COMMENTS ON DIFFERENTIAL INVARIANTS

By

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The subject of differential invariants is possibly as old as the algebraic invariant theory itself. The first differential invariant discovered was the Schwarzian derivative $(y'''/y'') - (3/2)(y''/y')^2$ of a function $y(t)$ of one variable. It has several invariance properties; two of them are under a fractional linear change of an independent variable and, separately, under a fractional linear change of the dependent variable.

The theory of differential invariants has never achieved the degree of maturity of the algebraic invariant theory. There seem to be several reasons for that. One of them is that by the turn of the century the theory of differential invariants was able to tackle (almost) only functions of one variable. The development of differential geometry required handling functions of more than one variable and such were developed in particular cases. The theory of torsion and curvature of a connection is an example.

Another possible reason is that by the turn of the XX-th century mathematical values changed. In particular, Hilbert's finiteness theorem became a worthier model than explicit results in particular cases. The topic of differential invariants was not ready for a similar kind of conceptualization.

We give below several examples representing, we hope, some important features of the subject. Then we prove a differential analog of Hilbert's theorem on finite generation of invariants. We conclude by pointing out several problems. In some of our comments we are rather informal.

Professor Boris Weisfeiler disappeared on January 5, 1985, in a mountain zone near Chillan in Chile in circumstances which are still not completely explained.

Conversations with J. Bernstein, R. Herman, V. Kac, D. Kazhdan, and V. S. Varadarajan helped me enormously in understanding the subject of this paper. I am grateful to them.

1. Examples.

To start with let us give several examples. More examples can be found in literature, see annotated bibliography in the end of this paper. We always assume that we are working over $\mathbb{C}$, the complex numbers.

Example 1. Consider a system of ordinary linear differential equations of the first order:

$$X' = AX \text{ where } X = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}, \; A = (a_{ij}(t))_{i,j=1,\dots,n},$$

where the $x_i(t)$ and the $a_{ij}(t)$ are functions of t from a given class $\mathfrak{F}$ and $X' = dX/dt$.

A substitution $X = BY$, where $B = B(t)$ is an invertible $n \times n$ matrix and Y is the column of new variables, gives rise to

$$Y' = (B^{-1}AB - B^{-1}B')Y$$

Thus to describe the equivalence classes of systems with respect to linear changes of dependent variables is the same as describing orbits of matrices $A \in \mathrm{Mat}_n(\mathfrak{F})$ under the action of $GL_n(\mathfrak{F})$ given by

$$B(A) = B^{-1}AB - B^{-1}B' \text{ for } A \in \mathrm{Mat}_n(\mathfrak{F}),\ B \in GL_n(\mathfrak{F}).$$

This is a classical problem, see [BV1] for a historical sketch. The group $GL_n(\mathfrak{F})$ is sometimes called the group of gauge transformations. We assume, of course, that $\mathfrak{F}$ is closed under subtraction, multiplication, taking the inverse, and differentiation; in other words we assume that $\mathfrak{F}$ is a differential field.

As in the geometric invariant theory to describe the orbits we need, in particular, to construct functions P of the a_{ij} and their derivatives such that the values of P for A and B(A) are the same. We restrict our attention now to the case when the P are polynomials. In other words we are looking at the ring $R = \mathcal{F}[a_{ij}, a'_{ij}, a''_{ij}, \ldots]_{i,j=1,\ldots,n}$ of the so called differential polynomials in the a_{ij}, see [K] and [R], and the action of $GL_n(\mathcal{F})$ on it, and we want to describe $R^{GL_n(\mathcal{F})}$, the ring of invariants of $GL_n(\mathcal{F})$ in R.

