dimensional. Let P, k, K, ∂ be as in section 2. Let K be algebraically closed of characteristic zero.

5.1. In this and the next sections we will study Lie $K[P]$-algebra structures on the linearly compact Lie K-algebra $G = S \hat{\otimes} K[[x_1, \ldots, x_n]]$, with S simple linearly compact. Since K is algebraically closed, every such Lie K-algebra can be given a split $K[P]$-structure (1.8)—that is, there is a linearly compact k-algebra S° such that $G = K \otimes_k G^\circ$, where $G^\circ =$

$S^\circ \otimes k[[x_1, \ldots, x_n]]$. Then by 2.2.5, every $K[P]$ structure $\tilde{\varphi}$ on G is obtained from the split structure φ via $\tilde{\varphi} = \varphi + \omega$, where $\omega : P \to \mathrm{Der}_k G$ is a differential 1-form satisfying

$$d\omega = -\frac{1}{2}[\omega, \omega] \tag{5.1.1}$$

We are interested in the equivalence class of $K[P]$-structures under the group $\mathrm{Aut}_K G$. By 2.2.7, we have

$$g^{-1}(\omega)(p) = g^{-1}(pg) + (g^{-1}\omega g)(p) \tag{5.1.2}$$

Remark. P. J. Cassidy in [Cas] calls the action given by 5.1.2 the Loewy action.

Recall (Theorem 4.6(ii)) that $G = S \hat{\otimes} K[[x_1, \ldots, x_n]]$ is given as $G = \mathrm{Hom}_{K[\tilde{P}]}(K[P], S)$, where $\tilde{P}$ is an appropriate Lie K/k-subalgebra of P of codimension n, and S is a linearly compact Lie $K[\tilde{P}]$-algebra. Therefore (cf. 4.6(iii)) to describe the $K[P]$-structures on G, it suffices to describe the $K[\tilde{P}]$-structures on S.

Thus we can assume that $P = \tilde{P}$ and $G = S$.

5.2. Let $\mathcal{G} \subseteq GL(n)$ be a connected linear algebraic k-group. Let $G^\circ = (\mathrm{Lie}\ \mathcal{G})(k)$ be the set of points of its Lie algebra over k. We set $G = K \otimes_k G^\circ = (\mathrm{Lie}\ \mathcal{G})(K)$, and we give G the split $K[P]$-structure.

Let A be the coordinate ring of $\mathcal{G}$. Then A is a bialgebra. For any k-algebra R, $\mathcal{G}(R)$ consists of the k-algebra maps from A to R, and $(\mathrm{Lie}\ \mathcal{G})(R)$ consists of the k-linear maps $X : A \to R$ satisfying $X(ab) = X(a)\epsilon(b) + \epsilon(a)X(b)$, $a, b \in A$, where ϵ is the counit of A. The split action of $K[P]$ on $(\mathrm{Lie}\ \mathcal{G})(K)$ is given by $(pX)(a) = p(X(a))$, $p \in P$, $X \in (\mathrm{Lie}\ \mathcal{G})(K)$.

THEOREM. *Let $\omega \in \Omega^1(P, G)$ define a $K[P]$-algebra structure on G. Let $K \otimes_k A$ be given a $K[P]$-Hopf algebra structure which agrees with that structure on G. There is a field L, which is a $K[P]$-algebra extension of K, and for which there exists $g \in \mathcal{G}(L)$ with $\omega(p) = g^{-1}(pg)$, $p \in P$. (The product $g^{-1}(pg)$ is taken in the $K[P]$-algebra* $\mathrm{Hom}_K(K \otimes_k A, L) \simeq \mathrm{Hom}_k(A, L)$, cf. 1.10.7).

Proof. For $p \in P$, $\lambda \in K$, $a \in A$, define

$$p(\lambda \otimes a) = p(\lambda) \otimes a + \lambda \sum \omega(p)(a_2) \otimes a_1 - \lambda \sum \omega(p)(a_1) \otimes a_2.$$

Then for $\lambda' \in K$ we have

$$
\begin{aligned}
p(\lambda'(\lambda \otimes a)) &= p(\lambda'\lambda \otimes a) \\
&= p(\lambda'\lambda) \otimes a + \lambda'\lambda \, \Sigma \, \omega(p)(a_2) \otimes a_1 \\
&\quad - \lambda'\lambda \, \Sigma \, \omega(p)(a_1) \otimes a_2 \\
&= \lambda' p(\lambda \otimes a) + p(\lambda')(\lambda \otimes a) \\
&= (p\lambda')(\lambda \otimes a).
\end{aligned}
$$

For $p, q \in P$, we have

$$
\begin{aligned}
&p(q(1 \otimes a) - q(p(1 \otimes a)) \\
&\quad = p(\Sigma \, \omega(q)(a_2) \otimes a_1 - \Sigma \, \omega(q)(a_1) \otimes a_2) \\
&\qquad - q(\Sigma \, \omega(p)(a_2) \otimes a_1 - \Sigma \, \omega(p)(a_1) \otimes a_2) \\
&\quad = \Sigma \, p(\omega(q)(a_2)) \otimes a_1 + \Sigma \, \omega(q)(a_3)\omega(p)(a_2) \otimes a_1 \\
&\qquad - \Sigma \, \omega(q)(a_3)\omega(p)(a_1) \otimes a_2 - \Sigma \, p(\omega(q)(a_1)) \otimes a_2 \\
&\qquad - \Sigma \, \omega(q)(a_1)\omega(p)(a_3) \otimes a_2 + \Sigma \, \omega(q)(a_1)\omega(p)(a_2) \otimes a_3 \\
&\qquad - \Sigma \, q(\omega(p)(a_2)) \otimes a_1 - \Sigma \, \omega(p)(a_3)\omega(q)(a_2) \otimes a_1 \\
&\qquad + \Sigma \, \omega(p)(a_3)\omega(q)(a_1) \otimes a_2 + \Sigma \, q(\omega(p)(a_1)) \otimes a_2 \\
&\qquad + \Sigma \, \omega(p)(a_1)\omega(q)(a_3) \otimes a_2 - \Sigma \, \omega(p)(a_1)\omega(q)(a_2) \otimes a_3 \\
&\quad = \Sigma(p(\omega(q)) - q(\omega(p)) + [\omega(p), \omega(q)])(a_2) \otimes a_1 \\
&\qquad - \Sigma(p(\omega(q) - q(\omega(p)) + [\omega(p), \omega(q)])(a_1) \otimes a_2 \\
&\quad = \Sigma \, \omega([p, q])(a_2) \otimes a_1 - \Sigma \, \omega([p, q])(a_1) \otimes a_2
\end{aligned}
$$

since $d\omega = -1/2[\omega, \omega]$ we finally have $= [p, q]\,(1 \otimes a)$. Thus, $K \otimes A$ is a $K[P]$-module.

For $a, b \in A$ we have

$$\begin{aligned} p((1 \otimes a)(1 \otimes b)) &= p(1 \otimes ab) \\ &= \Sigma\, \omega(p)(a_2 b_2) \otimes a_1 b_1 - \Sigma(\omega(p)(a_1 b_1) \otimes a_2 b_2) \\ &= \Sigma(\omega(p)(a_2)\epsilon(b_2) + \epsilon(a_2)\omega(p)(b_2)) \otimes a_1 b_1 \\ &\quad - \Sigma(\omega(p)(a_1)\epsilon(b_1) + \epsilon(a_1)\omega(p)(b_1)) \otimes a_2 b_2 \\ &= \Sigma\, \omega(p)(a_2) \otimes a_1 b + \Sigma\, \omega(p)(b_2) \otimes b_1 a \\ &\quad - \Sigma\, \omega(p)(a_1) \otimes a_2 b - \Sigma\, \omega(p)(b_1) \otimes b_2 a \\ &= (p(1 \otimes a))(1 \otimes b) + (1 \otimes a)(p(1 \otimes b)). \end{aligned}$$

Thus, $K \otimes A$ is a $K[P]$-algebra. We omit the (easier) verification that Δ, ϵ, and S are $K[P]$-module maps, and thus that A is a $K[P]$-Hopf algebra.

The induced action of $K[P]$ on (Lie $\mathcal{G}$)(K) is given, for $p \in P$, $X \in$ (Lie $\mathcal{G}$)(K), by $(pX)(\lambda \otimes a) = p(X(\lambda \otimes a)) - X(p(\lambda \otimes a))$, $\lambda \in K$, $a \in A$. Thus $(pX)(\lambda \otimes a) = p(\lambda X(a)) - X(p(\lambda) \otimes a + \lambda\, \Sigma\, \omega(p)(a_2) \otimes a_1 - \lambda\, \Sigma\, \omega(p)(a_1) \otimes a_2) = p(\lambda)X(a) + \lambda p(X(a)) - p(\lambda)X(a) - \lambda\, \Sigma\, \omega(p)(a_2)X(a_1) + \lambda\, \Sigma\, \omega(p)(a_1)X(a_2) = \lambda p(X(a)) + \lambda[\omega(p), X](a)$. Thus, $pX = p^\circ X + (\operatorname{ad} \omega(p))(X)$, where p° denotes the split action. This says that the induced structure is indeed the one defined by ω.

Since $\mathcal{G}$ is connected, it is irreducible, and thus $K \otimes_k A$ is a domain. Let L be its quotient field. Define $g: A \to L$ by: $g(a) = 1 \otimes a$, $a \in A$, $1 \in K$. Then $g \in \mathcal{G}(L)$; its inverse is given by $g^{-1}(a) = 1 \otimes S(a)$, where S is the antipode of A. For $p \in P$, $a \in A$ we have $g^{-1}(pg)(a) = \Sigma\, g^{-1}(a_1)(pg)(a_2) = \Sigma\, \omega(p)(a_3) \otimes S(a_1)a_2 = 1 \otimes \omega(p)(a)$. Thus $\omega(p) = g^{-1}(pg)$, as required.

