16. Pairs of centrally simple irreducible embedded subgroups.

Let k be an algebraically closed field of characteristic exponent $p := p(k)$.

(16.1) A pair $H \subseteq G$ of centrally simple non-commutative subgroups of $GL_n(k)$ will be called here <u>tight</u> if both H and G are irreducible. The following pairs of subgroups of $GL_n(\mathbb{C})$ are examples

(a) $G = Alt_{n+1}$ and H a doubly transitive group of permutations of $n+1$ letters (see W. Feit [Nice, §9.1] for concrete n and H) in the n-dimensional representation of Alt_{n+1},

(b) $G = Alt_{md}$, $m, d > 1$, $H = Alt_{md-1}$ and the representation of G corresponds to the rectangular Young diagram with height m

and width d (this example was explained to me by A. Regev), n is the dimension of the corresponding representation,

(c) $n = (q^m - 1)/2$, $G = Sp_{2m}(\mathbb{F}_q)$, $H = SL_2(\mathbb{F}_{q^m})$, q a power of an odd prime.

To explain (a) note that since H is doubly transitive its permutation representation is a direct sum of the trivial one and an irreducible one of dimension n

✓ (see, e.g.,).

Since the same holds for Alt_{n+1} we see that the restriction of the n-dimensional component of the permutation representation of G on $n+1$ letters is irreducible.

To explain (b) note that by the

✓ branching rule (see G. James [ ]), the restriction of an irreducible representation of Sym_r with Young diagram T to Sym_{r-1} consists of components whose

Young diagrams are obtained from T by removing exactly one square. For a rectangular diagram there is just one way to remove a square which leads to a Young diagram. Thus the restriction of our representation from Sym_r to Sym_{r-1} remains irreducible. That the same holds for Alt_r and Alt_{r-1} follows

v? from, e.g., G. James [ ].

For (c) we note that $SL_2(\mathbb{F}_{q^m})$ can be considered as acting on $\mathbb{F}_{q^m}^2 = \mathbb{F}_q^{2m}$. It preserves in this action a symplectic form whence imbedding $SL_2(\mathbb{F}_{q^m}) \subseteq Sp_{2m}(\mathbb{F}_q)$. $Sp_{2m}(\mathbb{F}_q)$ has an irreducible representation, (say 4)

v of dimension $n:=(q^m-1)/2$ (see, e.g.

Since $(q^m-1)/2$ is also the smallest degree of a non-trivial representation of $SL_2(\mathbb{F}_{q^m})$ we see $\varphi | SL_2(\mathbb{F}_{q^m})$ must be irreducible.

The following result says that example (c) is, in some sense, typical.

(16.2) **Theorem.** Let $G \supseteq H$ be a tight pair in $GL_n(k)$. Suppose that G is of Lie ℓ-type and H is of Lie r-type with $\ell \neq p$. Then $\ell = r$, unless $|H/\text{center}| \leq 1.5 \cdot 10^{33}$.

We also have

(16.3) **Theorem.** Let $G \supseteq H$ be a tight pair in $GL_n(k)$. Suppose that G is of Lie ℓ-type with $\ell \neq p$ and H is a covering group of Alt_m. Then, unless $m \leq 32$, G is classical and the image of Alt_m in the natural representation of G over $\overline{\mathbb{F}_\ell}$ is the smallest non-trivial irreducible representation of Alt_m.

(16.4) Proof. Suppose $r \neq \ell$, (G/center $\simeq {}^c\overline{X}_a(m^c)$, m^c a power of ℓ.) Take an irreducible non-trivial representation

$$G \longrightarrow GL_d(\overline{\mathbb{F}}_\ell) \quad \text{with}$$

$d = d(X_a) \leq$	$a+1$	$2a$	$2a+1$	7	26	27	56	248
if $X_a =$	A_a	C_a, D_a	B_a	G_2	F_4	E_6	E_7	E_8.

Since $|H/\text{center}| \leq f(d)$ where, as before,

$f(d) := (2d+1)^{2\log_3(2d+1)+1}$ we see that

only a finite number (independent of p) of H can join tight pairs with G having $d \leq 213$ or $d = 248$. (This excludes all exceptional G.) Assume therefore that

$d \geq 214$, $d \neq 248$

From Table T4.4 we see that $b \geq a$. Since ${}^2B_2(2)$, ${}^2G_2(3)$, ${}^2F_4(2)$ are excluded we also have $m \geq 2$.

Thus by (4.4.2) $n \geq (2^a - 1)/2$. Combining this with $a \geq (d-1)/2$ we obtain

$$n \geq (2^{(d-1)/2} - 1)/2 = 2^{(d-3)/2} - 0.5 .$$

One easily verifies that $2^{(d-3)/2} - 0.5 > f(d)$ for $d \geq 214$. Thus $n > |H/\text{center}|$.

By C. Curtis and I. Reiner [(53.16)] degrees of ordinary representations of H divide $|H/\text{center}|$. Since every representation in characteristic $p>0$ is a composition factor of a reduction mod p of an ordinary representation we see that the degrees of all representations of H are $\leq |H/\text{center}|$. Thus if $d \geq 214$, $d \neq 248$, our representation of H can not be irreducible, a contradiction.

Note, finally, that $f(248) \leq 1.5 \cdot 10^{33}$.

(16.4) Remark. Actually for almost all H we have $|H| \leq f(d)$. Then instead of estimate:

$$(\text{degree of a representation}) \leq |H/\text{center}|$$

we can use:

$$(\text{degree of a representation}) \leq \sqrt{|H|}$$

Then it is sufficient to take $d \geq 78$, $d \neq 248$. Additional restrictions are obtained if one takes into account (4.4.3)(b) but this only reduces 78 to 68.

(16.5) Corollary. Let p and l be odd primes, $p \neq l$.

Let $\varphi : SL_2(\mathbb{F}_l) \longrightarrow GL_{\frac{l-1}{2}}(\overline{\mathbb{F}}_p)$ be an irreduci

representation. Then

$\varphi(SL_2(\mathbb{F}_l))$ is maximal in $Sp_{\frac{l-1}{2}}(\mathbb{F}_p)$? if $l \equiv 1$ (mo

in $SO_{\frac{l-1}{2}}(\mathbb{F}_p)$?