This problem admits a geometric interpretation. Recall that a connection (of class $\mathcal{F}$) on the principle $GL_n(\mathbb{C})$-bundle $\mathbb{C}^1 \times GL_n(\mathbb{C})$ over $\mathbb{C}^1$ is a map $A: \mathbb{C}^1 \longrightarrow \mathrm{Lie}\, GL_n(\mathbb{C}) = \mathrm{Mat}_n(\mathbb{C})$, i.e. $A \in \mathrm{Mat}_n(\mathcal{F})$. Given a section $B: \mathbb{C}^1 \longrightarrow \mathbb{C}^1 \times GL_n(\mathbb{C})$ one can compare it pointwise with the section $c \longmapsto c \times \mathrm{Id}$ to obtain a function $B: \mathbb{C}^1 \longrightarrow GL_n(\mathbb{C})$, i.e. $B \in GL_n(\mathcal{F})$. This new section B determines a new map $B(A) \in \mathrm{Mat}_n(\mathcal{F})$ given by $B(A) = B^{-1}AB - B^{-1}B'$ (see [BV]). Thus the question we are looking at can also be considered as a question of classifying connections in the principal $GL_n(\mathbb{C})$-bundles over a line.

This is the problem we are concerned with here. The answer to this problem is trivial in our case (see [NW, Theorem 5.2] which is applicable because in the one-dimensional case curvature is always zero): *If* $\mathcal{F}$ *is differentially closed then* $GL_n(\mathcal{F})$ *acts transitively on* $\mathrm{Mat}_n(\mathcal{F})$ (action as above) *and, therefore,* $R^{GL_n(\mathcal{F})} = \mathbb{C}$.

However, if $\mathcal{F}$ is not differentially closed there may exist more than one orbit of $GL_n(\mathcal{F})$ on $\mathrm{Mat}_n(\mathcal{F})$. In the classically most interesting cases where $\mathcal{F}$ is the field of formal Laurent series in t or the field of convergent (outside 0) series the problem was solved recently by D. G. Babitt and V. S. Varadarajan (see [BV1], [BV2] and the forthcoming papers). They first classify the orbits for the action of $GL_n(\mathcal{F}_f)$, where $\mathcal{F}_f = \mathbb{C}((t))$ is the field of the formal Laurent power series. Then they study the action of $GL_n(\mathcal{F}_{conv})$, where $\mathcal{F}_{conv}$ is the subring of $\mathbb{C}((t))$ consisting of the series $s = s(t)$

convergent for $0 < |t| < \mathcal{E} = \mathcal{E}(s)$ some $\mathcal{E}(s) > 0$, on the convergent parts of orbits of $GL_n(\mathcal{F}_f)$. In other words, they study the orbits of $GL_n(\mathcal{F}_{conv})$ on the sets $\{B(A),\ B \in GL_n(\mathcal{F}_f)\} \cap Mat_n(\mathcal{F}_{conv})$, that is, the equivalence classes of germs of linear differential meromorphic (with at most an isolated pole in 0) equations at 0.

__Example__ 2. Consider now a system of ordinary linear differential equations of the second order: $X'' = 2A_1X' + A_2X$. Setting $X = BY$ we obtain that $Y'' = 2(B^{-1}A_1B - B^{-1}B')Y' + (B^{-1}A_2B + 2B^{-1}A_1B' - B^{-1}B'')Y$. Thus the action of $GL_n(\mathcal{F})$ on $M = Mat_n(\mathcal{F}) \oplus Mat_n(\mathcal{F})$ is given now by

$$B\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} = \begin{bmatrix} B^{-1}A_1B - B^{-1}B' \\ B^{-1}A_2B + 2B^{-1}A_1B' - B^{-1}B'' \end{bmatrix}$$

If $\mathcal{F}$ is a differentially closed field with the field of constants $\mathbb{C}$ then, again by [NW, Theorem 5.2], we see that $GL_n(\mathcal{F})$ acts transitively on $Mat_n(\mathcal{F})$ by $B(A_1) = B^{-1}A_1B - B^{-1}B'$. Thus to study the action of $GL_n(\mathcal{F})$ on M is the same as to study the action of the stabilizer of the subset $N: = 0 \oplus Mat_n(\mathcal{F})$ on N. The stabilizer is

$$\{B \in GL_n(\mathcal{F}) \mid B^{-1}B' = 0\} = GL_n(\mathbb{C}).$$

The action of $GL_n(\mathbb{C})$ on $Mat_n(\mathcal{F})$ is given by $B(A_2) = B^{-1}A_2B$ and we are led to considering the invariants $R^{GL_n(\mathbb{C})}$ where R is the algebra of differential polynomials in the entries of A_2, as in the previous example.