5.2.1. Let us put the above theorem in the context of 5.1. Given $\omega: P \to G = \operatorname{Lie}\mathcal{G}(K)$, the adjoint representation ad: $\operatorname{Lie}\mathcal{G}(K) \to \operatorname{Der}_K$ $(\operatorname{Lie}\mathcal{G}(K))$ gives us ad $\omega: P \to \operatorname{Der}_K \operatorname{Lie}\mathcal{G}(K) \subseteq \operatorname{Der}_L \operatorname{Lie}\mathcal{G}(L)$. Given $g \in \mathcal{G}(L)$ we have Ad $g \in \operatorname{Aut}_L \operatorname{Lie}\mathcal{G}(L)$. Then Theorem 5.2 asserts that $(\operatorname{ad} \omega)(p) = (\operatorname{Ad} g^{-1})(p(\operatorname{Ad} g))$. Under the action (5.1.2), we then have $(\operatorname{Ad} g)(\operatorname{ad} \omega) = 0$. The case of 5.1 is case in which G is a split simple Lie algebra, and $\mathcal{G} = \operatorname{Ad} G$.

5.3. Our proof is similar in spirit to a proof of a similar result by Kovacic [Kov, Proposition 6] and Cassidy [Cas, Proposition 38]. We were led to it by a suggestion of J. Tits.

6. $K[P]$-structures on Lie algebras of the form $S \hat{\otimes} K[[x_1, \dots, x_n]]$, S simple of Cartan type. Let P, k, K, ∂ be as in section 2. Let K be algebraically closed of characteristic zero.

6.1. We wish to determine all linearly compact $K[P]$-algebra structures on $G = S \otimes K[[x_1, \dots, x_n]]$. According to Theorem 4.6(iii), we must determine all open Lie K/k-subalgebras $\tilde{P}$ of P of codimension n, and all Lie $K[\tilde{P}]$-structures of cofinite kind on S.

Let us describe this more explicitly. Denote by Γ_n the set of all open Lie K/k-subalgebras of P of codimension n. Then Γ_n is a subset of the Grassmanian of all subspaces of codimension n; we will not describe it any further. Let $\Pi(G, P)$ be the set of all Lie $K[P]$-algebra structures of cofinite kind on G. Then by Theorem 4.6(iii) there is a map $\pi : \Pi(G, P) \to \Gamma_n$, with $\pi^{-1}(\tilde{P}) = \Pi(S, \tilde{P})$ for $\tilde{P} \in \Gamma_n$.

Thus we may (and will) assume for our classification that $G = S$, $P = \tilde{P}$.

6.2. Since K is algebraically closed, there exists a linearly compact Lie $\mathbf{Q}$-algebra G° with $G = K \otimes_{\mathbf{Q}} G^\circ$. We give G the split $K[P]$-structure (1.8); then G is a linearly compact $K[P]$-algebra (2.4.3). As in 1.8.3, we denote the split action of $p \in P$ by p°. By 2.2.5, any other Lie $K[P]$-algebra structure on G is given by a differential 1-form $\omega : P \to \operatorname{Der}_K G$ such that $d\omega = -\frac{1}{2}[\omega, \omega]$. The conditions for the structure defined by ω to be linearly compact are given in 2.4.7.

The algebra G can be considered in a natural way to be the completion of a graded Lie algebra (cf. [Gui 2]); we write $G = \Sigma_{i \geq -2} G_i$. Then we can express $\operatorname{Der}_K^c G$, the space of continuous derivations of G, as $\operatorname{Der}_K^c G = \Sigma_i (\operatorname{Der}_K G)_i$. We know that $\operatorname{Der}_K^c G$ is the semidirect product of ad G and T, where $T = 0$ if S is of type W_n or K_n, and $\dim T = 1$, with $t \in T$ acting as a scalar on each G_i, if G is of type H_n or S_n. In particular, $T \subseteq (\operatorname{Der}_K^c G)_0$, $\operatorname{Der}_K^c G = \Sigma_{i \geq -2} (\operatorname{Der}_K^c G)_i$, and $\Sigma_{i<0} (\operatorname{Der}_K G)_i = \Sigma_{i<0} (\operatorname{ad} G)_i \simeq \Sigma_{i<0} G_i$.

Let us write $\omega(p) = \Sigma_{i \geq -2} x_{i,p}$, with $x_{i,p} \in (\operatorname{Der}_K G)_i$. Set $\psi_\omega(p) = x_{-2,p} + x_{-1,p}$. Then ψ_ω maps P to $G_{-2} + G_{-1}$. Also we may write $x_{0,p} = x'_{0,p} + t_p$, where $x'_{0,p} \in (\operatorname{ad} G)_0$, $t_p \in T$. Recall that $t_p \neq 0$ implies that G is of type H_n or S_n.

6.2.1. LEMMA. *Let G be of type H_n or S_n. For each $p \in P$, let $\lambda_p \in K$ be the scalar by which $t_p \in T$ acts on G_{-1}. Suppose that the system of equations $\{-\lambda_p = \mu^{-1}p(\mu)\}_{p\in P}$ is solvable in K for μ* (cf. 5.2). *Then there exists $g \in \mathrm{Aut}_K^c G$* (the continuous K-automorphisms) *such that $g^{-1}(\omega)(p) = g^{-1}pg + g^{-1}\omega(p)g \in \mathrm{ad}\, G$.*

Proof. Define $\tau \in \mathrm{Aut}_K^c G$ by: $\tau(x) = (-\mu)^i x$ for $x \in G_i$. For $x \in G_i$, we have $(\tau^{-1}p\tau)(x) = (-\mu)^{-i}p(-\mu)^i x$ by 2.3.2(iii). Since $(-\mu)^{-i}p(-\mu)^i = (-\mu)^{-i}i(-\mu)^{i-1}p(-\mu) = i\mu^{-1}p\mu = i\lambda_p$, we have $\tau^{-1}p\tau = -t_p$. Then $(\tau^{-1}\omega)(p) = -t_p + \mathrm{ad}\,\tau^{-1}(x_p)$, and our result follows.

6.2.2. Using 6.2.1 and the fact that $G \simeq \mathrm{ad}\, G$, we have that every linearly compact $K[P]$-algebra on G is given by a form $\omega: P \to G$.

PROPOSITION. *Let $\omega: P \to G$ be a form with $d\omega = -\frac{1}{2}[\omega, \omega]$. Then the $K[P]$-algebra structure on G given by ω is linearly compact iff ω is continuous.*

Proof. Suppose that ω is continuous. We must establish (i) and (ii) of Proposition 2.4.7. Condition (i) is automatic. Let U be an open subspace of G. By [Gui 1, Proposition 2.2], there is an open subspace V of G with $[V, G] \subseteq U$. Since $\omega^{-1}(V) \subseteq P_U$, P_U is open.

Conversely, assume that (i) and (ii) of 2.4.7 hold. Let U be an open subspace of G. Then U contains an open subspace V of the form $V = \Sigma_{i\geq n} G_i$, some n. Note that if $[x, G] \subseteq V$, then $x \in \Sigma_{i\geq n+1} G_i \subseteq V \subseteq U$. Thus $P_V \subseteq \omega^{-1}(U)$, so $\omega^{-1}(U)$ is open. Thus ω is continuous.

6.2.3. COROLLARY. *Let $\omega: P \to G$ determine a linearly compact $K[P]$-algebra structure on G. Then* ker ψ_ω *is open.*

Proof. Note that ker $\psi_\omega = \omega^{-1}(\Sigma_{i\geq 0} G_i)$, and apply 6.2.2.

6.2.4. Let $\omega: P \to G$ determine a linearly compact $K[P]$-algebra structure on G. Let us re-formulate the action on ω of $g \in \mathrm{Aut}_K^c G$.

Let $x, y \in G$. With our identification of G with Ad G, we have $(g^{-1}xg)(y) = g^{-1}([x, g(y)]) = [g^{-1}(x), y] = g^{-1}(x)(y)$. Thus, $g^{-1}xg = g^{-1}(x)$.

Thus we have: $g^{-1}(\omega)(p) = g^{-1}pg + g^{-1}(\omega(p))$.

Suppose that $g = \exp \mathrm{ad}\, y$, $y \in G$. Then by 2.3.1 we have

$$g^{-1}pg = (dg)(p) = -\sum_{i>0} \frac{(-1)^i(\mathrm{ad}(\mathrm{ad}\, y))^{i-1}}{i!}(p(\mathrm{ad}\, y)).$$

Now $p(\operatorname{ad} y) = \operatorname{ad} p^\circ y$, and for any $z \in G$ we have $(\operatorname{ad}(\operatorname{ad} y))(\operatorname{ad} z) = [\operatorname{ad} y, \operatorname{ad} z] = \operatorname{ad}[y, z] = \operatorname{ad}((\operatorname{ad} y)(z))$. Thus upon identifying G with $\operatorname{ad} G$, we have

$$g^{-1}pg = -\sum_{i>0} (-1)^i \frac{(\operatorname{ad} y)^{i-1}}{i!} (p^\circ y).$$

6.3. Since the proofs of the results of this section are cumbersome and computational, we will state some of them here and prove them later.

By 6.2.2, the linearly compact Lie $K[P]$-algebra structures on G are given by the continuous forms $\omega: P \to G$ satisfying $d\omega = -\frac{1}{2}[\omega, \omega]$.

