On the Maximal Subgroups of the Finite Classical Groups

Boris Weisfeiler

Department of Mathematics
The Pennsylvania State University
University Park, PA 16802

* Supported by NSF

1. Introduction

The present paper is a branch off of [ ] and was provoked by a paper of M. Aschbacher [ ]. Consider a finite simple classical group G_0 (i.e. a group of Lie type $A_n, B_n, C_n, D_n, {}^2A_n, {}^2D_n$ over a finite field $k \simeq F_{p^m}$.) Let G be a finite group such that $G_0 \subseteq G \subseteq \mathrm{Aut}\, G_0$. Let φ denote a projective representation of smallest dimension for G_0 (usually referred to as a natural representation) over an algebraic closure $\bar{k}$ of k. M. Aschbacher describes a family $\mathcal{C}_G$ of subgroups of G (recalled in (5.3) below) and proves then

(1.1) Theorem. If G_0 is of type D_4 assume that G contains no triality automorphism of G_0. Let H be a maximal subgroup of G such that $G = HG_0$. Then either H is a subgroup from $\mathcal{C}_G$ or

(i) the socle H_0 of H is simple,

(ii) $\varphi | H_0$ is absolutely irreducible,

(iii) the characters of $\varphi | H_0$ and $\varphi | G_0$ generate the same field.

Our result in [ ] is particularly relevant to the case when H satisfies (i) and (ii). It is

(1.2) Theorem. Let H_0 be a simple absolutely irreducible subgroup of $PGL_n(\bar{k})$. Then one of the following holds:

(i) H_0 is of Lie p-type,

or

(ii) $H_0 \simeq Alt_d$ for some d and H_0 lifts to a linear representation of Alt_d in $GL_n(\bar{k})$,

or

(iii) $|Aut\ H_0| \le n^{2\ \log_3 n+5}$ and the field generated by the character values of H_0 is $\mathbb{F}_{p^s}$, $s \le n^2$.

Since in (1.2)(ii) above only a finite number of Alt_d can appear (see [Weisfeiler]), we see that there exists a finite list $\mathfrak{F}_n$ (independent of k) of isomorphism classes of finite simple groups such that we have

(1.3) Corollary. In the assumptions of (1.1) one of the following holds (with n being the degree of φ):

(i) $H \in \mathcal{C}_G$,

(ii) $H_0 \in \mathfrak{F}_n$,

(iii) H_0 is of Lie p-type.

Since in the case (1.2)(ii) above the character values generate at most a quadratic extension of $\mathbb{F}_p$ (see [G. James]), (1.2) and (1.1)(iii) also imply

(1.4) Corollary. Under the assumptions of (1.1), if $k \simeq \mathbb{F}_{p^m}$, $m > n^2$, where n is the degree of φ, then either

(i) $H \in \mathcal{C}_G$,

or

(ii) H_0 is of Lie p-type.

Both (1.3) and (1.4) say, in essence, that most of the maximal subgroups of most of the finite classical groups are of Lie p-type. Thus it is definitely of interest to study the maximal subgroups of G of Lie p-type. To this end we use the Steinberg tensor product theorem [Theorem 41] and his result [, Theorem 43] about the possibility of extending the imbedding of H_0 into $PGL_n(\bar{k})$ to a representation of the corresponding algebraic group. This permits us (at the cost of an extension of the family $\mathcal{C}_G$ to a wider one $\mathcal{EC}_G$ (see (5.4) below)) to strengthen (1.1)(ii) as follows (see (6.2) for a better statement):

(1.5) Theorem. In the assumptions of (1.1) assume in addition that the socle H_0 of H is of Lie p-type and $p \neq 2,3$. Then either H is a subgroup from $\mathcal{EC}_G$ or $\varphi|H_0$ is absolutely infinitesimally irreducible (see [Steinberg] for the latter notion).

The question arises, of course, whether the conditions that $\varphi|H_0$ is infinitesimally irreducible, H does not belong to $\mathcal{EC}_G$, and $HG_0 = G$ are sufficient for H to be maximal. This is definitely not so. A look at E.B. Dynkin's paper [ ] shows that there are infinitely many representations $\mathfrak{H} \to \mathfrak{G}$ of the connected algebraic group corresponding to H_0 into the connected algebraic group corresponding to G_0 whose image is infinitesimally irreducible (for large p) but not maximal. By restriction to groups of rational points over finite fields we get counter-examples to the question above. Nevertheless the class of examples we described turns out to be fairly typical. We first show (in (2.1)) that the imbedding $H_0 \to G_0$ does extend to a homomorphism of the corresponding connected algebraic groups $\mathfrak{H} \to \mathfrak{G}$. Then we have

(1.6) Theorem. Let G, G_0 and φ be as in (1.1). Assume that $p \neq 2,3$ and $m \geq 3$. Let H be a subgroup of G with socle H_0 of Lie p-type, $\varphi|H_0$ absolutely infinitesimally irreducible, and $G = HG_0$. If the image of $\mathcal{H}$ in $\mathcal{G}$ (of algebraic groups) is a maximal algebraic subgroup of $\mathcal{G}$ and H_0 is not isomorphic to one of a finite list $\mathfrak{R}$ (independent of p and $\mathcal{G}$) of simple groups, then H is maximal in G.

The proof of this claim uses the classification of finite simple groups to compile the list $\mathfrak{R}$.

There is a gap between (1.5) and (1.6). First, it may happen that $\mathcal{H}$ is not maximal in $\mathcal{G}$ but, because of extra automorphisms, H is maximal in G. This question, it appears to me, can be settled only after E.B. Dynkin's paper [ ] is redone in positive characteristics. Second, the finite list mentioned in the statement is probably almost empty. Third, the condition $m \geq 3$ is imposed because I can obtain only fragmentary results on whether there exist imbeddings of H into $Sym_d \subseteq G$ (see, however, ( )). Fourth, although the condition that $p \neq 2,3$ can be relaxed, after modifying the statement (see ( )) the case when $p = 2$ seems in an unsatisfactory condition.

2. Extension of homomorphisms between finite groups of Lie p-type to algebraic groups

(2.1) We consider here two finite universal groups of Lie p-type G and M, a homomorphism $\tau: G \to M$ and a representation $\omega: M \to GL_n(\bar{\mathbb{F}}_p)$. We assume that $\varphi := \omega \circ \tau$ is an irreducible representation of G. Let $\mathfrak{G}$ and $\mathfrak{M}$ be algebraic absolutely almost simple simply connected groups associated to G and M respectively. By a result of R. Steinberg [, Theorem 43] φ and ω extend to representations $\tilde{\varphi}$ and $\tilde{\omega}$ of $\mathfrak{G}$ and $\mathfrak{M}$. Moreover the highest weights of $\tilde{\varphi}$ and $\tilde{\omega}$ can be chosen to satisfy the additional conditions given in R. Steinberg [, Theorem 43], (see also (4.1.3) below). Under these additional conditions $\tilde{\varphi}$ (resp. $\tilde{\omega}$) is determined by φ (resp. ω) uniquely up to equivalence of representations. This implies that

(2.1.1) $\tilde{\varphi}(\mathfrak{G})$ (resp. $\tilde{\omega}(\mathfrak{M})$) is determined by φ (resp. ω) uniquely as a subgroup of GL_n.

Indeed, two different $\tilde{\varphi}$ and $\tilde{\varphi}'$ would give that $\tilde{\varphi}(g) = A\tilde{\varphi}'(g)A^{-1}$ for $g \in G$, i.e. $\varphi(g) = A\varphi(g)A^{-1}$ whence A is scalar in view of the irreducibility of φ.

We are interested in when the following holds (it makes sense by (2.1.1)):

(2.1.2) Conjecture. $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

Our results in the direction of this conjecture are given in (2.2), (2.3.3), (2.4), (10.3), (10.4).

(2.1.3) Remarks. (i) This statement is similar to the main result [,(A), p.500] of A. Borel and J. Tits. The main difference is that we are concerned with finite groups and, therefore, one of their powerful instruments, the Zariski closure, is not available.

(ii) Our proofs are based on R. Steinberg's [, Theorem 43]. This proof is non-constructive. The only case when we are able to use it effectively in the case when $\mathfrak{M}$ is the stabilizer of a non-degenerate bilinear form. But even in this case (see below) the proof is not direct.

(2.2) Proposition. (2.1.2) holds if $\mathfrak{M}$ is classical and $\tilde{\omega}$ is its natural representation, except possibly in the case when $p = 2$, $\mathfrak{M}$ is of type D_m and $\tilde{\omega}$ is a twist by the Frobenius endomorphism of an infinitesimally irreducible representation of $\mathfrak{G}$.

(2.2.1) Proof. Let $\tilde{\omega}(\mathfrak{M})$ be a stabilizer of a non-degenerate bilinear form f. Then $\varphi(G)$ also preserves f. Since $\varphi(G)$ is irreducible f is the unique (up to a scalar factor) bilinear form preserved by $\varphi(G)$. Thus φ is equivalent to its contragredient $\check{\varphi}$. By R. Steinberg [, Lemma 78] the highest weight of $(\tilde{\varphi})^{\vee}$ satisfies the same inequalities as one of φ from R. Steinberg [, Theorem 43]. Since $(\tilde{\varphi})^{\vee}$ and $\tilde{\varphi}$ are equivalent on G this implies by R. Steinberg [, Theorem 43] that $\tilde{\varphi}$ and $(\tilde{\varphi})^{\vee}$ are equivalent. Thus $\tilde{\varphi}(\mathfrak{G})$ preserves a non-degenerate bilinear form. This form, by unicity of f for $\varphi(G)$, must be proportional to f whence our claim.

(2.2.2) Let now $\tilde{\omega}(\mathfrak{M})$ be the stabilizer of a non-singular quadratic form F with associated bilinear form f. If $p \neq 2$ then F is uniquely determined by f and we are done by (2.2.1). Since ω is irreducible this implies that $n = 2m$ and $\tilde{\omega}(\mathfrak{M}) \simeq SO_{2m}$, i.e. $\mathfrak{M}$ is of type D_m.

Write $\tilde{\varphi} \simeq \otimes \tilde{\varphi}_i \circ Fr^i$ where the $\tilde{\varphi}_i$ are infinitesimally irreducible (see R. Steinberg [, Theorem 4.1] or ( ) below). Let $I = \{i | \tilde{\varphi}_i \simeq Id\}$. Suppose that $|I| \geq 2$. The condition that $\tilde{\varphi}$ preserves a bilinear form (it does preserve f) is the condition of symmetry on the highest weight of $\tilde{\varphi}$ (see R. Steinberg [, Lemma 78]). This condition is then satisfied by the highest weights of every $\tilde{\varphi}_i$. Thus each $\tilde{\varphi}_i(\mathbb{G})$ preserves a non-degenerate bilinear form. This form is necessarily alternating (well-known and proved in (2.2.3) below). Then by M. Aschbacher [, 9.1(4)] $\tilde{\varphi}(\mathbb{G})$ preserves a unique, up to scalar multiple, quadratic form F'. We must thus have that F and F' are proportional whence $\tilde{\varphi}(\mathbb{G}) \subseteq \tilde{\omega}(\mathfrak{M})$. This concludes our proof of (2.2).

(2.2.3) Lemma. Let H be an irreducible subgroup of $GL_n(\overline{\mathbb{F}}_2)$. If H preserves a non-degenerate symmetric bilinear form f then f is alternating, i.e., $f(x,x) = 0$ for all $x \in \overline{\mathbb{F}}_2^n$.

Proof. Set $V = \overline{\mathbb{F}}_2^n$. By the irreducibility of H there exists $v \in V$ and $g_1, \ldots, g_n \in H$ such that $\{v_i := v + g_i v\}$ is a basis of V. We have $f(v+gv, v+gv) = f(v,v) + f(v,gv) + f(gv,v) + f(gv,gv)$. This $=0$ by symmetricity and invariance of f. Thus $f(v_i, v_i) = 0$ whence $f(w,w) = 0$ for all $w = \sum a_i v_i$.