A more detailed study of this case and some of its generalizations is conducted in [W,Ch. IV] and Section 2 of the present paper.

In the nextexample we give a description of a result from [W] which is, undoubtedly, the most interesting study of differential invariants from the "algebraic" point of view. A more modern formulation of this result is given in [M1], [M2].

Example 3. Consider the linear differential operators

$$L(y) = y^{(n)} + a_1(t)y^{(n-1)} \dots a_n(t)y$$

The changes of variables (both dependent and independent) of the form

$$\begin{cases} y_1 = \lambda(t)y \\ t_1 = \mu(t) \end{cases}$$

form a group if $\lambda \neq 0$ and $\mu' \neq 0$. This group preserves the equation $L(y) = 0$. We are interested in the polynomials in $a_1, a_2, \dots, a_n$ and their derivatives which are invariant under this group. It turns out that such invariants are very few. We are led then to considering <u>relative differential invariants</u>. These are, in essence, differential forms, i.e. such differential polynomials P in the a_i that the substitution $y_1 = \lambda(t)y$, $t_1 = \mu(t)$ gives P times some power of the Jacobian of our substitution. Using our substitution (and solving a couple of differential equations, the procedure corresponding to an extension of the differential field) one can bring the equation $L(y) = 0$ into the form where $a_1 = 0 = a_2$. Since the most interesting action occurs in this case we assume now that $a_1 = a_2 = 0$. The only transformations from our group which preserve linearity of $L(y)$, the equation $L(y) = 0$, and the condition $a_1 = a_2 = 0$ are of the form $y_1 = ky/(ct+d)^{n-1}$, $t_1 = (at+b)/(ct+d)$ with $k \neq 0$, $ad-bc \neq 0$, $a,b,c,d,k \in \mathbb{C}$. The group $G(\mathbb{C})$ of such transformations is isomorphic to a quotient of $SL_2(\mathbb{C}) \times \mathbb{C}^*$ by a finite central subgroup. If we write $(gL)(y) = 0$ for $g \in G(\mathbb{C})$ as $y_1^{(n)} + (g(a_3))y_1^{(n-3)} + \dots + (g(a_n))y_1 = 0$ then the problem of finding (relative) differential invariants is the same as finding for all $m \in \mathbb{Z}$ the differential polynomials $P(a_3,\dots,a_n)$ such that $P(a_3,\dots,a_n)dt^m = P(g(a_3),\dots,g(a_n))dt_1^m$. Thus we consider the algebra $R := \bigoplus_{m\geq 0} \mathcal{F}\{a_3,\dots,a_n\}dt^m$ (where $\mathcal{F}\{a_3,\dots,a_n\}$ is the algebra of differential polynomials) and we want a description of $R^{G(\mathbb{C})}$. Set for $i\geq 3$, $i\leq n$

$$\Theta_i := \frac{1}{2} \sum_{i\geq s\geq 0} \frac{(-1)^s (i-2)!\, i!\, (2i-s-2)!}{(i-s-1)!\,(i-s)!\,(2i-3)!\, s!} a_{i-s}^{(s)}$$

$$\Theta_{j\cdot 1} := 2j\Theta_j\Theta_j'' - (2j+1)\,(\Theta_j')^2$$

and, for $P \in \mathcal{F}\{a_3,\dots,a_n\}dt^m$,

$$[\Theta_3,P]: = 3\Theta_3P' - m\Theta_3'P.$$

Theorem ([W,p. 53] and also [M1, Lemma 2]). *The* Θ_i, $\Theta_{i\cdot 1}$, $i=3,\dots,n$ *and their repeated brackets with* Θ_3 *generate the algebra* $R^{G(\mathbb{C})}$.