6.3.1. THEOREM. *Suppose that P has a commuting basis. Then the $K[P]$-module G is of cofinite kind iff ψ_ω is surjective: i.e., $\psi_\omega(P) = \Sigma_{i<0} G_i$.*
The proof will be given in 6.5–6.7.

6.3.2. Let ω and ω' be continuous 1-forms with values in G. Suppose that ω and ω' determine Lie $K[P]$-structures on G, and that $\psi_\omega(P) = \Sigma_{i<0} G_i = \psi_{\omega'}(P)$.

THEOREM. *If $\psi_\omega(p) = \psi_{\omega'}(p)$ for all $p \in P$, then there exists $g \in \operatorname{Aut}_K^c G$ such that $g(\omega) = \omega'$.*
The proof will be given in 6.8.

6.3.3. For G of type W_n, the above results lead to a description of the "moduli" spaces for the Lie $K[P]$-algebra structures on G such that $\psi_\omega(P) = G_{-1}$. For the types H_n, S_n, K_n, we obtain much weaker results.

For a Lie algebra G of Cartan type, let $\Pi(G, P)$ be the set of Lie $K[P]$-algebra structures on G for which $\psi_\omega(P) = \Sigma_{i<0} G_i$ (where ω is as in 6.2.2), considered up to continuous isomorphism. This $\Pi(G, P)$ consists of fibers of that of 6.1. By 6.2.1 and 5.1, the elements of $\Pi(G, P)$ can be considered as G-connections.

For each 1-form ω, set $P_\omega = \{p \in P: \psi_\omega(p) = 0\}$, $P'_\omega = \{p \in P: \omega(p)_{-2} = 0\}$. By [Rud, Theorem 1], each continuous K-automorphism g of G preserves the filtration. Thus, $g^{-1}pg$ preserves the filtration, and so must lie in $\Sigma_{i\geq 0} (\operatorname{Der}_K G)_i$. Since $g^{-1}(\omega)(p) = g^{-1}pg + g^{-1}(\omega(p))$, it follows that $P_{g(\omega)} = P_\omega$, $P'_{g(\omega)} = P'_\omega$. Therefore the correspondence $\omega \to P_\omega$ determines a map $\pi: \Pi(G, P) \to \Gamma_n$, $n = \dim \Sigma_{i<0} G_i$, for cases W_n, S_n, H_n. For the case K_n, the correspondence $\omega \to (P'_\omega, P_\omega)$ determines a map $\pi: \Pi(K_n, P) \to \Gamma_{1,n}$, where $\Gamma_{1,n}$ is the subset of the set of incomplete flags consisting of a K-subspace of P of codimension 1, containing an open Lie K/k-subalgebra of codimension n.

THEOREM.

(i) $\Pi(W_n, P) = \Gamma_n$.
(ii) *If G is of type* S_n *or* H_n, *then* $\pi^{-1}(\lambda)$, $\lambda \in \Gamma_n$, *is a subset of* $K^*/\tilde{k}^*$, *or of* $GL(n, K)/Sp(n, K) \cdot \tilde{k}^*$ *respectively, where* $\tilde{k} = \{\lambda \in K : p\lambda = 0 \text{ all } p \in P\}$.
(iii) *If G is of type* K_n, *then* $\pi^{-1}(\lambda)$, $\lambda \in \Gamma_{1,n}$, *is a subset of* $GL(n-1, K)/Sp(n-1, K)$.

This theorem will be proved in 6.9.

6.4. In this section we will prove an existence theorem for solutions of certain equations in Lie algebras of Cartan type. Let G be a simple Lie algebra of Cartan type. Let R be a subspace of G_{-1}, and let $c \in G_{-2}$. If G is of type K_n, assume $c \neq 0$; otherwise we must have $c = 0$. Now set $[r_1, r_2] = \Phi(r_1, r_2)c$ for $r_1, r_2 \in R$.

Let $q \geq 0$. Suppose that we are given $f_{q-1,c} \in G_{q-1}$, and also a linear map $R \to G_q$, with $r \to f_{q,r}$ for $r \in R$.

THEOREM. *Suppose that*

$$[r, f_{q,r'}] - [r', f_{q,r}] = \Phi(r, r')f_{q-1,c} \quad \textit{for} \quad r, r' \in R.$$

$$[r, f_{q-1,c}] = [c, f_{q,r}].$$

Then there exists $f_{q+1} \in G_{q+1}$ *such that* $f_{q,r} = [r, f_{q+1}]$ *for* $r \in R$, *and* $f_{q-1,c} = [c, f_{q+1}]$.

6.4.1. We will first reduce this to the case $R = G_{-1}$. Let $r_1, \ldots, r_m$ be a basis for R, and $r_1, \ldots, r_n$ a basis for G_{-1}. We assume that we are given $f_{q,r_1}, \ldots, f_{q,r_m}$ satisfying the hypotheses of the theorem, and we would like to extend this list to $f_{q,r_1}, \ldots, f_{q,r_n}$. It suffices to find $f_{q,r_{m+1}}$.

Let $z = [r_{m+1}, f_{q-1,c}]$. Let $z_i = [r_{m+1}, f_{q,r_i}] + \Phi(r_i, r_{m+1})f_{q-1,c}$, for $1 \leq i \leq m$. Then for $1 \leq i, j \leq m$ we have

$$\begin{aligned}
[r_j, z_i] &= [r_j, [r_{m+1}, f_{q,r_i}]] + \Phi(r_i, r_{m+1})[r_j, f_{q-1,c}] \\
&= [r_{m+1}, [r_j, f_{q,r_i}]] + \Phi(r_j, r_{m+1})[c, f_{q,r_i}] \\
&\quad + \Phi(r_i, r_{m+1})[r_j, f_{q-1,c}] \\
&= [r_{m+1}, [r_i, f_{q,r_j}]] + \Phi(r_j, r_i)[r_{m+1}, f_{q-1,c}]
\end{aligned}$$

$$+ \Phi(r_j, r_{m+1})[c, f_{q,r_i}] + \Phi(r_i, r_{m+1})[r_j, f_{q-1,c}]$$

$$= [r_i, [r_{m+1}, f_{q,r_j}]] + \Phi(r_{m+1}, r_i)[c, f_{q,r_j}] + \Phi(r_j, r_i)z$$

$$+ \Phi(r_j, r_{m+1})[c, f_{q,r_i}] + \Phi(r_i, r_{m+1})[r_j, f_{q-1,c}]$$

$$= [r_i, z_j] - [r_i, \Phi(r_j, f_{m+1})f_{q-1,c}] + \Phi(r_j, r_i)z$$

$$+ \Phi(r_j, r_{m+1})[c, f_{q,r_i}]$$

$$= [r_i, z_j] + \Phi(r_j, r_i)z.$$

Also,

$$[r_j, z] = [r_j, [r_{m+1}, f_{q-1,c}]]$$

$$= [\Phi(r_j, r_{m+1})c, f_{q-1,c}] + [r_{m+1}, [r_j, f_{q-1,c}]]$$

$$= [c, \Phi(r_j, r_{m+1})f_{q-1,c}] + [r_{m+1}, [c, f_{q,r_j}]]$$

$$= [c, \Phi(r_j, r_{m+1})f_{q-1,c}] + [c, [r_{m+1}, f_{q,r_j}]]$$

$$= [c, z_j].$$

Thus, by induction we can find $f_{q,r_{m+1}}$ with $[r_i, f_{q,r_{m+1}}] = z_i = [r_{m+1}, f_{q,r_i}] + \Phi(r_i, r_{m+1})f_{q-1,c}$ for $1 \leq i \leq m$, and $[c, f_{q,r_{m+1}}] = [r_{m+1}, f_{q-1,c}]$, as required.

6.4.2. We now prove the theorem for the cases $G = W_n, S_n, H_n$. By 6.4.1, we may assume $R = G_{-1}$. Let $r_1, \ldots, r_m$ be a basis of G_{-1}. Let $y_1, \ldots, y_m$ be elements of G_q, with $[r_i, y_j] = [r_j, y_i]$ for $1 \leq i, j \leq m$. Let $\varphi: G_{-1} \to G_q$ be the linear map given by $\varphi(r_i) = y_i$, for $1 \leq i \leq m$. Since $[r_j, \varphi(r_i)] = [r_j, y_i] = [r_i, y_j] = [r_i, \varphi(r_j)]$, it follows by [Rud, p. 711] that $\varphi = \operatorname{ad} y$ for some $y \in G_{q+1}$.

6.4.3. Our proof for the case K_n is a direct computation using the realization of K_n and the following well-known result.

LEMMA. *Let $f_1, \ldots, f_m$ be homogeneous polynomials of degree t in $x_1, \ldots, x_n$, $n \geq m$, such that $\partial f_i/\partial x_j = \partial f_j/\partial x_i$ then there exists a homogeneous polynomial f of degree $t + 1$ with $f_i = \partial f/\partial x_i$, $i = 1, \ldots, m$.*

We shall write ∂_i in place of $\partial/\partial x_i$.