(2.3) Let $\bar{\mathfrak{G}}$ be the simple Lie algebra over $\mathbb{C}$ of the same type as $\mathfrak{G}$. Let λ be the highest weight of $\tilde{\varphi}$. Then $\tilde{\varphi}$ is a unique quotient of the reduction modulo p of an irreducible representation $\bar{\varphi}$ of $\bar{\mathfrak{G}}$ with "the same" highest weight λ. Let $d_0(\lambda)$ denote the degree of the latter representation. It is given by H. Weyl's formula

$$(2.3.1)\quad d_0(\lambda) = \Pi_{\alpha\in\Sigma^+}((\lambda+\rho,\alpha)/(\rho,\alpha))$$

where Σ^+ is the set of positive roots of $\mathfrak{G}$, ρ is the sum of fundamental weights, and $(\cdot,\cdot)$ is a scalar product invariant under the Weyl group.

(2.3.2) Let now $G \simeq {}^cX_a(\tilde{q}^c)$ (in the notation of R. Steinberg [ ]). Set $q := \tilde{q}$ except when ${}^cX_a = {}^2B_2, {}^2G_2, {}^2F_4$ when we set $q := \tilde{q} \cdot p^{-1/2}$.

(2.3.3) Proposition. (2.1.2) holds if $p \neq 2$ and $q > 2d_0(\lambda)$.

Proof. Let $\bar{V}$ be the space of $\bar{\varphi}$, V its reduction modulo p and V' a maximal submodule of V such that the $\mathfrak{G}$-module V/V' is isomorphic $\tilde{\varphi}$. We consider $\mathfrak{G}$ acting on E := End V/V' via Ad $\circ\ \tilde{\varphi}$. Then E is the quotient of reduction modulo p of End $\bar{V}$. Since End $\bar{V} \simeq \bar{V} \otimes \bar{V}^V$ (as $\bar{\mathfrak{G}}$-modules) the action of $\bar{\mathfrak{G}}$ on End $\bar{V}$ satisfies: $((\mathrm{ad} \circ \bar{\varphi})(\bar{X}_\alpha))^i = 0$ for $i \geq 2 \dim \bar{V} = 2d_0(\lambda)$ for every root element X_α. Therefore in A. Borel [5.13] we have for every absolute root subgroup $X_\alpha(t)$ of $\mathfrak{G}$ that

$$(2.3.4)\quad (\mathrm{Ad} \circ \tilde{\varphi})(x_\alpha(t)) = \Sigma_{i\leq 2d_0(\lambda)} t^i X_{\alpha,i}.$$

For all cases, with the exception of $^2B_2, ^2G_2, ^2F_4$ and , for an appropriately chosen root subgroup $x_\alpha(t)$, we have that $x_\alpha(\mathbb{F}_q) \subseteq G$. We consider then (2.3.4) as a system of linear equations with Vandermonde determinant (if $q > 2d_0(\lambda)$) whose constant terms lie in $\mathrm{Ad} \circ \varphi(G) \subseteq \mathrm{Ad} \circ \tilde{\omega}(\mathfrak{M})$. Solving it we obtain that

(2.3.5) $X_{\alpha,i} \in \overline{\mathbb{F}}_p \cdot \mathrm{Ad} \circ \tilde{\omega}(\mathfrak{M})$.

In particular, each $X_{\alpha,i}$ preserves $\mathrm{Lie}\,\tilde{\omega}(\mathfrak{M}) \subseteq \mathrm{End}\,V$. Therefore

(2.3.6) $\mathrm{Ad} \circ \tilde{\varphi}(\mathfrak{G})$ preserves $\mathrm{Lie}\,\tilde{\omega}(\mathfrak{M})$.

If $p \neq 2,3$ then $\mathrm{Aut}\,\mathrm{Lie}\,\tilde{\omega}(\mathfrak{M})$ has the same type as $\mathfrak{M}$ (see R. Steinberg [ ] or G. M. D. Hogeweij [ ]). Therefore, in this case $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$. The same is true in characteristic 3 except when $\mathfrak{M}$ is of type A_2 and in characteristic 2 except when $\mathfrak{M}$ is of type D_n, $n \geq 3$, or G_2.

(2.3.7) These latter cases can be treated as in B. Weisfeiler [, Section 4]. We do only the case $p = 3$. If $\mathfrak{M}$ is of type A_2 then $\mathfrak{G}$ can be of type A_2 or G_2. If $\mathfrak{G}$ is of type G_2 then the 3-dimensional irreducible representation of $\mathfrak{M}$ gives rise to such a representation of $G \simeq G_2(q)$. Since this latter group does not have any non-trivial irreducible representations of degree ≤ 6 this case is impossible (one can also use a description of all finite subgroups of $SL_3(\overline{\mathbb{F}}_p)$).

If $\mathfrak{G}$ is of type A_2 (and $p = 3$, $\mathfrak{M}$ of type A_2) then both $\mathfrak{G}$ and $\mathfrak{M}$ extend the representation φ of G. Therefore $\tilde{\varphi}(\mathfrak{G}) = \tilde{\omega}(\mathfrak{M})$ by (2.1.1).

(2.3.8) It remains to consider the case $p = 3$ and $G \simeq {}^2G_2(3^{2s+1})$. Let $\theta := Fr^s$. Let α, β be simple roots of $\mathcal{G}$. We can assume that G contains all elements $x_{2\alpha+3\beta}(v)x_{\alpha+2\beta}(v^\theta)$ with v any element of $k := \mathbb{F}_{3^{2s+1}}$. We have by (2.3.4)

$$\tilde{\varphi}(x_{2\alpha+3\beta}(v)x_{\alpha+2\beta}(v^\theta)) = \Sigma_{i,j\le 2d_0(\lambda)} v^{i+p^s j} X_{2\alpha+3\beta,i} X_{\alpha+2\beta,j}.$$

If $q\ (=p^s) > 2d_0(\lambda)$ we obtain as above in (2.3.5), that $X_{2\alpha+3\beta,i}X_{\alpha+2\beta,j}$ preserve Lie $\tilde{\omega}(\mathfrak{M})$, for all i and j. In particular, $X_{2\alpha+3\beta,i} = X_{2\alpha+3\beta,i}X_{\alpha+2\beta,0}$ and $X_{\alpha+2\beta,j}$ preserve Lie $\tilde{\omega}(\mathfrak{M})$. Thus (2.3.6) holds in the case under consideration. The proof is completed as in (2.3.7).

(2.4) Proposition. (2.1.2) holds if $p \neq 2$, $\mathfrak{M}$ is of type G_2 and $\tilde{\omega}$ is its representation of dimension 7.

Proof. Let V be the space of $\tilde{\omega} \oplus \mathrm{id}$ so that V can be considered (when $p \neq 2$) as an algebra of octonions $\mathcal{O}$ and $\mathfrak{M} = \mathrm{Aut}\,\mathcal{O}$. We consider the orthogonal group $\mathcal{H}$ of the norm form of $\mathcal{O}$; $\mathcal{H}$ is of type D_4. We let $\mathcal{H}$ act on $\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}$ via its different fundamental representations corresponding to the extreme ends of the Dynkin diagram of $\mathcal{H}$. Then the diagonal action of $\mathfrak{M}$ determines an embedding $\mathfrak{M} \longrightarrow \mathcal{H}$ so that $\mathfrak{M}$ is the set of fixed points of the triality automorphism σ of $\mathcal{H}$. Since $p \neq 2$ and the action of $\mathfrak{M}$ on $\mathcal{O}$ is completely reducible it follows from (2.2) that each projection $\tau_i \circ \omega \circ \tau$ (on i-th factor, $i = 1,2,3$) extends to $\tilde{\pi}_i : \mathcal{G} \longrightarrow \mathcal{H}$. Since $\omega \circ \tau(G) \subseteq \mathfrak{M}$ it follows from (2.1.1) that $\sigma \circ \tilde{\pi}_1 =$

π_2, $\sigma \circ \tilde{\pi}_2 = \tilde{\pi}_3$, $\sigma \circ \tilde{\pi}_3 = \pi_1$, i.e. $\sigma(\mathfrak{G}) \subseteq \mathfrak{G}$. Thus σ induces an endormorphism of $\mathfrak{G}$. Since it is an automorphism of $\mathfrak{H}$ it is also (?traversality?) an automorphism of $\mathfrak{G}$. Since it is trivial on G it must be trivial on $\mathfrak{G}$. Thus $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$ as claimed.

3. Tensor product decomposability

The "if" part of the following result is a Corollary to Theorem 41 of R. Steinberg [ ]. The "only if" part and its proof are modelled after E.B. Dynkin [, Theorems 3.1 and 3.2].

(3.1) Theorem. Let k be an algebraically closed field of characteristic exponent p, $\mathbb{G}$ an almost simple algebraic group over k, φ and ψ infinitesimally irreducible representations of $\mathbb{G}$ with highest weights λ and μ respectively. Then $\varphi \otimes \psi$ is irreducible if and only if

(i) $\operatorname{supp} \lambda \cap \operatorname{supp} \mu = \emptyset$,

(ii) λ and μ can not be connected by a p-chain.

Here $\operatorname{supp} \lambda$ is the set of simple roots α of $\mathbb{G}$ such that $(\lambda, \check{\alpha}) \neq 0$ (in $\mathbb{Z}$). Let Δ (resp. Σ, resp. Σ^+) be the set of simple (resp. all, resp. positive) roots of $\mathbb{G}$. The p-chains are defined by

(3.2) Definition. Let λ and μ be weights of $\mathbb{G}$. A subset $\alpha_1, \ldots, \alpha_m \subseteq \Delta$ is a (minimal) p-chain connecting λ and μ if

(mc1) $(\lambda, \check{\alpha}_1) \neq 0$

(mc2) $(\mu, \check{\alpha}_m) \neq 0$

and for $m > 1$

(mc3) $(\lambda, \check{\alpha}_2) = \ldots = (\lambda, \check{\alpha}_m) = 0$

(mc4) $(\mu,\check{\alpha}_1) = \ldots = (\mu,\check{\alpha}_{m-1}) = 0$

(mc5) $(\alpha_i,\check{\alpha}_{i+1})(\alpha_{i+1},\check{\alpha}_i) \notin p\mathbb{Z}$

(mc6) $(\alpha_i,\check{\alpha}_j) = 0$ if $|i-j| \geq 2$.

(3.3) Remark. Let Δ_1, Δ_2 be two subsets of Δ such that $\Delta_1 \cap \Delta_2 = \emptyset$ and Δ_1 cannot be connected to Δ_2 by a p-chain. By inspection of the Dynkin diagrams one verifies that this can happen only if either $p = 3$ and $\mathfrak{G}$ is of type G_2 or $p = 2$ and $\mathfrak{G}$ is of type B_n, C_n, F_4. In both cases the roots in each Δ_1 and Δ_2 have the same length but the roots from Δ_1 have length different from those from Δ_2. This remark explains why the "if" part follows from the Corollary to Theorem 41 in [ ].

(3.5) Lemma. Let φ and ψ be infinitesimally irreducible representations with highest weights λ and μ. Let $\alpha_1,\ldots,\alpha_m$ be a minimal chain connecting supp λ with supp μ ($m = 1$ if supp $\lambda \cap$ supp $\mu \neq \emptyset$). Then $\varphi \otimes \psi$ is reducible.