Morikawa gives in [M1, Lemma 2] also relations between the $2(n-2)$ relative invariants Θ_i and Θ_{ik1}. One also has

Theorem (see [W, p. 39]). *Let* $R\cdot R^{-1}$ *be the quotient field of* R. *Then* $(R\cdot R^{-1})^{G(\mathbb{C})}$ *is the differential field of rational functions in the* Θ_i, $i=3,\dots,n$, *and* $\Theta_{3\cdot 1}$ *with respect to derivation* $\frac{\Theta_3}{\Theta_4}\frac{d}{dt}$.

There are many results in [W] which describe in much detail decompositions of the level sets of the differential invariants into orbits under $G(\mathbb{C})$.

Finally, let us note that the invariants $\Theta_i,\Theta_{j\cdot 1}$ were used (first, I believe, by G. H. Halphen) to study geometry of the curves in $\mathbb{P}^{n-1}(\mathbb{C})$. Halphen used them to describe the moduli space of such curves in $\mathbb{P}^3(\mathbb{C})$; his proof, though, had gaps.

Let $L(y)$ be as above and let $y_1(t),\dots,y_n(t)$ be a basis of solutions of $L(y) = 0$. Then the curve $C: \mathbb{C}^1 \longrightarrow \mathbb{C}^n$ is given by the parametric representation $(y_1(t),\dots,y_n(t))$. If $\bar{y}_s = \Sigma\, a_s^i y_i$ is another basis then $\det |a_s^i| \neq 0$ and, therefore, $L(y) = 0$ is the same for all curves $A(C)$ where $A \in GL_n(\mathbb{C})$. Conversely, if a curve C is given parametrically by $(y_1(t),\dots,y_n(t))$ and is not contained in any $\mathbb{C}^{n-1}$ then

$$L(y): = \begin{bmatrix}\text{Wronskian of}\\ y_1,\dots,y_n\end{bmatrix}^{-1} \cdot \begin{bmatrix}\text{Wronskian of}\\ y,y_1,\dots,y_n\end{bmatrix}$$

gives rise to C.

Thus the classes of linear equivalence of curves $C \subseteq \mathbb{C}^n$ which are not contained in any $\mathbb{C}^{n-1}$ correspond bijectively to equations $L(y) = 0$ of n-th order.

The image of our curve in $\mathbb{P}^{n-1}(\mathbb{C})$ is given by the homogeneous coordinates $(y_1(t): y_2(t): \ldots: y_n(t))$. Then it is clear that the curve in $\mathbb{P}^{n-1}(\mathbb{C})$ does not change if we replace y by $\lambda(t)y$ and t by $\mu(t)$. Thus we have

Theorem. *The equivalence classes of the curves in* $\mathbb{P}^{n-1}(\mathbb{C})$ (not contained in any $\mathbb{P}^{n-2}(\mathbb{C})$) *correspond bijectively to the equivalence classes of* $L(y) = 0$ *under the group of variables' changes* $y \longmapsto \lambda(t)y$, $t \longmapsto \mu(t)$.

The geometric properties of such curves are, therefore, expressible through the properties of the invariants Θ_i, $\Theta_{i \cdot 1}$.

Example 4. Consider the differential invariants of the action of $G \simeq \mathbb{Z}/2$, $G = \{1,\sigma\}$, on $\mathbb{C}$ given by $\sigma(y) = -y$. Thus we consider $\mathcal{F}\{y\}^G$ where $\mathcal{F}$ is an apropriate (differential) field of functions on $\mathbb{C}$. It is clear that $\mathcal{F}\{y\}^G = \mathcal{F}[y^2,(y')^2,\ldots,(y^{(n)})^2,\ldots]$.