The type K_n is realized [cf. Rud] by $K[[x_1, \ldots, x_n]]$, $n = 2m + 1$, with the bracket given by $[f, g] = (2g - \Sigma_{i \le n-1} x_i \partial_i g)\partial_n f + (2f - \Sigma_{i \le n-1} x_i \partial_i f)\partial_n g - \Sigma_{i \le m} (\partial_i f \partial_{i+m} g - \partial_i g \partial_{i+m} f)$. The space G_i is spanned by the homogeneous polynomials of the form $\Sigma P_\alpha x_n^\alpha$, where P_α is a homogeneous polynomial of degree $i - 2\alpha + 2$ in the variables $x_1, \ldots, x_{n-1}$. In particular, G_{-2} is spanned by 1 (and consists of constants), and $G_{-1} = \Sigma_{i \le n-1} Kx_i$. For any polynomial P not involving x_n, we have $[x_i, Px_n^\alpha] = \alpha x_i P x_n^{\alpha-1} - \partial_{i+m} P x_n^\alpha$ if $i \le m$, $[x_i, Px_n^\alpha] = \alpha x_i P x_n^{\alpha-1} + \partial_{i-m} P x_n^\alpha$ if $m < i < n$, and $[1, Px_n^\alpha] = 2\alpha P x_n^{\alpha-1}$.

By 6.4.1, we may assume $R = G_{-1}$. We will take $c = 1$; then $[x_i, x_j] = \Phi(x_i, x_j)$ for $i, j \le n - 1$.

Suppose that $y_i = \Sigma P_{i,\alpha} x_n^\alpha \in G_q$, $1 \le i \le n - 1$, and $y = \Sigma P_\alpha x_n^\alpha \in G_{q-1}$, satisfy the equations of our theorem. Let us first observe that for any $Q \in G_{q+1}$, the elements $\tilde{y}_i = y_i - [x_i, Q]$, $1 \le i \le n - 1$, $\tilde{y} = y - [1, Q]$ will also satisfy the equations of the theorem, and that the conclusion holds for y_i, y iff it holds for $\tilde{y}_i$, $\tilde{y}$.

Let β be the largest power of x_n which effectively enters in at least one y_i or in y. There are two cases to consider.

(i) x_n^β enters effectively in y.

Comparing coefficients of x_n^β in the equation $[x_i, y] = [1, y_i]$, we find $-\partial_{i+m} P_\beta = 0$, $1 \le i \le m$, and $\partial_{i-m} P_\beta = 0$, $m + 1 \le i \le n - 1$. Thus P_β is a constant. This shows that $q = 2\beta - 1$, and thus that $\beta > 0$ in this case. We can find a constant Q_β so that $[1, Q_\beta x_n^{\beta+1}] = P_\beta x_n^\beta$. Note that $[x_i, Q_\beta x_n^{\beta+1}] = (\beta + 1) x_i Q_\beta x_n^\beta$. The elements $\tilde{y}_i = y_i - [x_i, Q_\beta x_n^{\beta+1}]$, $1 \le i \le n - 1$, $\tilde{y} = y - [1, Q_\beta x_n^{\beta+1}]$ also satisfy our equations. Since x_n^β does not occur in $\tilde{y}$, and x_n occurs in $\tilde{y}_i$, $\tilde{y}$ only to powers less than or equal to β, we are reduced to the second case.

(ii) x_n^β does not enter in y.

Comparing coefficients of x_n^β in the equation $[x_i, y_j] - [x_j, y_i] = [x_i, x_j]y$, we find that $\partial_{i+m} P_{j,\beta} = \partial_{j+m} P_{i,\beta}$ for $i, j \le m$, $\partial_{i-m} P_{j,\beta} = \partial_{j-m} P_{i,\beta}$ for $m \le i, j \le n - 1$, and $\partial_{i+m} P_{j,\beta} = -\partial_{j-m} P_{i,\beta}$ for $i \le m$, $m \le j \le n - 1$. Setting $Q_i = P_{i+m,\beta}$ for $i < m$, $Q_i = -P_{i-m,\beta}$ for $m < i \le n - 1$, we have $\partial_i Q_j = \partial_j Q_i$ for $1 \le i, j \le n - 1$. By the lemma, we can find a homogeneous polynomial Q in $x_1, \ldots, x_{n-1}$ of degree $q + 1 - 2\beta$ with $-\partial_{i+m} Q = -Q_{i+m} = P_{i,\beta}$ for $i \le m$, and $\partial_{i-m} Q = Q_{i-m} = P_{i,\beta}$ for $m < i \le n - 1$. Now set $\tilde{y}_i = y_i - [x_i, Qx_n^\beta]$, $1 \le i \le n - 1$, $\tilde{y} =$

$y - [1, Qx_n^\beta]$. Then the $\tilde{y}_i$, $\tilde{y}$ satisfy the equations of the theorem, but do not involve x_n^β. We are done by induction on β.

6.5. The following is needed to simplify treatment of the case K_n.

LEMMA. *Let G be of type K_n. Write $\omega(p) = \Sigma x_{i,p}$. Suppose that $x_{-2,p} \neq 0$ for some $p \in P$. Then there exists $g \in \mathrm{Aut}_K^c G$ such that $g^{-1}(\omega)(p) = x_{-2,p}$ (for that particular p).*

Proof. We construct g inductively. Suppose that we have found $g_m \in \mathrm{Aut}_K^c G$ so that $g_m^{-1}(\omega)(p) = x_{-2,p} \bmod \Sigma_{i \geq m} G_i$. Say $g_m^{-1}(p) = x_{-2,p} + z \bmod \Sigma_{i \geq m+1} G_i$, with $z \in G_m$. Since $[x_{-2,p}, G_{m+2}] = G_m$, we can find $y \in G_{m+2}$ such that $[x_{-2,p}, y] = -z$. Let $h = \exp \operatorname{ad} y \in \mathrm{Aut}_K^c G$. Now $h^{-1}(g_m^{-1}(\omega)(p) = h^{-1}ph + h^{-1}(g_m^{-1}(\omega)(p))$ by 6.2.3. We have $h^{-1}ph \in \Sigma_{i \geq m+2} G_i$, and $h^{-1}(g_m^{-1}(\omega)(p)) = x_{-2,p} + z + [-y, x_{-2,p}] \bmod \Sigma_{i \geq m+1} G_i$. Thus with $g_{m+1} = g_m h$, we have $g_{m+1}^{-1}(\omega)(p) = h^{-1}(g_m^{-1}(\omega))(p) = x_{-2,p} \bmod \Sigma_{i \geq m+1} G_i$. Since G is complete, the g_m's converge to an automorphism g with the required property.

6.6. We take another step towards the proof of Theorem 6.3.1.

Let $\{b_i\}$ be a commutative basis for P. Changing notation if necessary, we may assume that $b_1, \ldots, b_r$ forms a basis for a complement P' to $P_\omega = \{p \in P: \psi_\omega(p) = 0\}$. Since P' is a Lie K/k-subalgebra, we can consider G to be a $K[P']$-module. Suppose that $\psi_\omega(P') \neq \Sigma_{i<0} G_i$. We shall show in 6.7 that in this case for any $q \geq 0$ there exists $0 \neq y \in \Sigma_{i \geq q} G_i$ with $p'y = 0$ for all $p' \in P'$. Let Y_q be the subspace of G spanned by all such y's. Note that if b_i is a basis element and $P'y = 0$, then for $1 \leq j \leq r$ we have $b_j(b_i y) = b_i(b_j y) = 0$; thus, $P'(b_i y) = 0$. This shows that if $P'y = 0$, $y \in \Sigma_{i \geq q} G_i$, then $Py \subseteq Y_q$. Since Y_q is the span of such y's, we have $PY_q \subseteq Y_q$.

For every open subspace N of G, we have $\Sigma_{i \geq q} G_i \subseteq N$ for some q. Since $K[P]Y_q \subseteq Y_q \subseteq N$, the fact that $Y_q \neq 0$ shows by 2.6 that G cannot be of cofinite kind.

Conversely, if $\psi_\omega(P) = \Sigma_{i<0} G_i$, then $N = \Sigma_{i \geq 0} G_i$ satisfies the conditions of Proposition 6.2.1, so G is of cofinite kind.

6.7. Let us proceed with the proof of 6.3.1.

By 6.6, we may assume that $P = P'$—that is, that ψ_ω is injective. As in 6.5, we write $\omega(p) = \Sigma x_{i,p}$, $x_{i,p} \in G_i$. By 6.5, we may also assume that *if* $x_{-2,p} \neq 0$ for some $p \in P$, then there exists $e \in P$ with $x_{i,e} = 0$ for $i \geq -1$, $x_{-2,e} = c$ (with c as in 6.4). We write $T = \{t \in P: x_{-2,t} = 0\}$. Clearly

$P = T \oplus Ke$. For $p_1, p_2 \in P$, write $[p_1, p_2] = t(p_1, p_2) + \lambda(p_1, p_2)e$, where $t(p_1, p_2) \in T$, $\lambda(p_1, p_2) \in K$.

6.7.1. Lemma. *For $t_1, t_2 \in T$, we have $[x_{-1,t_1}, x_{-1,t_2}] = x_{-2,[t_1,t_2]} = \lambda(t_1, t_2)c$.*

Proof. By 2.2.5, we have $[\omega(t_1), \omega(t_2)] = \omega([t_1, t_2]) - t_2^\circ\omega(t_2) - t_2^\circ\omega(t_1)$. The result follows.

Note that this gives $\lambda(t_1, t_2) = \Phi(x_{-1,t_1}, x_{-1,t_2})$ for $t_1, t_2 \in T$, with Φ as in 6.4.

6.7.2. Lemma. *Suppose that $\psi_\omega(P)$ is a proper subspace of $\Sigma_{i<0} G_i$. Then for each $q \geq 0$, there exists $0 \neq y_q \in G_q$ with $[\psi_\omega(P), y_q] = 0$.*

Proof. We give a case by case proof, using the realizations.

The type W_n is realized as the Lie algebra of derivations of $K[[x_1, \ldots, x_n]]$. The derivations from G_i are those of the form $\Sigma P_j \partial_j$, where each P_j is a homogeneous polynomial in $x_1, \ldots, x_n$ of degree $i + 1$. After a linear change of variables, we may assume $\psi_\omega(P) \subseteq K\partial_1 + \cdots + K\partial_{n-1}$. Then we may take $y_q = x_n^{q+1}\partial_n$.