(3.5.1) Proof when supp $\lambda \cap$ supp $\mu \neq \emptyset$. Let $\alpha \in$ supp $\lambda \cap$ supp μ. Let x and y be the highest weight vectors of φ and ψ respectively. We have

(*) $e_\alpha e_{-\alpha} x = (\lambda,\check{\alpha})x.$

Indeed $e_\alpha e_{-\alpha} x = e_{-\alpha} e_\alpha x + [e_\alpha, e_{-\alpha}]x = h_\alpha x = \lambda(h_\alpha)x = (\lambda,\check{\alpha})x$ (we have $e_\alpha x = 0$ since x is a highest weight vector). Here $\{e_\alpha\}_{\alpha\in\Sigma} \cup \{h_\alpha\}_{\alpha\in\Delta}$ is a Chevalley basis of the Lie algebra Lie $\mathbb{G}$ of $\mathbb{G}$. Since φ is infinitesimally irreducible it follows that $0 \leq (\lambda,\check{\alpha}) \leq p - 1$ and since $\alpha \in \operatorname{supp} \lambda$ it follows that $(\lambda,\check{\alpha}) \neq 0$. Thus $(\lambda,\check{\alpha}) \neq 0$ in k. Then (*) implies that $e_{-\alpha}x \neq 0$.

Let $z := (\mu,\check{\alpha})e_{-\alpha}x \otimes y - (\lambda,\check{\alpha})x \otimes e_{-\alpha}y$. Then $z \neq 0$ by the above, but $e_{\alpha}z = 0$ by (*). For $\beta \in \Delta$, $\beta \neq \alpha$, we also have $[e_{\beta},e_{-\alpha}] = 0$ $e_{\beta}x = 0$, $e_{\beta}y = 0$. Therefore $e_{\beta}z = 0$ for all $\beta \in \Delta$. Let A be the set of $\gamma \in \Sigma^+$ such that $e_{\gamma}z = 0$; $A \supseteq \Delta$. If $A \neq \Sigma^+$ there exists $\gamma \in \Sigma^+ - A$ such that $[e_{\gamma},e_{-\alpha}] = ce_{\delta}$, $\delta \in A$. It follows that $e_{\gamma}z = 0$ i.e. $\gamma \in A$, a contradiction. Thus $A = \Sigma^+$ and z is a highest weight vector of $\varphi \otimes \psi$ for Lie $\mathbb{G}$.

If $\varphi \otimes \psi$ were irreducible for $\mathbb{G}$ it would be for Lie $\mathbb{G}$ a direct sum of a number of equivalent irreducible representations. Then the highest weight $\lambda + \mu$ of $\varphi \otimes \psi$ should be congruent modulo p$\cdot$(lattice of weights) to $\lambda + \mu - \alpha$, or, the same $(\beta,\check{\alpha}) \equiv 0 \bmod p$ for all $\beta \in \Sigma$. This can happen only when $p = 2$, $\mathbb{G}$ is of type C_n, $n \geq 1$, α is the long simple root, say α_n.

In this latter case let λ_n be the fundamental weight of C_n paired to α. Set $\lambda' := \lambda - \lambda_n$, $\mu := \mu - \lambda_n$. These are dominant weights by our assumption that $\alpha \in \operatorname{supp} \lambda \cap \operatorname{supp} \mu$. Since $p = 2$ and λ and μ are the highest weights of infinitesimally irreducible representations, it follows that $\alpha \notin \operatorname{supp} \lambda'$, $\alpha \notin \operatorname{supp} \mu'$. Let φ', ψ', φ_n be the infinitesimally irreducible representations of $\mathbb{G}$ with highest weights λ', μ', λ_n. Then by R. Steinberg [, Corollary to Theorem 41], $\varphi_1 \otimes \varphi_n$ and $\psi' \otimes \varphi_n$ are infinitesimally irreducible and, therefore, equivalent to φ and ψ respectively. Thus $\varphi \otimes \psi$ is equivalent to $\varphi' \otimes \psi' \otimes \varphi_n \otimes \varphi_n$. But φ_n is the representation of C_n with highest weight λ_n and so $\varphi_n \otimes \varphi_n$ has a highest weight $2\lambda_n$ which is the highest weight of $\varphi_n \circ \mathrm{Fr}$ (in characteristic 2). The representation $\varphi_n \circ \mathrm{Fr}$ has the same dimension as φ_n and so $\varphi_n \otimes \varphi_n$ can not be irreducible.

(3.5.2) Assume now that $\mathrm{supp}\,\lambda \cap \mathrm{supp}\,\mu = \emptyset$ but that λ and μ can be connected by a p-chain $\{\alpha_1,\alpha_2\}$ of length 2. Let x and y be the highest weight vectors of φ and ψ. Then

$$x_0 := x, \quad x_1 := e_{-\alpha_1}x, \quad x_2 := e_{-\alpha_2}e_{-\alpha_1}x$$

$$y_0 := y, \quad y_1 := e_{-\alpha_2}y, \quad y_2 := e_{-\alpha_1}e_{-\alpha_2}y.$$

From (mc3) and (mc4) we have

(**) $e_{-\alpha_2}x = 0 = e_{-\alpha_1}y.$

As in (*) in (3.5.1) we have

$$e_{\alpha_1}x_1 = (\lambda,\check{\alpha}_1)x_0$$

$$e_{\alpha_2}x_2 = (\lambda-\alpha_1,\check{\alpha}_2)x_1 = -(\alpha_1,\check{\alpha}_2)x_1.$$

Under our assumptions these imply that $x_1 \neq 0$, $x_2 \neq 0$. From (**):

$$e_{\alpha_2}x_1 = 0 = e_{\alpha_1}x_2.$$

Similarly

$$e_{\alpha_2} y_1 = (\mu_1 \check{\alpha}_2) y_0$$

$$e_{\alpha_1} y_2 = (\mu - \alpha_2, \check{\alpha}_1) y_1 = -(\alpha_2, \check{\alpha}_1) y_1$$

$$e_{\alpha_2} y_2 = 0 = e_{\alpha_1} y_1$$

and

$$y_1 \neq 0, \quad y_2 \neq 0.$$

Let

$$z = (\lambda, \check{\alpha}_1)(\alpha_1, \check{\alpha}_2) x_0 \otimes y_2 + (\alpha_1, \check{\alpha}_2)(\alpha_2, \check{\alpha}_1) x_1 \otimes y_1 + (\mu, \check{\alpha}_2)(\alpha_2, \check{\alpha}_1) x_2 \otimes y_0.$$

Then $z \neq 0$. We have

$$e_{\alpha_1} z = -(\lambda, \check{\alpha}_1)(\alpha_1, \check{\alpha}_2)(\alpha_2, \check{\alpha}_1) x_0 \otimes y_1 + (\alpha_1, \check{\alpha}_2)(\alpha_2, \check{\alpha}_1)(\lambda, \check{\alpha}_1) x_0 \otimes y_1 = 0$$

and

$$e_{\alpha_2} z = (\alpha_1, \check{\alpha}_2)(\alpha_2, \check{\alpha}_1)(\mu, \check{\alpha}_2) x_1 \otimes y_0 - (\mu, \check{\alpha}_2)(\alpha_2, \check{\alpha}_1)(\alpha_1, \check{\alpha}_2) x_1 \otimes y_0 = 0.$$

(Remark: If $(\alpha_2, \check{\alpha}_1) \in p\mathbb{Z}$ then $z = (\lambda, \check{\alpha}_1)(\alpha_1, \check{\alpha}_2) x_0 \otimes y_2$ and we still have $e_{\alpha_1} z = 0 = e_{\alpha_2} z$. But we do not know anymore that $y_2 \neq 0$).

Now we have $e_\beta z = 0$ for all $\beta \in \Delta$. Let $A = \{\gamma \in \Sigma^+ | e_\gamma z = 0\}$. We have $\Delta \subseteq A$. Since $[e_\alpha, e_\beta] = \pm e_{\alpha+\beta}$, $\alpha, \beta \in \Delta$ and $\alpha + \beta \in \Sigma$ it follows that $\Sigma^+ \bigcap (\Delta+\Delta) \subseteq A$. Let $\gamma \in \Sigma^+ - A$ be such that $\gamma - \alpha_1 - \alpha_2 \in A$. Then $e_\gamma x_i = 0 = e_\gamma y_i$, whence $\gamma \in A$. Since Δ and $\Sigma \bigcap (\Delta+\Delta) \subseteq A$ it follows that $A = \Sigma^+$. Thus $\varphi \otimes \psi$ is not infinitesimally irreducible.

On the other hand since φ and ψ are infinitesimally irreducible and supp $\lambda \bigcap$ supp $\mu = \emptyset$, it follows that $\lambda + \mu$ is a highest weight of an infinitesimally irreducible representation, whence $\varphi \otimes \psi$ if irreducible must be infinitesimally irreducible, a contradiction.

(3.5.3) We leave the general case as an exercise.

4. Fields of rationally of representations of finite groups of Lie p-type

The arguments below are based mostly on R. Steinberg [,§12,13] and on ideas of M. Liebeck [,§2] or myself [,§5].

(4.1) If $\varphi: G \to GL_n(\overline{\mathbb{F}}_p)$ is a representation of a group G we call the field $\mathbb{F}_p(\varphi)$ generated over $\mathbb{F}_p$ by the $\mathrm{Tr}\,\varphi(g)$, $g \in G$, a rationality field of φ. It is important for us, in view of (1.1)(iii) to know the fields of rationality of irreducible representations of groups of Lie p-type. It is known (see, e.g., [Aschbacher (3.2)] that

(4.1.1) For finite groups G and absolutely irreducible φ the representation φ is equivalent to a representation $G \to GL_n(\mathbb{F}_p(\varphi))$.

The main problem for us here is to describe $\mathbb{F}_p(\varphi)$ in terms of the highest weights of the corresponding algebraic group.

(4.1.2) If $\mathbb{G}$ is an algebraic semisimple group defined over a finite field $k \subseteq \overline{\mathbb{F}}_p$ then there is an action of $\Gamma := \mathrm{Gal}(\overline{\mathbb{F}}_p/k)$ on the weights of $\mathbb{G}$; this action preserves a system of simple roots Δ of $\mathbb{G}$ (with respect to appropriate choices), see, e.g., J. Tits [,3.1]. Namely we associate to every $\alpha \in \Delta$ a conjugacy class $\mathcal{P}_\alpha$ of minimal non-solvable parabolics. Then for $\gamma \in \Gamma$ we have that $\gamma(\mathcal{P}_\alpha)$ is also such a class, say $\mathcal{P}_\beta$, and we set then $\gamma(\alpha) := \beta$ and extend this action by $\mathbb{Q}$-linearity to all weights. Let $\Gamma_{\mathbb{G}} := \{\gamma \in \Gamma \mid \gamma(\alpha) = \alpha \text{ for all } \alpha \in \Delta\}$ and let $k_{\mathbb{G}} := \overline{\mathbb{F}}_p^{\Gamma_{\mathbb{G}}}$, the fixed field of $\Gamma_{\mathbb{G}}$. If $\mathbb{G}$ is absolutely almost simple then $k_{\mathbb{G}} = k$ if $\mathbb{G}$ is split over k, $[k_{\mathfrak{G}}:k] = 2$ if $\mathfrak{G}$ is quasi-split of type $^2A_n, {}^2D_n, {}^2E_6$ and $[k_{\mathfrak{G}}:k] = 3$ if $\mathfrak{G}$ is quasi-split of type $^3D_4$.

If $\mathfrak{G}$ is just almost k-simple then $\mathfrak{G}$ is centrally k-isogeneous to $R_{K/k}\mathfrak{H}$ for some finite extension K of k and for an absolutely almost simple K-group $\mathfrak{H}$ (see [ ] for a definition of the Weyl field restriction functor $R_{K/k}$ and for the above claim). In this case $\Gamma_{\mathfrak{G}} = \Gamma_{\mathfrak{H}}$ so that $k_{\mathfrak{G}} = K_{\mathfrak{H}}$.