Claim. $\mathcal{F}\{y\}^G$ *is not a finitely generated differential algebra.*

Indeed, if $z_1,\ldots,z_m$ are (differential) generators of $\mathcal{F}\{y\}^G$ then we can assume that $z_1,\ldots,z_m$ have no constant term. Let $\bar{R}$ be the differential subalgebra without 1 of $\mathcal{F}\{y\}^G$ generated by the z_i. Then the ideal $\bar{R}\cdot\mathcal{F}\{y\}$ of $\mathcal{F}\{y\}$ will be a finitely generated (by the z_i) ideal of $\mathcal{F}\{y\}$. This, however, is not the case by [R, §I.15].

2. Formalism and finite generation theorems.

Consider a commutative algebra K over $\mathbb{C}$. The algebra $\mathrm{Der}_{\mathbb{C}}$ K is a vector space over K and a Lie algebra over $\mathbb{C}$ with the bracket defined by

$$[p_1,p_2]\ (k) = p_1(p_2(k)) - p_2(p_1(k))$$

for $k \in K$, $p_1, p_2 \in \mathrm{Der}_{\mathbb{C}} K$.

We fix a pair (K,P) where K is a commutative algebra over $\mathbb{C}$ and P is a K-subspace of $\mathrm{Der}_{\mathbb{C}} K$ such that $[P,P] \subseteq P$. The <u>algebra</u> $K[P]$ of <u>differential</u> operators is, by definition, the associative algebra over $\mathbb{C}$ generated by its subalgebra K and the K-subspace P with relations

$$pk = p(k) + kp$$
$$pp_1 - p_1 p = [p,p_1]$$

for $k \in K$, $p,p_1 \in P$.

Let $K\{x_1,\dots,x_n\}$ denote the algebra of differential polynomials over K, i.e. the symmetric K-algebra over the free $K[P]$-module $\bigoplus_{1\leq i\leq n} K[P]x_i$. $K[P]$ acts on $K\{x_1,\dots,x_n\}$ in such a way that for $a,b \in K\{x_1,\dots,x_n\}$ and $p \in P$ we have

$$p(ab) = p(a)b + ap(b)$$

Generally, any K-algebra A with an action of $K[P]$ on it is called a <u>differential</u> <u>algebra</u> if it satisfies the above condition.

Even a finitely generated differential algebra over a differential field need not be Noetherian. An example exists already in $K\{x\}$ when $\dim_K P = 1$; it is described in [R,§I.15]. (Actually a stronger counter-example is established in [R,§I.15].) However $K\{x\}$ has a weaker Noetherian property. Recall that for an ideal I of a ring A its <u>radical</u>, $\mathrm{Rad}(I)$, is the ideal of A given by

$$\mathrm{Rad}(I) := \{a \in A \mid a^m \in I \text{ for some } m = m(a) \in \mathbb{N}\}.$$

<u>Definition</u>. A differential K-algebra A will be called <u>rdN</u> (: = <u>radically</u> <u>differentially</u> <u>Noetherian</u>) if every radical differential ideal is the radical of a finitely generated ideal.

In the terminology of [K] radical ideals are called "perfect" and rdN above is expressed in [K] by saying that perfect differential ideals form a "conservative Noetherian system". As with the

Noetherian property, rdN implies that a strictly increasing sequence of radical differential ideals is finite.

As Example 4 shows, passing to the rings of invariants may result in non-finitely generated rings even if one starts with the finitely generated rings. We still, however, have that the weaker finiteness property, rdN, is preserved if the action of the group is sufficiently simple.

Theorem. *Let* A *be an* rdN *algebra over a differential field* K (with derivations P) *and let* G *be a reductive (possibly disconnected) algebraic group over* $\mathbb{C}$. *Suppose that* $G(\mathbb{C})$ *acts on* A *by* K-*algebra automorphisms and that*

(i) $pP(g(a)) = g(p(a))$ *for* $g \in G(\mathbb{C})$, $p \in P$, $a \in A$.

Assume also that A *has a filtration* $A_0 = K \subseteq A_1 \subseteq \dots \subseteq A_s \subseteq \dots$ *such that*

(ii) $A_i \subseteq A_{i+1}$, $A = \cup A_i$,

(iii) $G(\mathbb{C})(A_i) \subseteq A_i$

(iv) $\dim_K A_i < \infty$, $i=0,1,\dots$

Then $A^{G(\mathbb{C})}$ *is a differential subalgebra of* A *and it is* rdN.