The type S_n is realized as the subalgebra of W_n consisting of those $g = \Sigma P_j \partial_j$ satisfying $\Sigma \partial_j P_j = 0$. With $\psi_\omega(P) \subseteq K\partial_1 + \cdots + K\partial_{n-1}$ as above, take $y_q = x_n^{q+1}\partial_{n-1}$.

The type H_n is realized (cf. [Rud]) as $K[[x_1, \ldots, x_n]]/K$, $n = 2m$, with the Hamiltonian bracket as its operation: $[P, Q] = \Sigma_{i \leq m} \partial_i P \partial_{i+m} Q - \partial_i Q \partial_{i+m} P \bmod K$. The space G_i consists of the polynomials of degree $i + 2 > 0$. After a linear symplectic change of variables, we can assume $\psi_\omega(P) \subseteq \Sigma_{i<n} Kx_i$, and take $y_q = x_m^{q+2}$.

Finally, suppose that G is of type K_n. We will use the notation from the beginning of 6.7. If $\psi_\omega(T) = G_{-1}$, then by 6.7.1 $\psi_\omega(P) = \Sigma_{i<0} G_i$. Thus we may assume that $\psi_\omega(T) \subsetneq G_{-1}$. The realization of K_n was given in 6.4.3; arguing as for the case of H_n, we may assume that $\psi_\omega(T) \subseteq \Sigma_{i \leq n-2} Kx_i$. Then we may take $y_q = x_{(n-1)/2}^{q+2}$.

6.7.3. We now establish the needed extension result.

As above, for $p \in P$ write $\omega(p) = \Sigma x_{i,p}$, where $x_{i,p} \in G_i$. Recall that $x_{-2,e} = c$. For any $y \in G$, write $y = \Sigma y_i$, where $y_i \in G_i$, and $by = \Sigma y_{i,b}$, where $b \in K[P]$, $y_{i,b} \in G_i$. Thus, $y_{k,b} = b^\circ y_k + \Sigma_{i+j=k} [x_{i,b}, y_j]$.

Proposition. *Let $y = \Sigma_{i=-2}^{m} y_i$, where $y_i \in G_i$, $i = -2, \ldots, m$. Suppose that $y_{i,t} = 0$ for $i < m$, $t \in T$, and $y_{i,e} = 0$ for $i < m-1$. Then there exists $y_{m+1} \in G_{m+1}$ so that, with $y' = y + y_{m+1}$, we have $y'_{i,t} = 0$ for $i < m + 1$, $t \in T$, and $y'_{i,e} = 0$ for $i < m$.*

Proof. For $t_1, t_2 \in T$ we have $[x_{-1,t_1}, y_{m,t_2}] = y_{m-1,t_1t_2}$. Thus, $[x_{-1,t_1}, y_{m,t_2}] - [x_{-1,t_2}, y_{m,t_1}] = y_{m-1,[t_1,t_2]} = y_{m-1,t(t_1,t_2)} + y_{m-1,\lambda(t_1,t_2)e} = \lambda(t_1, t_2)y_{m-1,e} = \Phi(x_{-1,t_1}, x_{-1,t_2})y_{m-1,e}$. We also have $[c, y_{m,t}] - [x_{-1,t}, y_{m-1,e}] = y_{m-2,et} - y_{m-2,te} = y_{m-2,[e,t]} = 0$. By Theorem 6.4, there exists $y_{m+1} \in G_{m+1}$ with $[x_{-1,t}, y_{m+1}] = -y_{m,t}$ for all $t \in T$, and $[c, y_{m+1}] = -y_{m-1,e}$.

Now set $y' = y + y_{m+1}$. For $i < m$ we have $y'_{i,t} = y_{i,t} = 0$; we have $y'_{m,t} = y_{m,t} + [x_{-1,t}, y_{m+1}] = y_{m,t} = 0$. Also, for $i < m - 1$ we have $y'_{i,e} = y_{i,e} = 0$, and $y'_{m-1,e} = y_{m-1,e} + [x_{-2,e}, y_{m+1}] = 0$.

6.7.4. It is now easy to complete the proof of 6.3.1. Given $q \geq 0$, take $y = y_q$ exhibited in 6.7.2. By applying 6.7.3 inductively, we construct $y = \Sigma_{i\geq q}\, y_q$ with $py = 0$, all $p \in P$. By 6.6, the proof is complete.

6.8. Let us now prove 6.3.2.

Write $\omega(p) = x_p = \Sigma x_{i,p}$, $\omega'(p) = x'_p = \Sigma x'_{i,p}$, $p \in P$, $x_{i,p}, x'_{i,p} \in G_i$. We are assuming that $\psi_\omega(P) = G_{-2} + G_{-1} = \psi_{\omega'}(P)$, and that $x_{-2,p} = x'_{-2,p}$, $x_{-1,p} = x'_{-1,p}$, all $p \in P$. We will write $H = P_\omega = P_{\omega'} = \{p \in P\colon x_{-2,p} = 0 = x_{-1,p}\}$, $T = \{t \in P\colon x_{-2,t} = 0\}$, $e \in P$ defined (in the case K_n) by $x_{-1,e} = 0$, $x_{-2,e} = c$, c from 6.4.5. Let $\tilde{T}$ be a complement of H in T.

We will construct inductively an element $g \in \mathrm{Aut}^c_K$ with $g^{-1}(\omega') = \omega$. We may assume that there exists $g_m \in \mathrm{Aut}^c{}_K G$ such that $(g_m^{-1}(\omega')(t))_i = \omega(t)_i$ for $i < m$, $t \in T$, and $(g_m^{-1}(\omega')(e))_i = \omega(e)_i$ for $i < m - 1$. These assumptions hold for $m = 0$ with $g_0 = Id$.

In constructing g_{m+1}, we may replace ω' by $g_m^{-1}(\omega')$. Then we have $x_{i,t} = x'_{i,t}$ for $i < m$, $t \in T$, and $x_{i,e} = x'_{i,e}$ for $i < m - 1$.

Let us consider G to be a $K[P]$-module via ω; that is, for $p \in P$, $y \in G$, we will write $py = p^\circ y + [x_p, y]$. Since $\gamma = \omega' - \omega$ defines another structure, we have by 2.2.5 that $d\gamma = -\tfrac{1}{2}[\gamma, \gamma]$. Writing $\gamma(p) = \Sigma x''_{i,p}$, $x''_{i,p} \in G_i$, we have $x''_{i,t} = 0$ for $i < m$, $t \in T$, and $x''_{i,e} = 0$ for $i < m - 1$.

In particular, for $\tilde{t}_1, \tilde{t}_2 \in \tilde{T}$ we have $\gamma([\tilde{t}_1, \tilde{t}_2]) = \tilde{t}_1\gamma(\tilde{t}_2) - \tilde{t}_2\gamma(\tilde{t}_1) + [\gamma(\tilde{t}_1), \gamma(\tilde{t}_2)]$. Let us write $[\tilde{t}_1, \tilde{t}_2] = \lambda e \bmod T$. Comparing terms of degree $m - 1$, we have $\lambda x''_{m-1,e} = [x_{-1,\tilde{t}_1}, x''_{m,\tilde{t}_2}] - [x_{-1,\tilde{t}_2}, x''_{m,\tilde{t}_1}]$.

We also have, for all $\tilde{t} \in \tilde{T}$, $\gamma([e, \tilde{t}]) = e\gamma(\tilde{t}) - \tilde{t}\gamma(e) + [\gamma(\tilde{t}), \gamma(e)]$. Comparing terms of degree $m - 2$ yields $0 = [x_{-2,e}, x''_{m,\tilde{t}}] - [x_{-1,\tilde{t}}, x''_{m-1,e}]$.

It now follows by 6.4 that there exists $y_{m+1} \in G_{m+1}$ with $[x_{-1,\tilde{t}}, y_{m+1}] = x''_{m,\tilde{t}}$, $\tilde{t} \in \tilde{T}$, and $[x_{-2,e}, y_{m+1}] = x''_{m-1,e}$. We would like to show that

$[x_{-1,t}, y_{m+1}] = x''_{m,t}$ for all $t \in T$. It suffices to show that $X''_{m,h} = 0$ for all $h \in H$. Now since $\omega([h, t]) = h^\circ\omega(t) - t^\circ\omega(h) + [\omega(h), \omega(t)]$ for $h \in H$, $t \in T$, we clearly have $[H, T] \subseteq T$. Comparing terms of degree $m - 1$ in the equation

$$\gamma([h, \tilde{t}]) = h\gamma(\tilde{t}) - \tilde{t}\gamma(h) + [\gamma(h), \gamma(\tilde{t})],$$

where $h \in H, \tilde{t} \in \tilde{T}$, yields $0 = -[x_{-1,\tilde{t}}, x''_{m,h}]$. Since $G_{-1} = \{x_{-1,\tilde{t}}: \tilde{t} \in \tilde{T}\}$, the result $x''_{m,h} = 0$ follows.

Now let $f = \exp \operatorname{ad}(-y_{m+1})$. By 6.2.3, we have $f^{-1}(\omega')(p) = f^{-1}(\omega'(p)) \bmod \Sigma_{i \geq m+1} G_i$. Then for $t \in T$, we have $f^{-1}(\omega')(t) = \omega'(t) + [y_{m+1}, x_{-1,t}] \bmod \Sigma_{i \geq m+1} G_i$. Since $[y_{m+1}, x_{-1,t}] = -x''_{m,t} = x_{m,t} - x'_{m,t}$, we have $(f^{-1}(\omega')(t))_i = x_{i,t}$ for $i \leq m$. Similarly, $f^{-1}(\omega')(e) = \omega'(e) + [y_{m+1}, x_{-2,e}] \bmod \Sigma_{i \geq m} G_i$ yields $(f^{-1}(\omega')(e))_i = x_{i,e}$ for $i \leq m - 1$.