(4.1.3) Let $\mathfrak{G}$ be an absolutely almost simple algebraic group over k and $G := \mathfrak{G}(k)$. Set also $q := |k|$, $a := [k:\mathbb{F}_p]$, $c := [k_{\mathfrak{G}}:k]$. Let φ be an absolutely almost irreducible representation of G. By R. Steinberg [, Theorem 43] φ extends to an absolutely irreducible representation $\tilde{\varphi}$ of $\mathfrak{G}$. By R. Steinberg [, Theorem 41 and 43] $\tilde{\varphi}$ is equivalent to $\otimes_{i=0}^{a-1} \tilde{\omega}_i \circ F_r{}^i$ where the $\tilde{\omega}_i$ are infinitesimally irreducible.

(4.1.4) Let λ_ψ be the highest weight (with respect to an appropriate simple root system) of an absolutely irreducible representation ψ of $\mathfrak{G}$. Let $\lambda_1,\ldots,\lambda_r$ (where $r = \mathrm{rank}(\mathfrak{G})$) be the fundamental highest weights. Then we know that $\lambda_{\psi\circ Fr} = p\cdot\lambda_\psi$ and $\lambda_\omega = \Sigma_{i=1}^r a_i\lambda_i$ with $0 \le a_i < p$ if and only if ω is infinitesimally irreducible.

(4.1.5) (see R. Steinberg [,§13] or C.W. Curtis [ ].) To a representation φ of G (from 4.1.3) there is also uniquely associated a Curtis highest weight. Let $\mathfrak{B}$ be a Borel k-subgroup of $\mathfrak{G}$, $\mathfrak{U} = [\mathfrak{B},\mathfrak{B}]$ the unipotent radical of $\mathfrak{B}$, and $\mathfrak{T}$ a maximal k-torus of $\mathfrak{B}$. Set $T := \mathfrak{T}(k)$, $B := \mathfrak{B}(k)$, $U := \mathfrak{U}(k)$. Then, associated to φ, there is a unique tuple $(\chi_\varphi,\mu_1,\ldots,\mu_{r'})$ where

$\chi_\varphi \in \mathrm{Hom}(T,\overline{\mathbb{F}}_p^*)$

$\mu_i = \pm 1$

$r' := \mathrm{rank}_k\, \mathbb{G}$.

We call $(\chi_\varphi, \mu_1, \ldots, \mu_{r'})$ the Curtis highest weight of φ. One sees easily that $\chi_\varphi(T) \subseteq k_{\mathbb{G}}$. Defining an action of $\mathrm{Gal}(k_{\mathbb{G}}/\mathbb{F}_p)$ on $\mathrm{Hom}(T, k_{\mathbb{G}})$ via the action on the second factor only we obtain an action of $\mathrm{Gal}(k_{\mathbb{G}}/\mathbb{F}_p)$ on the set of representations of G. Our purpose is to relate this action to the tensor product representation of φ, see (4.1.3).

(4.1.6) Let Δ be the simple root system of $\mathbb{G}$ (from (4.1.3),(4.1.4),(4.1.5)). Assume, in addition, that $\mathbb{G}$ is simply connected. Then $\mathcal{T}$ is a direct product over k of k-subtori $\mathcal{T}_1, \ldots, \mathcal{T}_{r'}$, each corresponding to a relative simple root, i.e., to an orbit of $\Gamma_{\mathbb{G}}$ on Δ. Moreover, over $\overline{k}$ (or $k_{\mathbb{G}}$) each $\mathcal{T}_i$ is a direct product of 1-dimensional tori corresponding to the (absolute) simple roots in the corresponding orbit.

(4.1.7) We now make the following observation. If x is a generator of $\mathbb{Z}/(q-1)$ then every element $\neq 1$ of $\mathbb{Z}/(q-1)$ has a unique representation in the form $x^{\sum_{i=0}^{a-1} a_i p^i}$ with $0 \le a_i < p$ and $1 \in \mathbb{Z}/(q-1)$ has two such representations: $a_0 = \ldots = a_{a-1} = 0$ and $a_0 = \ldots a_{a-1} = p - 1$.

(4.1.8) We note that if $\mathcal{T}_i$ corresponds to a trivial orbit, i.e., to a simple root $\alpha \in \Delta$, then for the corresponding fundamental weight λ we have that $\lambda | T_i$ (where $T_i = \mathcal{T}_i(k)$) is a generator, say χ, of $\mathrm{Hom}(T_i, \overline{\mathbb{F}}_p^*)$. Then (4.1.7) shows that $\sum_{i=0}^{a-1} a_i p^i \lambda$, $0 \le a_i < p$, corresponds to

$\chi^{\Sigma a_i p^i}$ except when $\Sigma a_i p^i = q - 1$, i.e. when $a_1 = \ldots = a_{a-1} = p - 1$. Considering $\Sigma a_i p^i \lambda$ as a highest weight of the rank 1 k-group $\mathfrak{G}_\alpha$ corresponding to α we see thereby (and by Steinberg [, Theorem 45]) that the representation of $\mathfrak{G}_\alpha$ with highest weight $\Sigma_{i=0}^{a-1} a_i p^i \lambda$, $0 \le a_i < p$, restricts to a representation of $\mathfrak{G}_\alpha(k)$ with the Curtis highest weight $(\chi^{\Sigma a_i p^i},1)$, unless $\Sigma a_i p^i = q - 1$.

When $\Sigma a_i p^i = q - 1$ then the dimension of the representation of $\mathfrak{G}_\alpha$ with highest weight $(q-1)\lambda$ is q and by [, Corollary (d) to Theorems 44 and 45] we get that the representation of $\mathfrak{G}_\alpha$ with highest weight $(q-1)\lambda$ corresponds to the Curtis highest weight $(1,-1)$.

(4.1.9) If now $\mathfrak{T}_i$ corresponds to a non-trivial orbit (of length $c = [k_{\mathfrak{G}}:k]$ then), let $\bar{\alpha} = \{\alpha_1,\alpha_2,\ldots,\alpha_c\}$ be that orbit. Set $T_i := \mathfrak{T}_i(k)$. Then $T_i \simeq k_{\mathfrak{G}}^*$. Let $\mathfrak{G}_\alpha$ be the group of relative rank 1 corresponding to $\bar{\alpha}$ and let $\lambda_1,\ldots,\lambda_c$ be the fundamental (absolute) weights of $\mathfrak{G}$ corresponding to $\alpha_1,\ldots,\alpha_c$. By the definition of twisting we have (see [ ]) that $\lambda_j | T_i = \lambda_1 \circ Fr^{(j-1)a}$ for $j = 1,\ldots,c$. We can take $\lambda_1 | T_i$ for a generator say χ, of $\mathrm{Hom}(T_i,\bar{\mathbb{F}}_p^*)$ and then (4.1.7) together with the above remark about $\lambda_j | T_i$ show that $\Sigma_{j=1}^{c} \Sigma_{i=0}^{a-1} a_{ij} p^i \lambda_j$, $0 \le a_{ij} < p$, corresponds to $\chi^{\Sigma a_{ij} p^{i+(j-1)a}}$ except when all $a_{ij} = p - 1$. Now the argument of (4.1.8) shows that the representation of $\mathfrak{G}_{\bar{\alpha}}$ with the highest weight $\Sigma_{j=1}^{c} \Sigma_{i=0}^{a-1} a_{ij} p^i \lambda_j$, $0 \le a_{ij} < p$, restricts to a representation of

$\mathbb{G}_{-\alpha}(k)$ with the Curtis highest weight $(\chi^{\Sigma a_{ij}p^{i+(j-1)a}},1)$ except when all a_{ij} are $p-1$ in which case the corresponding Curtis weight is $(1,-1)$.

(4.1.10) As mentioned in (4.1.5) $\mathrm{Gal}(k_{\mathbb{G}}/k)$ acts on the Curtis highest weights of $G = \mathbb{G}(k)$ where $\mathbb{G}$ is an absolutely almost simple simply connected k-group. For each Curtis highest weight $h := (\chi,\mu_1,\ldots,\mu_{r'})$ let $\lambda = \lambda(h)$ be the highest weight of the corresponding as in (4.1.8) and (4.1.9) representation of $\mathbb{G}$. Write $\lambda = \Sigma_{j=1}^{r} \Sigma_{i=0}^{a-1} a_{ij}p^{i}\lambda_j$ where the λ_j are the fundamental highest weights of $\mathbb{G}$ and $0 \le a_{ij} < p$. Denote by $\delta : \mathrm{Gal}(k_{\mathbb{G}}/\mathbb{F}_p) \longrightarrow \mathrm{Aut}\,\Delta$ the action of $\mathrm{Gal}(k_{\mathbb{G}}/\mathbb{F}_p)$ on Δ defining the twisting (see (4.1.2)).

We have $\mathrm{Fr}(\chi) = \chi^p$ and, using the correspondence described in (4.1.8) and (4.1.9) we see that on every $\mathbb{G}_{-\alpha}$, $\alpha \in \Delta$, we have that

$$\lambda(\chi^{p},\mu_1,\ldots,\mu_{r'}) = \Sigma_{j=1}^{r} \Sigma_{i=0}^{a-2} a_{ij}p^{i+1}\delta(\mathrm{Fr})\lambda_j) + \Sigma_{j=1}^{r} a_{a-1j}\delta(\mathrm{Fr})\lambda_j).$$

(for the case when the corresponding $\mu_j = -1$ we have $a_{ij} = p-1$ for all i and the above equality is equally clear).

Thus it holds for weights of $\mathbb{G}$ as the whole. To give a somewhat nicer description we define the section $s\colon \mathbb{Z}/a \longrightarrow \mathbb{Z}$ by taking for $s(n \bmod a)$ the integer between 0 and $a-1$ congruent to n modulo a. Then we have the following action of $\mathrm{Gal}(k/\mathbb{F}_p) \simeq \mathbb{Z}/a$ on the set of p^{ra} highest weights of $\mathbb{G}$ of the form $\Sigma_{j=1}^{r} \Sigma_{i=0}^{a-1} a_{ij}p^{i}\lambda_j$

$$\mathrm{Fr}(\lambda) = \Sigma_{j=1}^{r} \Sigma\, a_{ij}\, p^{s(i+1 \bmod a)}\lambda_j.$$

Our major contention can now be stated as

(4.1.11) Theorem. Let φ be an absolutely irreducible representation of $G = \mathbb{G}(k)$ and let λ and μ be the highest weights of representations of $\mathbb{G}$ corresponding to φ and $\varphi \circ Fr$ as in (4.1.3). Then

$$\mu = \delta(Fr)Fr(\lambda).$$

Remark. This, of course, can be verified ad hoc, using just R. Steinberg's extension theorem [, Theorem 43]. However, the "details" seem to be equivalent to our discussion. The statement of (4.1.11) was, undoubtfully, known. For example, M. Liebeck in [,§2] states substantially the same.

(4.2) For each highest weigth λ of the form $\Sigma_{j=1}^{r} \Sigma_{i=0}^{a-1} a_{ij}p^{i}\lambda_{j}$ we call $\lambda(i) := p^{i}\Sigma_{j=1}^{r} a_{ij}\lambda_{j}$, $i = 0,1,\ldots,a-1$, the parts of λ. Let $\Lambda(i)$ be the intersection of an orbit of $\lambda(i)$ under $Gal(k/\mathbb{F}_p)$ (as described in (4.1.10)) with the set of parts of λ.

(4.2.1) Example. $\mathbb{G}$ split of type A_2, $a = 20$, $p > 3$,

$$\lambda = \lambda_1 + 2p^2\lambda_1 + 3p^6\lambda + p^{10}\lambda_1 + p^{15}\lambda_1 + 3p^{16}\lambda_1.$$

Then the non-zero parts are:

$$\lambda_1, 2p^2\lambda_1, 3p^6\lambda_1, p^{10}\lambda_1, p^{15}\lambda_1, 3p^{16}\lambda_1.$$

We have:

$$\Lambda(0) = \Lambda(5) = \Lambda(10) = \Lambda(15) = \{\lambda_1, p^5\lambda_1, p^{10}\lambda_1, p^{15}\lambda_1\},$$

$\Lambda(2) = 2p^2\lambda_1$,

$\Lambda(6) = \Lambda(16) = \{3p^6\lambda_1, 3p^{16}\lambda_1\}$.