Note that this result applies to Example 4 and to Example 2 (see the concluding remarks in this example). We shall give a wide class of examples satisfying the above theorem later on.

That $A^{G(\mathbb{C})}$ is a differential subalgebra of A follows directly from (i). The proof that it is rdN is essentially the same as the proof of Hilbert's theorem on finite generation of invariants. We shall outline this proof.

Since $G(\mathbb{C})$ is Zariski-dense in the algebraic group G we see that G itself acts on A by K-automorphisms. Since G is reductive (and char K = 0) it is completely reducible on each A_i and there is a canonical projection E_i: $A_i \longrightarrow A_i^G = A_i^{G(\mathbb{C})}$ of A_i on its subspace of fixed vectors. We have that $E_i | A_j = E_j$ if $j \leqslant i$. Thus we obtain a canonical E: $A \longrightarrow A^{G(\mathbb{C})}$. We have

$$E(ab) = E(a)b \text{ for } a \in A,\ b \in A^G.$$

Indeed if $a,b \in A_j$ then we write $a = \bar{a} + a_0$, $a_0 \in A_j^{G(\mathbb{C})}$ and $\bar{a}$ belongs to the (canonical again) complement C_j of $A_j^{G(\mathbb{C})}$ in A_j. Then $ab = \bar{a}b + a_0b$. Clearly $a_0b \in A^{G(\mathbb{C})}$ and $\bar{a}b \in C_j \cdot A_j^{G(\mathbb{C})}$. The latter is (as a $G(\mathbb{C})$-module) a quotient of $C_j \otimes A_j^{G(\mathbb{C})}$ and since the latter is, as a $G(\mathbb{C})$-module, a multiple of C_j it follows that $C_j \otimes A_j^{G(\mathbb{C})}$ has no trivial $G(\mathbb{C})$-submodules whence $(C_j \cdot A_j^{G(\mathbb{C})})^{G(\mathbb{C})} = \{0\}$ whence the required property of E.

A similar argument based on commutativity of actions of $G(\mathbb{C})$ and P shows that

$$pE(a) = E(pa) \text{ for } p \in P,\ a \in A.$$

Now let J be a radical differential ideal of A^G. Then Rad(JA) is a radical differential ideal of A. rdN implies (in the same way as in the Noetherian case) that a finite set of radical generators of Rad(AJ) can be chosen from J. Let it be $a_1,\ldots,a_d \in J$. Let $a \in J$. Then $a \cdot 1 \in \mathrm{Rad}(AJ)$ and, therefore, $a^t \in \Sigma_{1 \leqslant i \leqslant d}\ A \cdot K[P] a_i$. Write $a^t = \sum b_j f_j(p)(a_{i_j})$. Then by the above $a^t = E(a^t)$

$= \Sigma_j E(b_j) \cdot f_j(p)(e(a_{i_j}))$ $\subseteq \Sigma A^{G(\mathbb{C})} K[P] a_i$ whence

$a^t \in \mathrm{Rad}(\Sigma A^{G(\mathbb{C})} K[P] a_i)$, i.e. $J = \mathrm{Rad}(\Sigma A^{G(\mathbb{C})} K[P] a_i)$. Thus J is the radical of a finitely generated ideal of $A^{G(\mathbb{C})}$ as claimed.

A version of the above result for the field of rational differential invariants is much simpler.

<u>Theorem</u>. *Let* Γ *be a group acting on a finitely generated differential field extension* L *of* K (with derivations P) *by automorphisms. If* $p(\gamma(\ell)) = \gamma(p(\ell))$ *and*

$\gamma(k) = k$ *for all* $\gamma \in \Gamma$, $p \in P$, $\ell \in L$, *and* $k \in K$, *then* L^{Γ} *is a finitely generated differential field extension of* K.