In terms of our original ω', we thus have $(f^{-1}(g_m^{-1}(\omega'))(t))_i = \omega(t)_i$, $i \leq m$, $t \in T$, and $(f^{-1}(g_m^{-1}(\omega'))(e))_i = \omega(e)_i$, $i \leq m - 1$. We set $g_{m+1} = g_m f$. Since G is complete, $g = \lim_{m \to \infty} g_m$ is a well-defined continuous automorphism, and $g^{-1}(\omega') = \omega$, as required.

6.9. Proof of 6.3.3.

6.9.1. We first consider the case W_n. We wish to show that π induces a bijection from the set of Lie $K[P]$-algebra structures of cofinite kind on $G = W_n$, to the set of open Lie K/k-subalgebra of P of codimension n.

(i) We first show that if $\pi(\omega) = \pi(\omega')$, then ω and ω' define isomorphic Lie $K[P]$-algebra structures on G. Let $p_1, \ldots, p_n$ be a basis of a complement to $P_\omega = P_{\omega'}$ in P. Then by 6.3.1, $\omega(p_1)_{-1}, \ldots, \omega(p_n)_{-1}$ and $\omega'(p_1)_{-1}, \ldots, \omega'(p_n)_{-1}$ are both bases of G_{-1}. By [Rud, Theorem 2] there is a continuous automorphism of G taking one of these bases into the other. Then ω, ω' define the same structure by 6.3.2.

(ii) Next, we wish to show that for every open Lie K/k-subalgebra P' of P of codimension n, there exists $\alpha \in \Pi(W_n, P)$ with $\pi(\alpha) = P'$. Given such a P', let $A = \operatorname{Hom}_{K[P']}(K[P], K)$, with the associative $K[P]$-algebra structure of 1.10.5. As a K-algebra, A is isomorphic to $K[[x_1, \ldots, x_n]]$. Thus, $\operatorname{Der}_K A$ represents W_n. When $\operatorname{Der}_K A$ is endowed with the Lie $K[P]$-algebra structure of 1.9.5, it has the required properties.

6.9.2. We now consider the cases S_n and H_n.

Fix an open Lie K/k-subalgebra P' of P. We wish to describe the set $\pi^{-1}(P')$ of equivalence classes of 1-forms $\omega: P \to G$ which determine linearly compact Lie $K[P]$-algebra structures of cofinite kind on G, for which $P_\omega = P'$.

Fix a basis $p_1, \ldots, p_n$ of a complement to P' in P. By 6.3.1, $\omega(p_1), \ldots, \omega(p_n)$ is a basis of G_{-1}. By 6.3.2, the assignment $\omega \to \omega(p_1), \ldots, \omega(p_n)$ sends non-equivalent forms to different bases. Thus it suffices to describe equivalence classes of bases.

In the cases S_n and H_n, the group of continuous automorphisms is the semidirect product of the inner automorphisms and the multiplicative group $\mathbf{G}_m$ (cf. 6.2). It follows from 6.2 that only the subgroup $\mathbf{G}_m(\tilde{k})$ of $\mathbf{G}_m(K)$, where $\tilde{k} = \{\lambda \in K : p\lambda = 0 \text{ all } p \in P\}$ preserves the property that $\omega(P) \subseteq \operatorname{ad} G$. Thus, two bases will represent the same element of $\Pi(G, P)$ only if one can be moved to the other by an element of the group $SL(m, K)\tilde{k}^*$ (in the case S_n), or $Sp(n, K)\tilde{k}^*$ (in the case H_n). Since $GL(n, K)$ acts simply transitively on the set of all bases, we can represent the equivalence classes as elements of the quotient $GL(n, K)/SL(n, K)\tilde{k}^* = K^*/\tilde{k}^*$ (in the case S_n) or $GL(n, K)/Sp(n, K)\tilde{k}^*$ (in the case H_n).

6.9.3. We now consider the case K_n.

We fix an open Lie K/k-subalgebra $\tilde{P}$ of P of codimension n, and K-subspace P' of codimension 1 containing $\tilde{P}$. We wish to describe the set of equivalence classes of 1-forms ω as above, with $P_\omega = \tilde{P}$, $P'_\omega = P'$. Select a basis $p_1, \ldots, p_n$ of a complement of $\tilde{P}$ in P, in such a way that $p_1, \ldots, p_{n-1}$ forms a basis of a complement of $\tilde{P}$ in P'. Then each 1-form ω is assigned to the basis $\omega(p_1), \ldots, \omega(p_n)$ of G_{-1}.

The group of all linear transformations of $G_{-2} + G_{-1}$ which preserve G_{-1} is the semidirect product $K^{n-1} \rtimes GL(n-1, K) \cdot K^*$. The group induced by $\operatorname{Aut}^c_K G$ on $G_{-2} + G_{-1}$ is the group of transformations which preserve G_{-1}, which preserve a skew-symmetric form on G_{-1} up to a scalar, and which act as the same scalar on the one-dimensional quotient $(G_{-2} + G_{-1})/G_{-1}$. This group is $K^{n-1} \rtimes GSp(n-1, K)$. This establishes 6.3.3(iii).

6.10. We owe to J. Hrabowski the observation that if P is finite-dimensional and contained in $\operatorname{Der}_k K$, then it will have a commutative basis. Indeed, $\operatorname{Hom}_K(P, K)$ is spanned by the maps $\hat{\alpha}$ defined for $\alpha \in K$, $p \in P$ by $\hat{\alpha}(p) = p(\alpha)$. If $\hat{\alpha}_i, \ldots, \hat{\alpha}_n$ is a basis of $\operatorname{Hom}_K(P, K)$, then the dual basis of P is commutative.

7. Algebraization. We wish to consider the question of whether the formal groups of Ritt of section 3 can be considered to be the completions at the identity of some sort of "algebraic groups."

7.1. *Definition.* Let B be a cocommutative K/k-bialgebra. Let A be a commutative B-algebra. Then we say that Spec A is an *algebraic B-scheme.* When A is a commutative B-Hopf-algebra, we say that Spec A is an *algebraic B-group.*

When B is an algebra of differential operators, we will say that Spec A is *affine* if A is generated by a B-module of cofinite kind.

7.1.1. *Remark.* The above definitions differ from those of differential algebraic varieties and groups given by Ritt, Kolchin, and Cassidy. The difference is that they consider only the points of varieties with coefficients in K. We consider all of their "derivatives" as well. Thus, the object that they consider is the projection of our object onto the "real" (as opposed to the "differentiated") space. To restore the balance, they are forced to make certain adjustments.

7.1.2. Consider the case $B = K[P]$. We assert that a Lie B-algebra of Cartan type cannot be the Lie algebra of an algebraic B-group. Indeed, since any commutative Hopf algebra is the union of its finitely-generated sub-Hopf-algebras, its Lie algebra will be the projective limit of finite-dimensional Lie algebras; thus, no infinite-dimensional simple Lie algebras can occur. This suggests that our definition of algebraic B-group might have to be extended.

7.2. Let $\mathcal{G}$ be an affine algebraic k-group, and Lie $\mathcal{G}$ its Lie algebra. We showed in 5.2 that if (Lie $\mathcal{G}$)(K) is simple (so that (Lie $\mathcal{G}$)(K) $\simeq$ Der_K (Lie $\mathcal{G}$)(K)), then every Lie $K[P]$-algebra structure on (Lie $\mathcal{G}$)(K) comes from a corresponding $K[P]$-Hopf algebra structure on $K \otimes A$. Thus, Spec ($K \otimes A$) is an algebraic $K[P]$-group whose Lie $K[P]$-algebra is (Lie $\mathcal{G}$) (K). We would like to show that if a Lie $K[\tilde{P}]$-algebra (Lie $\mathcal{G}$)(K) comes from an (affine) algebraic $K[P]$-group, then so does $\mathrm{Hom}_{K[\tilde{P}]}(K[P]$, (Lie $\mathcal{G}$)(K)), where $\tilde{P}$ and P are related as in section 4. This follows from rather general considerations, which we outline in the next subsection.

7.3. *Produced schemes.*

7.3.1. *Definition.* Let $\tilde{B} \subset B$ be cocommutative K/k-bialgebras. Let $\mathcal{G} = \mathrm{Spec}\, A$ be an affine algebraic $\tilde{B}$-scheme. Then a pair ($\mathfrak{F}$, φ) consisting of a B-scheme $\mathfrak{F}$ and a map $\varphi: \mathfrak{F} \to \mathcal{G}$ of $\tilde{B}$-schemes is said to be *produced* from $\mathcal{G}$ if for every pair ($\mathcal{H}$, τ) as above, there exists a unique map $\theta: \mathcal{H} \to \mathfrak{F}$ of B-schemes with $\tau = \varphi \circ \theta$.

7.3.2. *Free commutative B-algebras.* Let M be a B-module. The symmetric algebra $S(M)$ has a B-algebra structure, with $b(1) = \epsilon(b)1$,

$b(m, \ldots, m_r) = \Sigma_i(b_{1i}m_1) \cdots (b_{ri}m_r)$, for $b \in B$, $m_1, \ldots, m_r \in M$. For every commutative B-algebra D and every B-module map $f: M \to D$, there is a unique B-algebra map $\hat{f}: S(M) \to D$ which extends f.