(4.2.2) Let now $\Gamma(i)$ be the (setwise) stabilizer in $\mathrm{Gal}(k/\mathbb{F}_p)$ of $\Lambda(i)$ (for the action given in (4.1.10)) and let $k(i) := k^{\Gamma(i)}$ be the fixed field of $\Gamma(i)$. In the Example (4.2.1) we have

$\Gamma(0) = \langle Fr^5\rangle$, $k(0) = \mathbb{F}_{p^5}$,

$\Gamma(2) = \langle 1\rangle$, $k(2) = k$,

$\Gamma(6) = \langle Fr^{10}\rangle$, $k(6) = \mathbb{F}_{p^{10}}$.

The intended use of the above notions is based (see proof of (4.3) and ( ) below) on the following tautological

(4.2.3) Proposition. Let $\mathbb{G}$ be an almost simple k-group, $|k| = p^a$, d a divisor of a, φ its absolutely irreducible representation with the highest weight $\lambda = \Sigma_{i=1}^{a/d-1} p^{s+id} \Sigma_{j=1}^{r} b_j\lambda_j$ where $0 \le b_j < p^d$ and the λ_j are the fundamental highest weights of $\mathbb{G}$. Let $t := a/d$, $k' := \mathbb{F}_{p^t} \subseteq k$. Then φ considered as a representation of $(R_{k/k'}\mathbb{G})(k')$ (of course, $\simeq \mathbb{G}(k)$) lifts to an algebraic $\overline{\mathbb{F}}_p$-representation of $R_{k/k'}\mathbb{G}$ with highest weight $\Sigma_{i=0}^{a/d-1}\Sigma_{i=1}^{r} b_{j,i}\lambda'_{j+ai/d}$ where the λ'_m are the fundamental weights of $R_{k/k'}\mathbb{G}$ and $0 \le b_{ji} < p^d$.

Indeed the fundamental weights of $R_{k/k'}\mathbb{G}$ are elements of the orbit of the fundamental weights of $\mathbb{G}$ (identified with an almost simple component of $R_{k/k'}\mathbb{G}$ over $\overline{\mathbb{F}}_p$) under $\mathrm{Gal}(k/k')$).

(4.3) Let $\mathbb{G}$ and G be as in (4.1.10). It follows from C. Curtis [,(5.5)] that an irreducible $\bar{\mathbb{F}}_pG$-module φ with curtis highest weight $(\chi,\mu_1,\ldots,\mu_{r'})$ is isomorphic to a unique quotient of the left ideal in the group algebra $\bar{\mathbb{F}}_p[G]$ generated by a sum of the form $\bar{U}_w^- H_\chi w\bar{U}$ where $\bar{U},w,\bar{U}_w \in \mathbb{F}_p[G]$ and $H_\chi := \sum_{t\in T} \chi(t)t \in \mathbb{F}_p(\chi)[G]$. This implies that

(4.3.1) $$\mathbb{F}_p(\varphi) = \mathbb{F}_p(\chi).$$

This is, clearly, what is meant in R. Steinberg [, Corollary (a) to Theorem 46].

Let $\tilde{\varphi}$ be the corresponding representation of $\mathbb{G}$ with highest weight $\sum b_i\lambda_i$, $0 \le b_i < p^a$, and let Γ_φ and Γ'_φ its stabilizers in $\mathrm{Gal}(k_{\mathbb{G}}/\mathbb{F}_p)$ and $\mathrm{Gal}(k/\mathbb{F}_p)$ with the actions described in (4.1.11) and (4.1.10) respectively. Set $k_\varphi := k_{\mathbb{G}}^{\Gamma_\varphi}$ and $k'_\varphi := k^{\Gamma'_\varphi}$.

(4.3.2) Theorem. (i) $k_\varphi = \mathbb{F}_p(\varphi)$,

(ii) $k_\varphi = k'_\varphi$ if the action δ of $\mathrm{Gal}(k_{\mathbb{G}}/\mathbb{F}_p)$ on Δ is trivial,

(iii) $[k_\varphi : k'_\varphi] = [k_{\mathbb{G}} : k]$ otherwise.

Proof. (i) is a combination of (4.3.1) with (4.1.11). (ii) follows directly from (4.1.11) (and can also be deduced from J. Tits [, concluding statement of (3.3)]). If the highest weight of $\tilde{\varphi}$ (see (4.1.13)) is not invariant under $\mathrm{Gal}(k_{\mathbb{G}}/\mathbb{F}_p)$ neither is the corresponding Curtis highest weight. Thus $k_\varphi \not\subseteq k$, whence $k_\varphi k = k_{\mathbb{G}}$, i.e. $0 < [k_\varphi : k'_\varphi] \le [k_{\mathbb{G}} : k]$. But all the fields involved are finite, so $[k_\varphi : k'_\varphi] \mid [k_{\mathbb{G}} : k]$ whence (iii).

(4.4) To complete the picture consider again an algebraic absolutely irreducible representation $\tilde{\varphi}$ of an algebraic absolutely almost-simple k-group $\mathbb{G}$. Let λ be the highest weight of $\tilde{\varphi}$, $\Lambda(i)$ and $\Gamma(i)$ have the meaning described in (4.2). Set $\mu(i) := \Sigma_{\mu \in \Lambda(i)}\mu$. Let $\mu_1,\ldots,\mu_t$ be the set of different non-zero $\mu(i)$ and $\tilde{\varphi}_i,\ldots,\tilde{\varphi}_t$ the corresponding irreducible representations of $\mathbb{G}$.

Set $G := \mathbb{G}(k)$, $\varphi := \tilde{\varphi}|G$, $\varphi_i := \tilde{\varphi}_i|G$ for $i = 1,\ldots,t$, $q := |k|$. Suppose that φ is absolutely irreducible and that $\lambda = \Sigma_{i=1}^{\mathrm{rk}\mathbb{G}} a_i\lambda_i$ with fundamental weights λ_i of $\mathbb{G}$ and $0 \leq a_i < q$. Let Γ_φ be as in (4.3) and $\Gamma(i)$, $i = 1,\ldots,t$, as in (4.2). We clearly have $\Gamma_\varphi = \bigcap_{i=1}^{t} \Gamma(i)$ whence by (4.3.2) (i)

(4.4.1) Proposition. $\mathbb{F}_p(\varphi) = \mathbb{F}_p(\varphi_1)\ldots\mathbb{F}_p(\varphi_t)$.

Set $k' := \mathbb{F}_p\cdot(\varphi) \cap k$. Then by (4.2.3), (4.4.1) implies

(4.4.2) Proposition. $\tilde{\varphi}$ can be considered as an $\mathbb{F}_p(\varphi)$-representation of $R_{k/k'}\mathbb{G}$.

We next consider the case when t from (5.4.3) is 1, i.e. $\lambda = \mu_1$. In the case of an algebraic K'-group $\mathbb{G}$, its algebraic K'-representation φ and a subfield K of K' such that $[K':K] < \infty$, denote by $R^{\otimes}_{K'/K}\varphi$ the $\otimes_{\sigma\ldots}\sigma\circ\varphi$. It will be a K-representation of $R_{K'/K}\mathbb{G}$. In our case, if ψ is a non-trivial absolutely irreducible representation of $\mathbb{G}$ whose highest weight is a part of λ (see (4.2.3)). Assume that $\Lambda(s) \neq 0$ for definiteness. Then by (4.2.3)

By (5.1.1), (5.2.8) the representation with highest weight $\lambda(s)$ has a non-degenerate bilinear, quadratic or sesqui-linear form if so does φ. Let $\mathfrak{H}$ be the unitary group of that form. It is defined over k? Let $k' := k^{\Gamma(s)}$. Then by (4.2.3) $\tilde{\varphi}$ can be considered as a representation of $R_{k/k'}\mathfrak{G}$.

(5.4.5) Proposition. $\tilde{\varphi}(\mathfrak{G}) \subseteq R_{k/k'}\mathfrak{H}$.

?? SOME CONFUSION HERE

Possibly some pages are out of order in the manuscript

5. Invariant bilinear and sesquilinear forms and an extension of the family $\mathcal{C}_G$ of M. Aschbacher

(5.1) Let $\mathbb{G}$ be an absolutely almost simple algebraic group over a finite field k, $|k| = p^a$, φ an absolutely irreducible $\bar{\mathbb{F}}_p$-representation of $G := \mathbb{G}(k)$, λ the highest weight of the corresponding (as in (4.1.3)) representation $\tilde{\varphi}$ of $\mathbb{G}$ and $\lambda(0),\ldots,\lambda(a-1)$ the corresponding parts (so that $\lambda(i)/p^i$ is the coefficient of p^i in the p-adic decomposition of λ).

If $\mathbb{G}$ preserves a non-degenerate bilinear form, say f, then so does G. Conversely if G preserves a non-degenerate bilinear form then it means that the representation φ is equivalent to its dual $\varphi^{\vee}$. By the unicity of the correspondence (see (4.1.3)) the dual $\tilde{\varphi}^{\vee}$ of the corresponding representation of $\mathbb{G}$ should restrict to $\varphi^{\vee}$ and equivalence of φ and $\varphi^{\vee}$ implies that of $\tilde{\varphi}$ and $\tilde{\varphi}^{\vee}$.

Now by R. Steinberg [, Lemma 78] $\tilde{\varphi}$ has an invariant bilinear form if and only if its highest weight λ satisfies $\lambda = -w_0\lambda$, where w_0 is the longest element of the Weyl group. If this condition holds for λ it does so for its parts (see (4.2)) as well. Whence

(5.1.1) The components of the tensor-product decomposition of φ, as in (4.1.3), have non-degenerate invariant bilinear forms if so does φ.

In characteristic 2 for groups of type B_n, C_n, F_4 and in characteristic 3 for groups of type G_2 there are additional $\otimes$-decompositions of the representations, see R. Steinberg [, Corollary to theorem 41] and/or §3 above. For them the same argument as above yields

(5.1.2) The components of these additional $\otimes$-decompositions of φ have a non-degenerate invariant bilinear forms if so does φ.

Its handy to use (and worth mentioning) that (see R. Steinberg [, Exercise after Lemma 78])

(5.1.3) $-w_0 = 1$ unless $\mathbb{G}$ is of type A_n, D_{2n+1}, or E_6 in which case it is the non-trivial symmetry of the Dynkin diagram.

In particular,

(5.1.4) All irreducible representations of an algebraic group have a non-degenerate bilinear invariant except those representations of A_n, D_{2n+1}, E_6 whose highest weight is not invariant under the symmetry of the Dynkin diagram.

(5.1.5) Quadratic forms in characteristic $\neq 2$ are completely described by the associated bilinear forms. In characteristic 2 I do not know a criterium for the existence of a non-singular invariant quadratic form (similar to, say, the condition $\lambda = -w_0\lambda$ for the existence of a bilinear form). This lack of understanding can be, however, bypassed at this stage (see ( ) below) using (5.1.2) and M. Aschbacher [, 9.1(4)]. We also observe (see (2.2.3)) that if $\tilde{\varphi}$ preserves a symmetric bilinear form f in characteristic 2 then f is alternating, i.e. $f(x,x) = 0$.

(5.2) We now look at non-degenerate invariant Hermitian forms. Hermitian forms are associated to a quadratic extension K' of a field k'. For the purposes of our applications (that is in view M. Aschbacher [, p. 469], (see (5.3) below) we are interested only in the cases when

(5.2.1.) $k' = k(\varphi)$.