<u>**Proof**</u>. Since the action of P commutes with that of Γ, L^{Γ} is a differential subfield of L. But by [K,Proposition II.14] any subfield of a finitely generated differential field is a finitely generated differential field, whence our claim.

It is proper now to recall again that a very interesting study of the field of differential rational and algebraic functions on the differential manifolds of ∞-jets of maps of $U \subseteq \mathbb{C}^1$ into $M(\mathbb{C})$, where M is a homogeneous under G algebraic variety, was undertaken by M. Green [G]. His results give, in particular, a more detailed information on the structure of the differential quotient field of $A^{G(\mathbb{C})}$ for some A from the next to last Theorem. In addition, M. Green computes explicitly differential invariants in a large number of cases.

We shall now describe a class of examples to which the above theorems apply.

First, let us consider the algebra $\mathcal{H}$ of germs at 0 of the holomorphic functions $f: U \longrightarrow \mathbb{C}$ defined on a neighborhood U of 0 in $\mathbb{C}^n$. Set $\mathbb{C}^n := \oplus \mathbb{C}t_i$ and $\partial_i := \partial/\partial t_i$. Then $f(t_1,\dots,t_n) = \Sigma_{\bar{i}\geq 0} a_{\bar{i}} t^{\bar{i}}/\bar{i}!$ where, as usual, $\bar{i} := (i_1,\dots,i_n)$, $t^{\bar{i}} := t_1^{i_1}\dots t_n^{i_n}$, $\bar{i}! = i_1!\dots i_n!$ and $\bar{i} \geq 0$ means that the $i_j \geq 0$. We consider the $a_{\bar{i}}$ as the coordinates of f, i.e., as functions on $\mathcal{H}$. Then the coordinates of $\partial f/\partial t_m$ are $a_{i_1\dots i_{m-1},i_m+1,i_{m+1},\dots,i_n}$. Thus the coordinates of f are acted upon by $\mathbb{C}[\partial_1,\dots,\partial_n]$ as the elements of the free $\mathbb{C}[\partial_1,\dots,\partial_n]$-module generated by $a_{\bar{0}}$. To make our considerations independent of the choice of the uniformizers $t_1,\dots,t_n$ we are led to considering $P_{\mathcal{H}} := \oplus \mathcal{H}\partial_i \subseteq \mathrm{Der}_{\mathbb{C}} \mathcal{H}$ and the algebra $\mathcal{H}\{a_0\}$ of differential polynomials in one variable.

Note now that in actuality the $a_{\bar{i}}$ are also coordinates on the ring $\mathbb{C}[[t_1,\dots,t_n]]$ of formal power series. Therefore an algebraic study of objects arising in the algebra $\mathcal{H}\{a_{\bar{0}}\}$ will, by necessity, be only a study of <u>formal</u> objects. We thus replace $\mathcal{H}$ by $\mathcal{F} := \mathbb{C}[[t_1,\dots,t_n]]$ and $P_{\mathcal{H}}$ by $P_{\mathcal{F}} := \oplus\, \mathcal{F}\partial_i$ and then $\mathcal{F}$ by its field of quotients K and $P_{\mathcal{F}}$ by $P := \oplus\, K\partial_i$. (Note that there is a difficulty with our definition of K as the notion of formal Laurent series in many variables is not straightforward.)

Now let G be an algebraic $\mathbb{C}$-group acting, say, linearly on $M := \mathbb{A}^m$. We consider $\mathcal{M} := \mathcal{M}_n :=$ {the set of ∞-jets at 0 of holomorphic maps $f\colon U \longrightarrow M(\mathbb{C})$, U open in $\mathbb{C}^n$, $0 \in U$}. The action of $G(\mathbb{C})$ on M gives an action of $G(\mathbb{C})$ on $\mathcal{M}$ by

$$(g(m))(x) = g(m(x)) \text{ for } g\in G(\mathbb{C}),\ m\in\mathcal{M},\ x\in U\subseteq\mathbb{C}^n.$$