7.3.3. *Construction of the produced scheme.* Let M be a $\tilde{B}$-submodule of A which generates A as a $\tilde{B}$-algebra. Then we have a surjective map $h: S(M) \to A$ of $\tilde{B}$-algebras; write J for its kernel. The map $i: M \to B \otimes_{\tilde{B}} M$, $i(m) = 1 \otimes m$, extends to an algebra map $\hat{i}: S(M) \to S(B \otimes_{\tilde{B}} M)$. Write $\langle BJ \rangle$ for the B-ideal of $S(B \otimes_{\tilde{B}} M)$ generated by $\hat{i}(J)$. Then $C = S(B \otimes_{\tilde{B}} M)/\langle BJ \rangle$ is a B-algebra, and we have a natural map $\varphi^*: A \to C$ of $\tilde{B}$-algebras. We claim that $\mathfrak{F} = \operatorname{Spec} C$ is produced from $\mathcal{G}$.

Indeed, let D be a commutative B-algebra, and $\tau^*: A \to D$ a map of $\tilde{B}$-algebras. Since $A = S(M)/J$, τ^* extends uniquely to a $\tilde{B}$-algebra map from $S(M)$ to D. This map extends uniquely to a B-algebra map from $S(B \otimes_{\tilde{B}} M)$ to D, which clearly vanishes on $\langle BJ \rangle$ and thus passes to a B-algebra map $\theta^*: C \to D$. Then θ^* is the unique B-algebra map with $\theta^* \circ \varphi^* = \tau^*$, as required.

7.3.4. *Produced group schemes.* Suppose that $\mathcal{G}$ is a $\tilde{B}$-group scheme, with multiplication $\mu: \mathcal{G} \times \mathcal{G} \to \mathcal{G}$ and inverse $\iota: \mathcal{G} \to \mathcal{G}$. Then the $\tilde{B}$-scheme maps $\mu \circ (\varphi \times \varphi)$ and $\iota \circ \varphi$ lift, by the produced scheme property, to B-scheme maps $\mu: \mathfrak{F} \times \mathfrak{F} \to \mathfrak{F}$, $\iota: \mathfrak{F} \to \mathfrak{F}$, which make $\mathfrak{F}$ into a B-group scheme. The corresponding B-algebra homomorphisms $\Delta: C \to C \otimes C$, $S: C \to C$ satisfy $\Delta(b\varphi^*(a)) = \Sigma b_1\varphi^*(a_1) \otimes b_2\varphi^*(a_2)$, $S(b\varphi^*(a)) = b\varphi^*(S(a))$, $a \in A$, $b \in B$.

7.3.5. *Tangent spaces.* We now assume that B and $\tilde{B}$ are Hopf K/k-algebras. Let $x: C \to K$ be a B-algebra map. Then its kernel I_x is a B-ideal. The map $\varphi(x) = x \circ \varphi^*: A \to K$ is a $\tilde{B}$-algebra map, whose kernel $I_{\varphi(x)}$ is a $\tilde{B}$-ideal. We have a B-module map $f: B \otimes_{\tilde{B}} (I_{\varphi(x)}/I_{\varphi(x)}^2) \to I_x/I_x^2$ given by $f(b \otimes \bar{a}) = \overline{b\varphi^*(a)}$ for $b \in B$, $a \in I_{\varphi(x)}$.

Write $N = B \otimes_{\tilde{B}} I_{\varphi(x)}/I_{\varphi(x)}^2$. The B-module $D = K \oplus N$ has a K-algebra structure given by $(\lambda, n)(\lambda', n') = (\lambda\lambda', \lambda n' + \lambda' n)$ for $\lambda, \lambda' \in K$, $n, n' \in N$, making D into a B-algebra. The function $g: A \to D$, $g(a) = (\varphi(x)(a), 1 \otimes \overline{(a - \varphi(x)(a)1)})$ is a $\tilde{B}$-algebra map. Thus there is a $\tilde{B}$-algebra map $h: C \to D$ with $h \circ \varphi^* = g$.

It is easy to check that C is generated as a K-algebra by $B\varphi^*(A)$, and thus that I_x is generated by $B\varphi^*(I_{\varphi(x)})$. This means that $h(I_x) \subset N$, and that h induces a B-module map $f': I_x/I_x^2 \to N$. Then $f'(\overline{b\varphi^*(a)}) = h(b\varphi^*(a)) = bh(\varphi^*(a)) = bg(a) = b \otimes \bar{a}$ for $b \in B$, $a \in I_{\varphi(x)}$, so $f' = f^{-1}$.

The tangent space to $\mathfrak{F}$ at x is the B-module $T(\mathfrak{F})_x = \mathrm{Hom}_K(I_x/I_x^2, K) \simeq \mathrm{Hom}_K(B \otimes_{\bar{B}} I_{\varphi(x)}/I^2_{\varphi(x)}, K) \simeq \mathrm{Hom}_{\bar{B}}(B, \mathrm{Hom}_K(I_{\varphi(x)}/I^2_{\varphi(x)}, K))$. That is, $T(\mathfrak{F})_x \simeq \mathrm{Hom}_{\bar{B}}(B, T(\mathfrak{G})_{\varphi(x)})$ as B-modules.

When $\mathfrak{G}$ is a group scheme, $\mathfrak{F}$ is also a group scheme, and we have Lie algebras $(\mathrm{Lie}\ \mathfrak{G})(K) = T(\mathfrak{G})_e$, $(\mathrm{Lie}\ \mathfrak{F})(K) = T(\mathfrak{F})_e$. From the above, there is a B-module isomorphism $h: (\mathrm{Lie}\ \mathfrak{F})(K) \to \mathrm{Hom}_{\bar{B}}(B, (\mathrm{Lie}\ \mathfrak{G})(K))$ given by $(h(X)(b))(a) = (bX)(\varphi^*(a))$, for $X \in (\mathrm{Lie}\ \mathfrak{F})(K)$, $b \in B$, $a \in A$. Let us verify that h is a Lie algebra map. For $X, Y \in (\mathrm{Lie}\ \mathfrak{F})(K)$, $b \in B$, $a \in A$, we have

$$\begin{aligned}
(h([X, Y])(b))(a) &= (b(XY - YX))(\varphi^*(a)) \\
&= \Sigma((b_1X)(b_2Y))(\varphi^*(a)) - \Sigma((b_1Y)(b_2X))(\varphi^*(a)) \\
&= \Sigma((b_1X)(\varphi^*(a_1)))((b_2Y)(\varphi^*(a_2))) \\
&\quad - \Sigma((b_1Y)(\varphi^*(a_1)))((b_2X)(\varphi^*(a_2)) \\
&= \Sigma(h(X)(b_1))(a_1)(h(Y)(b_2))(a_2) \\
&\quad - \Sigma(h(Y)(b_1)(a_1))(h(X)(b_2)(a_2)) \\
&= \Sigma[h(X)(b_1), h(Y)(b_2)](a) \quad \text{since } B \text{ is cocommutative} \\
&= ([h(X), h(Y)](b))(a).
\end{aligned}$$

Thus $h([X, Y]) = [h(X), h(Y)]$, as asserted, and $(\mathrm{Lie}\ \mathfrak{F})(K) \simeq \mathrm{Hom}_{\bar{B}}(B, (\mathrm{Lie}\ \mathfrak{G})(K))$ as Lie B-algebras.

8. Open Questions.

8.1. We described the Lie algebras of formal groups of Ritt. The question: "Are there non-isomorphic formal groups of Ritt with the same Lie algebra?"—remains open.

8.2. For Lie B-algebras one could study their irreducible B-representations. The case when $B = K[P]$, $\dim P = 1$, is probably within reach. However the case when $\dim P > 1$ seems to be very difficult. One point is worth mentioning: if $\dim G < \infty$ and G is simple then it is not possible to assume even after a field extension that G has a split structure.

A representation of G may not extend to a representation of the group of inner automorphisms of G, and therefore we can not make a structure of G split by a group element which preserves the representation.

8.3. Study, for a given B-representation M of a Lie B-algebra G, extensions of M by G. The questions 8.3 and 8.2, if answered, would cover the third paper of Ritt.

8.4. Study the forms of a given Lie B-algebra G. That is, describe all Lie B-algebras G' such that $G \otimes L \cong G' \otimes L$ (as $B \otimes L$-algebras) for some B-extension $L \supseteq K$. If G is simple, $\dim G < \infty$ and K is algebraically closed then we know by Theorem 5.2 that G is a form of a split B-algebra.

Problems of forms are usually formulated in terms of Galois cohomology. But it is not clear to us whether any form is "split" over a "differential Galois" extension. If it is so then one has next to solve the problem of computing the "differential Galois" cohomology.

On the other hand, the problem can be formulated as the problem of finding the flat connections, i.e., the problem of describing orbits of Ad G on $\{\omega \in \Omega^1(P, G) \mid d\omega = -\frac{1}{2}[\omega, \omega]\}$.

8.5. One should be able to prove the results of section 6 without the assumption that P has a commuting basis (c.f., however 6.10). Moreover, our proof is not canonical at several points (for example, the choice of e in 6.7). We had a feeling that part of our difficulty is due to the fact that we really work with something dual to our situation, but we were not able to find out what this dual is. It is also possible that Blattner's techniques from [Bla 2] may be extendable to our case.

8.6. More generally, Blattner's [Bla 2] and Block's [Blo] techniques may permit one to obtain more general results; for example, they may give a description of not only Lie but also associate, and Jordan simple B-algebras.