(Otherwise our gruop G would fall into the family C_5 of M.

Aschbacher.)

Suppose that K' has a subfield k' of codegree 2. Let $b := [k':\mathbb{F}_p]$. The condition that φ has an invariant Hermitian form means that $\varphi\circ Fr^b$ is equivalent to $\breve{\varphi}$, or in the notation of (5.1) and (4.1.11) that

(5.2.2.) $$\delta(Fr^b)Fr^b(\lambda) = -w_0\lambda$$

where λ is the highest weight of $\mathbb{G}$ as in (4.1.3).

We have further

(5.2.3) φ is not equivalent to $\breve{\varphi}$.

Indeed, if φ is equivalent to $\breve{\varphi}$ then $\varphi\circ Fr^b$ is equivalent to φ and then by (4.3.1) $\mathbb{F}_p(\chi) \subseteq k'$, a contradiction.

Thus (see (5.1.4)) only the cases when $\mathbb{G}$ is of type A_n, D_{2n+1}, E_6 and the representation $\tilde{\varphi}$ has highest weight λ which is not invariant under the automorphisms of the Dynkin diagram are of interest to us. This together with (5.2.2) implies that

(5.2.4) $\delta(Fr^b)$ is non-trivial of order 2.

Thus

(5.2.5) $\mathbb{G}$ is a Steinberg (twisted) form of type ${}^2A_n, {}^2D_{2n+1}, {}^2E_6$.

Also from (5.2.2), in view of (5.2.4),

(5.2.6) $$Fr^b(\lambda) = \lambda.$$

Since $\delta(Fr^b) \neq 1$, it follows that $k' \subsetneq k$ and, since finite fields have a unique subfield of a given degree, it follows that

(5.2.7) $$[k:k'](=a/b) \text{ is odd}$$

(otherwise it would have been impossible to have $Kk = k_{\mathcal{G}}$, i.e. $\delta(Fr^b) \neq 1$).

In view of (5.2.7) and (5.2.4) it follows that $\delta(Fr^a)\lambda = -w_0\lambda$, a condition which holds, of course, for every part (as in (5.1) or (4.2)) of λ, or in analogy with (5.1.1).

(5.2.8) Proposition. If φ has a non-degenerate invariant Hermitian form (but no invariant bilinear form) then $\mathcal{G}$ is a Steinberg (twisted) form over k of type ${}^2A_n, {}^2D_{2n+1}, {}^2E_6$ and every component of the tensor product decomposition of φ, as in (4.1.3), has a non-degenerate invariant Hermitian or bilinear form.

(5.3) We can now describe M. Aschbacher's family $\mathcal{C}_H$. It consists of 8 subfamiles $C_1,\ldots,C_8$. Instead of copying his descriptions we give only the properties which we are going to use. We say below that a representation is $\otimes$-decomposable if it is isomorphic to a tensor product of representations and $\otimes$-imprimitive if a representation space can be realized as a tensor product of spaces with the group action permuting the factors. Thus usual notions of reducibility and imprimitivity of representations can be expressed as $\oplus$-decomposability and $\oplus$-imprimitivity.

The group H we are dealing with here is a classical group with simple socle; we consider it acting on its "natural" module via a projective representation. For a subgroup G of H we consider its lifting $\tilde{G}$ to the linear representation.

C_1 consists of G with reducible $\tilde{G}$,

C_2 consists of G with imprimitive $\tilde{G}$,

C_3 consists of G which belong to C_2 after a field extension,

C_4 consists of G with $\otimes$-decomposable $\tilde{G}$,

C_5 consists of groups over smaller subfields,

C_6 consists of normalizers of certain nilpotent groups,

C_7 consists of G with $\otimes$-imprimitive $\tilde{G}$,

C_8 consists of other classical groups on the same space, which happen to be subgroups of H.

(5.4) It seems proper to extend M. Aschbacher's family $\mathcal{C}_G$ to $\mathcal{E}\mathcal{C}_G$ ($\mathcal{E}$ for "extended") by adding one additional family C_9 (which is in the same relation to C_7 as C_3 is to C_2). We describe C_9 at the level of detail of M. Aschbacher. Let V be a vector space over a finite field K with a form f which is either

I. trivial
II. alternating
III. quadratic
IV. Hermitian

Let k be a subfield of K of (odd in case IV) prime codegree. We consider $V' := \otimes_{\sigma\in \mathrm{Gal}(K/k)} V^{\sigma}$ with the form $f' := \otimes_{\sigma} f^{\sigma}$. In the language which seems preferable we consider $(V', f') := R^{\otimes}_{K/k}(V, f)$. We use superscript $\otimes$ to denote that we are taking tensor products; $R^{\oplus}_{K/k}(V, f)$

would be, thus $\oplus_\sigma V^\sigma$ with the form $\oplus_\sigma f^\sigma$ and would produce, as noted, M. Aschbacher's family C_3. The condition that the degree $[K;k]$ is a prime is made in order to improve the chances of subgroups defined below to be maximal.

We now consider V' with the form

$f'' = f'$ unles $p = 2$ and f is alternating or quadratic

= the unique quadratic form associated by M. Aschbacher [, 9.1(4)] to f' if f is alternating.

The case when $p = 2$ and f is quadratic we do not consider (it cannot give a maximal subgroup).

Our family C_9 will consist of the stabilizer in the group of semi-similitudes $\Gamma(V',f'')$ of (V',f'') of the decomposition $(V',f') \simeq R^{\otimes}_{K/k}(V,f)$. It is clear that our groups from C_9 are normalizers in $\Gamma(V',f'')$ of groups isomorphic to $GU(V,f) \rtimes Gal(K:\mathbb{F}_p)$ where $GU(V,f)$ is the group of unitary (or orthogonal) similitueds of (V,f).

Finally we let C' be the set of groups given by the families $C_1 - C_8$ and C_9. For a classical simple group H_0 and a subgroup $H_0 \subseteq H \subseteq \text{Aut } H_0$ we let $\mathcal{EC}_H$ be the family of subgroups G of H whose preimage $\tilde{G}$ is of the form $N_H(M \cap \tilde{H})$ for $M \in C'$.

(5.3) Let $\varphi,\tilde{\varphi},\mathbb{G},G,k,a,$ etc. be as in (5.1). Consider $k' := k \cap \mathbb{F}_p(\varphi)$ and $K' := \mathbb{F}_p(\varphi)$ (consult (4.4),(4.3)). Then by (4.4.2) we can consider $\tilde{\varphi}$ as a K'-representation $\tilde{\varphi}'$ of the k'-group $\mathbb{G}' := R_{k/k'}\mathbb{G}$. Thus $\tilde{\varphi}' : \mathbb{G}' \to GL_{n,K'}$. In view of (5.2.4) and (5.2.7) we have

(5.3.1) Proposition. If $\tilde{\varphi}'(\mathbb{G}')$ preserves a non-degenerate Hermitian form f but no bilinear form then f can be chosen with coefficients in k' (in a basis in which $\tilde{\varphi}'$ is defined over K').

A similar statement holds also for bilinear forms.

(5.3.2) Proposition. If $\tilde{\varphi}'(\mathbb{G}')$ preserves a non-degenerate-bilinear or quadratic form then the latter can be chosen with coefficients in K' in a basis in which $\tilde{\varphi}'$ is defined over K'.

Proof. Let f' be a non-degenerate bilinear form invariant under $\tilde{\varphi}'(\mathbb{G}')$. Since it is unique up to scalar factor and $\tilde{\varphi}'$ is defined over K' we have $\sigma(f') = t_\sigma f'$ with $t_\sigma \in \overline{\mathbb{F}}_p^*$, $\sigma \in Gal(\overline{\mathbb{F}}_p/K')$. Then t_σ is a 1-cocycle and by Hilbert's theorem 90 t_σ is cohomologically trivial so that $\sigma(tf') = tf'$ for some $t \in \overline{\mathbb{F}}_p^*$. We take $f := tf'$.

(5.4) The innocent-looking statements of (5.3) have less innocent consequences.

(5.4.1) Set $K'' = K'$ if $\tilde{\varphi}(\mathbb{G}')$ has no non-zero invariant belinear or sesqui-linear form and if it has a non-degenerate invariant bilinear form. Let K'' be the unique quadratic subfield of k' if $\tilde{\varphi}(\mathbb{G}')$ has a non-degenerate Hermitian form. Let $\mathfrak{H}$ be the unitary group of the corresponding $\tilde{\varphi}'(\mathbb{G}')$-invariant bilinear form. Then $\mathfrak{H}$ is defined over K'' by (5.3). The following follows directly from (5.3):

(5.4.2) Proposition. $\tilde{\varphi}'(\mathbb{G}'(K'')) \subseteq \mathfrak{H}(K'')$.

Remark. The meaning of this is that unles $\tilde{\varphi}'(\mathbb{G}') = \mathfrak{H}$ or $K'' = k'$ we have that $\tilde{\varphi}'(\mathfrak{G}'(k')) = \tilde{\varphi}(\mathfrak{G}(k))$ is not a maximal subgroup of $\mathfrak{H}(K'')$.

(5.4.3) To proceed we need some notation. Let $\lambda(i), \Lambda(i)$, and $\Gamma(i)$ have the same meaning as in (4.2). Set $\mu(i) := \Sigma_{\mu\in\Lambda(i)}\mu$. Let $\mu_1,\ldots,\mu_t$ be the distinct non-zero $\mu(i)$ and let $\tilde{\varphi}_i$ be the corresponding representations of $\mathfrak{G}$. Let K_i', K_i'' and k_i' be the fields constructed as in (5.4.1) and (5.4.2) for each $\tilde{\varphi}_i$. Let $\mathfrak{G}_i' := R_{k/k_i'}(\mathfrak{G})$. Let f_i be an appropriate form on the space V_i of $\tilde{\varphi}_i$. We let

$f := \otimes_{i=1}^{t} f_i$ except when $p = 2$ and all f_i are bilinear (hence alternating)

= the unique quadratic form defined as in M. Aschbacher [,(9.1)(4)].

By (5.3.1) and (5.3.2) each f_i, $i = 1,\ldots,t$, is defined over K_i''. Moreover, by (5.1.1) each f_i has an associated non-degenerate bilinear form if $\tilde{\varphi}(\mathfrak{G})$ has one, and each f_i is either Hermitian or has an associated non-degenerate bilinear form if $\tilde{\varphi}(\mathfrak{G})$ has an invariant non-degenerate bilinear form.

(5.4.4) **Proposition.** Let $\mathfrak{U} := \times_{i=1}^{t} \mathfrak{G}_i$.

Let again $K' := \mathbb{F}_p(\varphi)$, $k' := k \cap K'$ and let K'' be defined as in (5.4.1). By (4.1.1) $k' \supseteq k_i'$, $K' \supseteq K_i'$, $i = 1,\ldots,t$. Thus we can consider the direct product $\mathfrak{M} := \times_{i=1}^{t} \mathfrak{G}_i'$ as a K''-group. Let $\mathfrak{H}$ be the unitary group of f and $\mathfrak{H}_i$ the unitary group of f_i. We have as in M. Aschbacher a natural map $\sigma : \prod_{i=1}^{t} \mathfrak{H}_i \to \mathfrak{H}$. It is defined over K'' and therefore we have

(5.4.4) Proposition $\sigma(\prod_{i=1}^{t} \varphi_i(\mathbb{G}'_i(K''))) \subseteq \sigma(\prod_{i=1}^{t} \mathcal{H}_i(K))$.

6. Necessary conditions on maximal Lie p-type subgroups of classical finite groups

All fields below are considered as subfields of a universal field.

(6.1.1) Let K be a finite field, θ an automorphism of K of order 1 or 2, V a vector space over K of dimension n, f a form on V which is as in M. Aschbacher [ ] (see (5.4)) of type I, II, III, or IV. We consider the connected algebraic group $\mathcal{H}$ associated to (V,f) so that $\mathcal{H}$ is defined over k and $\mathcal{H}(k) = SU(V,f)$, the special "unitary" group of (V,f).