This action is differential which in our case means that $p(g(m)) = g(p(m))$ for $p \in P_{\mathcal{H}}$, $g \in G(\mathbb{C})$, $m \in \mathcal{M}$. Our action gives rise to a differential action of $G(\mathbb{C})$ on the algebra $\mathcal{H}\{x_1,\dots,x_m\}$. As before, we pass over from $\mathcal{H}$ to $\mathcal{F} := \mathbb{C}[[t_1,\dots,t_n]]$, and then to $K := \mathcal{F}\cdot\mathcal{F}^{-1}$. Our Theorems apply to such actions (with G reductive for the first Theorem). They also apply to the case when M is taken to be just an algebraic variety over $\mathbb{C}$.

As a more concrete example, consider the case $G = SL_2$ acting via the natural representation on $M := \mathbb{A}^2$. We take $n < \infty$ and consider the action of $G(\mathbb{C})$ on $A := K\{x,y\}$. As $G(\mathbb{C})$ commutes with $P := \bigoplus_{1\leqslant i\leqslant n} K\partial_i$ we are looking at invariants of $G(\mathbb{C})$ acting on direct sums $(= \bigoplus_{\bar{i}<m} K\partial^{\bar{i}}(Kx \oplus Ky))$ of natural $SL_2(\mathbb{C})$-modules. By the classical invariant theory we know then that $A^{G(\mathbb{C})}$ is generated (as a K-algebra) by the determinants

$$D_{\bar{i};\bar{j}} := \det\begin{bmatrix} \partial_{\bar{i}}x & \partial_{\bar{j}}x \\ \partial_{\bar{i}}y & \partial_{\bar{j}}y \end{bmatrix},\ \bar{i} \neq \bar{j},\ \bar{i},\bar{j} \geqslant 0.$$

3. Problems

The above results most probably extend to the case (including Example 1 and its analogs) when an action of the group is such that P still acts on the algebra or the field of the invariants. However some actions of interesting groups on interesting algebras do not have this latter property, see, e.g., Example 3. I do not know how to define such actions, whence

Problem 1. Give a definition of a differential action covering Example 3 and other examples appearing in the literature (say in [V,pp. 143-152]).

The first two Theorems quoted in our discussion of Example 3 give some kind of finite generation of the type one should be looking for.

Such a definition should be given, it seems, in terms of factors (i.e., cocycles) associated to an action of a group in question on the base differential field K.

Also it may be more convenient to consider actions of the differential Lie algebras instead of groups. A formalism is introduced in [NW] for classification of certain differential Lie algebras; it should be applicable here.

Another argument for considering Lie algebras instead of groups is that the differential Lie algebras of Cartan type (see [NW]) do not seem to have any group analog, except in the class of convergent or holomorphic maps, which seem to be of different flavour, see comments after Problem 3.

Once we have a definition of the general differential action we may look again at differential invariants. These, however, will not be generally preserved by some differentiations from P.

Problem 2. **(J. Bernstein)** Show that for an appropriate class of differential actions of differential groups on rdN-algebras the rings of invariants are again rdN with respect to derivations from P which normalize the action of our group or its Lie algebra.

Example 3 seems to confirm this conjecture: the bracket with

Θ_3 plays the role of the desired derivation.

The algebraic differential invariants of the type we consider here are generally formal invariants, i.e. invariants of germs of functions, equations, groups, etc. at some point.

__Problem__ __3__. Find a way of putting the invariants together to obtain global results.

Some results of this type are given by M. Green [G, §3].

We return to the interesting question of describing even at some point the decomposition of formal orbits into orbits under the action of the group of analytic-, C^{∞} etc. transformations. This has been addressed on a number of occasions, but the discussion of this question is beyond the scope of the present note (see however concluding remarks in Example 3).

[BV] contains several references to problems of such kind.

__References__ (with the library call numbers for books)

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M for H the paper studies the action of H on curves on M (under suitable genericity conditions on such curves). Many interesting explicit differential invariants of such action are computed.]

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