Appendix I. Hopf algebras. Let K be a field. A K-algebra is a K-vector space A equipped with a K-linear map $\mu: A \otimes_k A \to A$. The algebra A is associative if $\mu(\mu \otimes id) = \mu(id \otimes \mu): A \otimes_K A \otimes_K A \to A$. A unit for A is a K-linear map $\eta: K \to A$ for which $id = \mu(\eta \otimes id): A = K \otimes_K A \to A$ and $id = \mu(id \otimes \eta): A = A \otimes_K K \to A$.

Dually, a K-coalgebra is a K-vector space C equipped with a K-linear map $\Delta: C \to C \otimes_K C$. Our notation is $\Delta(c) = \Sigma_i c_{1i} \otimes c_{2i}$. The coalgebra C is coassociative if $(\Delta \otimes id)\Delta = (id \otimes \Delta)\Delta: C \to C \otimes_K C$. A counit for C is a K-linear map $\epsilon: C \to K$ for which $id = (\epsilon \otimes id)\Delta: C \to C = K \otimes_K C$, and $id = (id \otimes \epsilon)\Delta: C \to C = C \otimes_K K$.

A bialgebra is an associative algebra B with unit, which is a coassociative coalgebra with counit in such a way that $\Delta: B \to B \otimes_K B$ and $\epsilon: B \to K$ are algebra maps. That is, we require $\Delta(1) = 1 \otimes 1$, $\Delta(ab) = \Sigma a_{1i}b_{1j} \otimes a_{2i}b_{2j}$ for $a, b \in B$, $\epsilon(1) = 1$, $\epsilon(ab) = \epsilon(a)\epsilon(b)$.

A bialgebra B is a Hopf algebra if there exists a K-linear map $S: B \to B$ with $\Sigma_i S(b_{1i})b_{2i} = \epsilon(b)1 = \Sigma_i b_{1i}S(b_{2i})$ for all $b \in B$. The map S is called the antipode, and, if it exists, is unique.

Let C be a coassociative coalgebra with counit. Then C is the union of its finite-dimensional subcoalgebras. The sum of the minimal subcoalgebras of C is called the coradical of C, and is denoted C_0. The coradical filtration of C is the sequence of subspaces $C_0 \subseteq C_1 \subseteq C_2 \subseteq \cdots$, where $C_n = \{c \in C: \Delta(c) \in C_0 \otimes C + C \otimes C_{n-1}\}$; we have $C = \bigcup_n C_n$, and $\Delta(C_n) \subseteq \Sigma_{i+j=n} C_i \otimes C_j$.

An element g of C is called grouplike if $\Delta(g) = g \otimes g$, $\epsilon(g) = 1$. The grouplike elements are linearly independent. The coalgebra C is pointed if C_0 is spanned by the grouplikes, and irreducible if it has a unique minimal subcoalgebra. In particular, a bialgebra B is irreducible iff $B_0 = K1$.

Let C be a coalgebra with a distinguished grouplike element 1. An element p of C is called primitive if $\Delta(p) = p \otimes 1 + 1 \otimes p$. The set of primitives is denoted P, or $P_1(C)$. The subspaces $P_1(C) \subseteq P_2(C) \subseteq \cdots$ are defined by $P_n(C) = \{p \in C: \Delta(p) - p \otimes 1 - p \otimes p \in \Sigma_{i=1}^{n-1} P_i(C) \otimes P_{n-1}(C)\}$. If C is pointed irreducible (i.e. $C_0 = K1$), then $P_n(C) = C_n \cap \operatorname{Ker} \epsilon$.

Appendix II. Some notions from differential geometry. Let M be a variety over a field k. The variety M may be differentiable with $k = \mathbf{R}$, or analytic with $k = \mathbf{R}$ or $\mathbf{C}$, or algebraic with characteristic $k = 0$. Let $\mathfrak{F} = \mathfrak{F}(M)$ be the ring of (differentiable, or analytic, or regular) functions on M.

A vector field X on M can be thought of as a derivation of $\mathfrak{F}$—the derivation is differentiation along X. Thus, the set $P = P(M)$ of vector fields on M is $\operatorname{Der}_k \mathfrak{F}$. We have that P is a Lie k-algebra and an $\mathfrak{F}$-module, with the commutation of vector fields (Poisson bracket) satisfying $[fX, Y] = f[X, Y] - Y(f)X$ for $X, Y \in D$, $f \in \mathfrak{F}$. If P has a commuting basis of complete vector fields, then M has a global co-ordinate system. In particular, M is a quotient of $\mathbf{G}_a^n$, and the basis can be taken to be the image of the set of standard differentiations $f \to \partial f/\partial x_i$.

Let V be a k-vector space. An m-form on M with values in V is an $\mathfrak{F}$-m-linear map $\omega: P^m \to \mathfrak{F} \otimes_k V$ such that, for each permutation τ on m symbols, $\omega(X_1, \ldots, X_m) = (\det \tau)\omega(X_{\tau(1)}, \ldots, X_{\tau(m)})$ for all $X_1, \ldots, X_m \in P$. The set of m-forms with values in V is denoted $\Omega^m(M, V)$; it is an $\mathfrak{F}$-module.

The exterior differential $d: \Omega^m(M, V) \to \Omega^{m+1}(M, V)$ is defined, for $X_0, \ldots, X_m \in P$, by

$$(d\omega)(X_0, \ldots, X_m) = \frac{1}{m+1} \sum_{i=0}^{m} (-1)^i X_i(\omega(X_0, \ldots, \hat{X}_i, \ldots, X_m))$$

$$+ \frac{1}{m+1} \sum_{0 \le i < j \le m} (-1)^{i+j}$$

$$\cdot\, \omega([X_i, X_j], X_0, \ldots, \hat{X}_i, \ldots, \hat{X}_j, \ldots, X_m)).$$

For $m = 1$, this reduces to

$$(d\omega)(X, Y) = \frac{1}{2}(X(\omega(Y)) - Y(\omega(X)) - \omega([X, Y])), \quad \text{for} \quad X, Y \in P.$$

Let G be a Lie group, complex analytic Lie group, or algebraic group, and consider a principal bundle over M with structure group G. Since the bundle locally looks like $M \times G$, we will give definitions for this case only; in general, one glues together local charts.

Consider the group $\mathrm{Map}(M, G)$—the maps are differentiable, analytic, or regular, as the case may be. The multiplication in this group is defined componentwise: for $f, g \in \mathrm{Map}(M, G)$, $m \in M$, we have $(fg)(m) = f(m)g(m)$. The group $\mathrm{Map}(M, G)$ can be interpreted in two ways: as the group of sections of the fiber bundle $M \times G$, or as the points over $\mathfrak{F}$ of the "same" group G. According to the second interpretation, we can consider $\mathrm{Lie}(\mathrm{Map}(M, G))$ as simply $\mathfrak{F} \otimes_k \mathrm{Lie}\, G = \mathrm{Map}(M, \mathrm{Lie}\, G)$. Now for each $g \in \mathrm{Map}(M, G)$, define the differential 1-form $dg \in \Omega^1(M, \mathrm{Lie}\, G)$ by $dg(\chi) = g^{-1}X(g)$. (Here $X(g)$ is a tangent to G at the "point" g, and $g^{-1}X(g)$ is its translate to the identity). The 1-form dg is called the logarithmic derivative of g.

Now define the action of $\mathrm{Map}(M, G)$ on $\Omega^1(M, \mathrm{Lie}\ G)$ by $g^{-1}(\omega) = dg(X) = g^{-1}X(g)$. (Here $X(g)$ is a tangent to G at the "point" g, and $g^{-1}X(g)$ is its translate to the identity). The 1-form dg is called the logarithmic derivative of g.

The above definition of a connection is not standard. In the standard definition, a connection is a 1-form $\tilde{\omega}$ on $M \times G$ with values in Lie G, satisfying certain additional conditions. Our definition relates to the standard one as follows. Let $\tilde{\omega} \in \Omega^1(M \times G, \mathrm{Lie}\ G)$ represent a connection. Let $s_e: M \to M \times G$ be the identity section—that is, $s_e(m) = m \times e$ for $m \in M$. Then $s_e^*(\tilde{\omega}) = \omega \in \Omega^1(M, \mathrm{Lie}\ G)$. Conversely, take any $\omega \in \Omega^1(M, \mathrm{Lie}\ G)$. Now every section $s: M \to M \times G$ is of the form $s(m) = m \times g(m)$, $m \in M$, for some $g \in \mathrm{Map}(M, G)$. There is a unique connection $\tilde{\omega} \in \Omega^1(M \times G, \mathrm{Lie}\ G)$ with $s^*(\tilde{\omega}) = g^{-1}(\omega) = dg + (\mathrm{Ad}\, g^{-1})(\omega)$ for all such s; indeed, $\tilde{\omega}$ is defined by this equation and the requirement that $\tilde{\omega}(A^*) = A$, where A^* is the vector field tangent to the fibers $m \times G$, $m \in M$, and with constant value $A \in \mathrm{Lie}\ G$. For details, see [Kob, Section 1, Chapter II] or [Spiv, p. 8–14].

Finally, the curvature of the connection represented by $\omega \in \Omega^1(M, \mathrm{Lie}\ G)$ is the orbit of a 2-form $\Omega = d\omega + \frac{1}{2}[\omega, \omega]$ (Cartan equation), under the action of the group $\mathrm{Map}(M, G)$ on $\Omega^2(M, \mathrm{Lie}\ G)$ given by $g(\Omega) = (\mathrm{Ad}\, g)(\Omega)$. The expression $[\omega, \omega]$ is a 2-form given by $[\omega, \omega](X, Y) = [\omega(X), \omega(Y)]$.

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