(6.1.2) Let k' be another finite field and $\mathcal{G}$ an algebraic absolutely almost simple simply connected group defined over k'. Let $\tilde{\varphi} : \mathcal{G} \to \mathcal{H}$ be a homomorphism of algebraic groups. We set $G := \mathcal{G}(k')$ and $\varphi := \tilde{\varphi}|G$. We assume that $\varphi(G) \subseteq \mathcal{H}(k)$.

Since k' is finite $\mathcal{G}$ is quasi-split over k' and is split by an extension $k'_{\mathcal{G}}$ of k' of prime degree over k' (see (4.1.2)).

(6.1.3) We assume that $\tilde{\varphi}(\mathcal{G})$ is absolutely irreducible (on $V \otimes \bar{\mathbb{F}}_p$) and denote by λ the highest weight of $\tilde{\varphi}$. Then the Galois group $Gal(\bar{\mathbb{F}}_p/k')$ acts on the weight as in (4.1.2); this action factors through $Gal(k'_{\mathcal{G}}/k')$. Let Γ_λ be the stabilizer of λ in $Gal(\bar{\mathbb{F}}_p/k')$ (subsequently, in our special situation of $\otimes$-indecomposable representations, we will have $\Gamma_\lambda = \Gamma_{\tilde{\varphi}}$ where $\Gamma_{\tilde{\varphi}}$ is from (4.3)). Set $k'_\lambda := \bar{\mathbb{F}}_p^{\Gamma_\lambda}$. Since the action of $Gal(\bar{\mathbb{F}}_p/k')$ factors through that of $Gal(k'_{\mathcal{G}}/k')$ we see that k'_λ is either k' or $k'_{\mathcal{G}}$. We say that λ is symmetric in the first case and non-symmetric in the second.

(6.1.4) Now let $\mathcal{H}^{\sim}$ be the universal covering group of $\mathcal{H}$, $\mathcal{H}^{-}$ the adjoint group of $\mathcal{H}$, and we denote by π both projections $\mathcal{H}^{\sim} \longrightarrow \mathcal{H}^{-}$ and $\mathcal{H} \longrightarrow \mathcal{H}^{-}$. All these are defined over k. We set $H_0 := \pi(\mathcal{H}^{\sim}(k))$. This group is almost always simple (see R. Steinberg [, Theorems 5 and 34]) so that our present notation almost always agrees with that of M. Aschbacher [ ]. In all cases $\mathrm{Aut}\ H_0/\mathrm{Int}\ H_0$ is solvable (see R. Steinberg [, Theorem 30 and 35] and use the fact that the center of $\mathcal{H}(k)$ is either cyclic or $\mathbb{Z}/2 \times \mathbb{Z}/2$).

(6.1.5) Assume henceforth in this Section 6 that both H_0 and G/center are simple. This is the only case of interest to us. We can and do identify then H_0 with $\mathrm{Int}\ H_0 \subseteq \mathrm{Aut}\ H_0$. Moreover since $\pi(\mathcal{H}(k))/H_0$ is solvable we have that $\overline{G} := \pi\circ\varphi(G) \subseteq H_0$. Consider $H \subseteq \mathrm{Aut}\ H_0$, $H \supseteq H_0$. Let $N := N_H(\overline{G})$.

(6.2) Theorem. Assume

(a) if $\mathcal{H}$ is split of type D_4 (i.e. $n = 8$, f is quadratic of maximal Witt index) then H contains only those algebraic outer automorphisms of $\mathcal{H}$ which preserve the given embedding $\mathcal{H} \longrightarrow GL_n$,

(b) $NH_0 = H$.

If N is a maximal subgroup of H, then either N is contained in some member of $\mathcal{EC}_H$ (see (5.3),(5.4)) or

(i) φ is $\otimes$-indecomposable

(ii) $K = k'_\lambda$, $k = k'$ (except when $n = 8$, f quadratic of maximal Witt index, $\tilde{\varphi}(\mathcal{G}) = \mathcal{H}$, $\mathcal{G}$ of type ${}^3D_4$, $k = k'_\lambda = k'_{\mathcal{G}}$)

(iii) φ is defined over K

(iv) $\tilde{\varphi}(\mathfrak{G})$ is maximal among connected k-subgroups of $\mathfrak{H}$ invariant under outer algebraic automorphisms of $\mathfrak{H}$ contained in H.

Can the following happen?

$\overline{G} := Ad\circ\pi\circ\tilde{\varphi}(\mathfrak{G}(k))$, $H_0 := Ad\circ\pi(\mathfrak{H}^{\sim}(k))$

$h \in (Ad\ \mathfrak{H}^{-})(k) \subseteq Aut\ \mathfrak{H}$

$h\overline{G}h^{-1} = \overline{G}$

but h induces on $Ad\circ\pi\circ\tilde{\varphi}(\mathfrak{G})$ an outer automorphism?

(6.2.1) Amplification of (6.2)(i) (see Section 3). φ is $\otimes$-indecomposable if and only if φ is equivalent to a representations $\varphi\circ Fr^{i}$ of G where

(i) φ' is infinitesimally irreducible,

(ii) if $p = 2$ and $\mathfrak{G}$ is of type B_n, C_n, F_4 or $p = 3$ and $\mathfrak{G}$ is of type G_2 then φ' does not admit additional $\otimes$-decompositions provided by R. Steinberg [,Corollary to Theorem 41].

(6.2.2) Amplification of (6.2)(ii) (see below).

(i) If $f = 0$ then $\mathfrak{G}$ is split of type $A_m (m\geq 2)$, $D_{2m+1} (m\geq 1)$, or E_6 and $K = k = k' = k'_{\mathfrak{G}}$,

(ii) If f is Hermitian, $n \geq 3$, then $\mathfrak{G}$ is non-split of type ${}^2A_m (m\geq 2)$, ${}^2D_{2m+1} (m\geq 1)$, or ${}^2E_6$, $K = k'_{\mathfrak{G}}$, $k = k'$, and λ is not symmetric,

(iii) If f is alternating or quadratic then

(a) $\mathbb{G}$ can be split of type B_m $(m\geq1)$, C_m $(m\geq1)$, D_{2m} $(m\geq2)$, G_2, F_4, E_7, E_8 and $K = k = k' = k'_{\mathbb{G}}$, or

(b) split or quasi-split of type A_m $(m\geq2)$, ${}^2A_m$ $(m\geq2)$, D_{2m+1} $(m\geq1)$, ${}^2D_{2m+1}$ $(m\geq1)$, E_6, ${}^2E_6$ with λ symmetric and $K = k = k' = k'_\lambda$, or

(c) quasi-split of type ${}^3D_4$ or ${}^2D_{2m}$ $(m\geq2)$ with λ symmetric and $K = k = k' = k'_\lambda$, or

(d) the particular situation excluded in (6.2)(ii) takes place.

(6.2.3) Corollary of (6.2)(ii). Let $\sigma : \mathrm{Aut}\, H_0 \longrightarrow \mathrm{Gal}(K/F_p)$ be the surjective map defined in R. Steinberg [ ]. Then

$$\sigma(N_{\mathrm{Aut}H_0}(\overline{G})) = \sigma(\mathrm{Aut}\ H_0).$$

Proof. Indeed since $\mathbb{G}$ and φ are defined over k it follows that the action of Fr on $\mathcal{H}$ restricts to the action of Fr on $\tilde{\varphi}(\mathbb{G})$ whence our claim.

Remark. it is remarkable that ${}^3D_4$ and ${}^2D_{2m}$ with non-symmetric λ can not appear as maximal subgroups, except for ${}^3D_4$ in a very special situation. This situation, with all justice, should have been included into the family $\mathcal{EC}_H$. Namely, instead of the family C_5 of M. Aschbacher we should consider subgroups arising as $\mathcal{H}_{k'}(k')$ for some structure $\mathcal{H}_{k'}$ of a k'-group on $\mathcal{H}$ and some subfield k' of k of prime codegree. An evident extension of this would also exclude subgroups ${}^2B_2(2^{2s+1})$ in $B_2\,(2^{2s+1})$ (case $n = 4$, f alternating).

(6.3) The statement of (6.2.1) holds inview of (3.1) and (3.3) once we prove the following

Lemma. φ is $\otimes$-indecomposable if and only if the representation $\tilde{\varphi}$ of $\mathbb{G}$ extending φ (as in (4.1.3)) is $\otimes$-indecomposable as a representation of $\mathbb{G}$.

Proof. By R. Steinberg [, Theorem 13] since φ is $\otimes$-indecomposable it is of the form $\omega \circ Fr^i$ where ω is infinitesimally irreducible. Let j be such that Fr^{i+j} is the identity on G. If φ is equivalent to $\varphi_1 \otimes \varphi_2$ then $\omega \simeq \varphi \circ Fr^j \simeq (\varphi_1 \circ Fr^j) \otimes (\varphi_2 \circ Fr^j)$. Thus we can assume that φ is infinitesimally irreducible. Let $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ be extensions of φ_1 and φ_2 to $\mathbb{G}$. Then the representation $\tilde{\varphi}_1 \otimes \tilde{\varphi}_2$ of $\mathbb{G}$ is irreducible as its restriction to G is irreducible. The highest weight of $\tilde{\varphi}_1 \otimes \tilde{\varphi}_2$ is the sum of the highest weights of $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ and, therefore, $\tilde{\varphi}_1$ and $\tilde{\varphi}_2$ are infinitesimally irreducible. It follows from (3.1) that then $\tilde{\varphi}$ is equivalent to $\tilde{\varphi}_1 \otimes \tilde{\varphi}_2$ whence we get one implication of (6.3). The reverse implication is obvious.

(6.4) Proof of (6.2)(i). Let $\Lambda(i)$ have the meaning described in (4.2). Set $\mu(i) = \Sigma_{\mu \in \Lambda(i)} \mu$, let $\mu_1, \ldots, \mu_t$ be the set of different non-zero $\mu(i)$, let $\tilde{\varphi}_1, \ldots, \tilde{\varphi}_t$ be the corresponding representations of $\mathbb{G}$ and $\varphi_i := \tilde{\varphi}_i / G$.

It follows by (4.4.1) that $\mathbb{F}_p(\varphi) = \mathbb{F}_p(\varphi_1)\ldots\mathbb{F}_p(\varphi_t)$ so that $\mathbb{F}_p(\varphi) \supseteq \mathbb{F}_p(\varphi_i)$ for $i = 1,\ldots,t$. In view of (5.1.1), (5.2.8) and [, 9.1(4)], $t > 1$ implies that G is contained in a member of C_4 (see (5.3) or M. Aschbacher [, p. 972]).

Suppose, therefore, that $t = 1$. if Λ is the only non-trivial $\Lambda(i)$ let $s = |\Lambda|$. If $s > 1$ then by (4.2.3) and again (5.1.1), (5.2.8) and [, 9.1(4)] G is contained in a member of C_9 (see (5.4)). Thus φ is equivalent to $\omega \circ Fr^i$ with ω infinitesimally irreducible. If ω permits additional $\otimes$-decompositions (as in (3.3)) then G falls into the family C_4 again.

(6.5) Proof of (6.2)(ii). We rather prove (6.2.2). The cases (6.2.2)(i) and (6.2.2)(ii) are contained in (5.1.4) and (5.2.8) respectively. We can, therefore, assume that f is alternating or quadratic. The claim of (6.2.2)(iii) is (in view of (5.1.4)) that the groups N for G of type ${}^3D_4$ and ${}^2D_{2m}(m\geq 2)$ can not be maximal if λ is non-symmetric except in one case. So suppose that λ is non-symmetric. Then, by J. Tits [, Theorem 3.3.], $\tilde{\varphi}$ is defined over $k'_\lambda = k'_{\mathbb{G}}$. In particular, the enveloping algebra $k\cdot\tilde{\varphi}(\mathbb{G}(k'_{\mathbb{G}}))$ is isomorphic to $\mathrm{Mat}_n(k'_{\mathbb{G}})$.

We have, unless N is in $\mathscr{E}C_H$ that $\mathbb{F}_p(\varphi) = k$ (by M. Aschbacher [, p. 469]), $\mathbb{F}_p(\varphi) = k'_\lambda$ by (6.2)(i) and (4.3.2)(i), and $k'_\lambda = k'_{\mathbb{G}}$ by the assumption that λ is non-symmetric. Thus $k'_{\mathbb{G}} = k$ and $k\tilde{\varphi}(\mathbb{G}(k'_{\mathbb{G}})) = k\tilde{\varphi}(\mathbb{G}(k'))$ whence $G' := \tilde{\varphi}(\mathbb{G}(k'_{\mathbb{G}})) \subseteq \mathscr{H}(k)$.

7. Primitivity of infinitesimally irreducible representations of groups of Lie p-type

(7.1) Theorem. Let G be a universal group of Lie p-type and $\varphi : G \longrightarrow GL_n(\overline{\mathbb{F}}_p)$ an infinitesimally irreducible representation. If G is not isomorphic to one of a finite set $\mathfrak{K}_1$ of groups then the action of G on $\overline{\mathbb{F}}_p^n$ via φ is primitive.

(7.2) Proof. Let $B \supseteq U$ be a Borel subgroup and the Sylow p-subgroup of G. Let $V := \overline{\mathbb{F}}_p^n$; we write gv for $\varphi(g)(v)$, $g \in G$, $v \in V$. Let v be a highest weight vector of G on V so that $Uv = v$. Then v is unique (up to an element of $\overline{\mathbb{F}}_p^*$) by R. Steinberg [Theorem 46(a),(b)].

Write $V := \oplus_{i=1}^m V_i$ for an imprimitivity system of V, let $\psi : G \longrightarrow Sym_m$ be the homomorphism such that $gV_i = V_{\psi(g)i}$. For a vector $w \in V$ we write $w = \oplus w_i$ for the decomposition of w with $w_i \in V_i$.

In particular, for the highest weight vector v let $I = \{i | v_i \neq 0\}$. The condition $uv = v$ for $u \in U$ means that $v = uv = \oplus(uv)_i = \oplus v_{\psi(u)i}$, that is, I is invariant under $\psi(U)$. Therefore $I' = [1,m] - I$ is also invariant under $\psi(U)$. Hence $V' := \oplus_{i \in I'} V_i$ is invariant under U. Since kv is the unique line in V invariant under U, since $kv \notin V'$ and since U has a non-trivial vixed line in every non-trivial space on which it acts, it follows that $V' = 0$, i.e. $I = [1,m]$.

Let J be an orbit of U on $[1,m]$. Then $\Sigma_{i \in J} v_i$ is an invariant vector of U on V. By the unicity of v it follows that

(7.2.1) U is transitive on $[1,m]$.

In particular,

(7.2.2) m is a power of p.

For every $g \in G$ there exists therefore $u_g \in U$ such that $\psi(g)(1) = \psi(u_g)1$. Let L be the stabilizer of 1 in G. We have that $u_g^{-1} \cdot g \in L$, that is,

(7.2.3) $G = UL$.

In particular, $G = BL$, or, in the terminology of D. G. Higman (see G. M. Seitz [ ]) L is flag-transitive. Thus (7.2.3) and (7.2.2) give

(7.2.4) L is flag-transitive of index a power of p.

By G. M. Seitz [, Theorem A and second paragraph on p.28] the set $\mathfrak{R}_1$ of isomorphism classes of groups in (7.2.4) for which $U \not\subset L$ is finite. If $U \subset L$ then $m = 1$ and G is primitive. This concludes our proof.

8. ⊗-primitivity of ⊗-indecomposable representations of groups of Lie p-type

See (5.3) for definitions of ⊗-primitivity and ⊗-indecomposability and (6.2.1), (6.2.2) on more details on ⊗-indecomposability.

(8.1) Theorem. Let G be a universal group of Lie p-type and $\varphi : G \to GL_n(\overline{\mathbb{F}}_p)$ its ⊗-indecomposable representation. If G is not isomorphic to one of a finite set $\mathfrak{R}_2$ of groups then φ is ⊗-primitive.

(8.2) Proof. Let $V := \overline{\mathbb{F}}_p^n$. Suppose there exist vector spaces $V_1, \ldots, V_m$ over $\overline{\mathbb{F}}_p$ and a homomorphism of G in the normalizer in $GL(\otimes_{i=1}^m V_i)$ of the evident action of $\prod_{i=1}^m GL(V_i)$ on $\otimes_{i=1}^m V_i$ such that the resulting action is equivalent to φ. We may assume then that $V = \otimes_{i=1}^m V_i$ and that $\varphi(G)$ does normalize the action of $\prod_{i=1}^m GL(V_i)$. This defines a homomorphism $\psi : G \to Sym_m$. Since φ is ⊗-indecomposable we have that

(8.2.1) $\psi(G)$ is transitive on $[1,m]$.

Therefore,

(8.2.2) $\dim V_1 = \ldots = \dim V_m$.

Set $d = \dim V_1$. Then

(8.2.3) $n = d^m$.

Let $2t$ be the number of roots of an absolutely almost simple algebraic group $\mathfrak{G}$ corresponding to G. Then the infinitesimally irreducible representations of G have dimension at most p^t (see [?]). Thus $n \leq p^t$ whence

(8.2.4) $m \leq t \log_d p \leq t \log_2 p$.

(8.2.5) This is, clearly, only rarely possible. For example, if G is of type $A_1(p^a)$ (the only case when d can be equal 2), G has, except in a few cases, non-trivial permutation representations only of degree $\geq p^a + 1$. Thus if $m \neq 1$ we must have $p^a + 1 \leq \log_2 p$ $(t = 1$ in this case). This inequality never holds.

(8.2.6) The previous discussion generalizes to all classical groups. With the exception of a finite list $\mathfrak{K}_2'$ of isomorphism classes of finite universal classical groups of Lie p-type the following table (8.2.7) holds (by B. N. Cooperstein [, Table 1]) and implies that $m = 1$ for almost all classical groups. This implies (8.1) for classical groups not from $\mathfrak{K}_2'$.

(8.2.7) Table

Group	t	minimal possible d>1 (see (9.2.2))	a lower bound on m≠1	conditions when (8.2.4) holds with m≠1
$A_n(p^a)$	$\frac{n(n+1)}{2}$	n+1	p^{an}	never
$B_n(p^a)$, p odd	n^2	2n+1	$p^{(2n-1)a}$	never
$C_n(p^a)$, $p^a>2$	n^2	2n	$p^{(2n-1)a}$	never
$C_n(2)$, $n\geq 3$	n^2	2n	$2^{n-1}(2^n-1)$	never
$D_n(p^a)$, $n\geq 4$	n(n-1)	2n	$p^{a(2n-2)}$	never
${}^2A_n(p^{2a})$, $n\geq 3$	$\frac{n(n+1)}{2}$	n+1	$p^{a(n+1)}$	never
${}^2D_n(p^{2a})$, $n\geq 4$	n(n-1)	2n	$p^{a(2n-2)}$	never

(8.2.8) For exceptional groups and groups of Suzuki and Ree an analogue of B.N. Cooperstein [, Table 1] does not seem to exist in the literature. We use instead V. Landazuri and G.M. Seitz [, p.419]. They give lower bounds on dimensions of non-trivial irreducible representations. Considering the linear permutation representation (over $\mathbb{C}$) of a non-trivial permutation group gives a trivial representation plus some number of other representations, at least one of which is non-trivial. Therefore from [, p.419] we derive that, with the exception of a finite list $\mathfrak{K}_2''$ of finite universal groups of Lie p-type which were not considered in (8.2.6) and (8.2.7), the following table (8.2.9) holds and implies (8.1) for the rest of groups of Lie p-type.

(8.2.9) Table

Group	t	minimal possible d>1 (see (9.2.2))	a lower bound on $m \neq 1$	conditions when (8.2.4) holds with $m \neq 1$.
$E_6(p^a)$	36	27	p^{10a}	never
$E_7(p^a)$	63	56	p^{16a}	never
$E_8(p^a)$	120	248	p^{28a}	never
$F_4(p^a)$	24	24	p^{9a}	never
$G_2(p^a)$	6	5	$p^a(p^{2a-1})$	never
${}^2E_6(p^{2a})$	36	27	p^{14a}	never
${}^3D_4(p^{3a})$	12	8	p^{4a}	never
${}^2B_2(2^{2a+1})$	4	4	$2^a(2^{2a+1}-1)$	if a=0
${}^2F_4(2^{2a+1})$	24	24	2^{12a}	if a=0
${}^2G_2(3^{2a+1})$	6	6	3^{2a+1}	if a=0

(8.2.10) It follows that (8.1) holds with $\mathfrak{K}_2$ being the union of $\mathfrak{K}_2'$, $\mathfrak{K}_2''$ and the groups ${}^2B_2(2)$, ${}^2F_4(2)$, ${}^2G_2(3)$.

9. Tight embeddings of groups of Lie p-type into groups of Lie p'-type

We consider quadruples (G,M,τ,ω) consisting of a universal group G of Lie p-type, a group of Lie p'-type M, a homomorphism $\tau : G \longrightarrow M$, and a representation $\omega : M \longrightarrow GL_n(\overline{\mathbb{F}}_p)$, $n > 1$. We assume above and throughout this section that G and M are associated to absolutely almost simple algebraic groups $\mathfrak{G}$ and $\mathfrak{M}$. We call a quadruple as above admissible. Our result is

(9.1) Theorem. There exists a finite list $\mathfrak{K}_3$ of isomorphism classes of finite groups of Lie r-type, r a prime, such that if G is a universal group of Lie p-type not isomorphic to a group from the list $\mathfrak{K}_3$ then there is no admissible quadruple (G,M,τ,ω) satisfying the additional condition that $\varphi := \omega\circ\tau$ is an infinitesimally irreducible representation of G.

(9.1.1) Remark. The above Theorem can be generalized to admit just irreducible φ and representations ω of M over fields of characteristic just different from that of M. Although it does not follow from the proof it is probable that a minimal $\mathfrak{K}_3$ as defined above is very small or even empty.

(9.1.2) Remark. If we replace the condition that M is of Lie p'-type by, say, the condition that M is centrally simple but not of Lie p-type then there are infinitely many examples of G which can participate in a quadruple satisfying the remaining assumptions of (9.1). The following are such examples (they and other doubly transitive groups are relevant to some restrictions in our final results, see (   ) and (   )).

Let G be $SL_2(p^a)$, τ a homomorphism of G into $M \simeq Alt_{p^a+1}$ such that $\tau(SL_2(p^a))$ is doubly transitive on $p+1$ points (the action of $SL_2(p^a)$ on the points $\mathbb{P}^1(\mathbb{F}_{p^a})$ of a projective line). Then the linear permutation representation over $\mathbb{C}$ of Alt_{p^a+1} is a direct sum of an irreducible representation ω' of dimension p and of a trivial representation. Since $\tau(SL_2(p^a))$ is doubly transitive $\omega' \circ \tau$ is an irreducible representation of $SL_2(p^a)$ (see, e.g. [ ]). If $p \neq 2$ then $SL_2(p^a)$ has only one, namely, the Steinberg irreducible linear representation of degree p^a (see, e.g., T.A. Springer [, Ch. II, §3]). Let ω be a reduction of ω' modulo p. Then it is known (and follows from the fact that ω' is of defect 0) that $\omega \circ \tau$ is irreducible. It is then a representation with corresponding highest weight $(p^a-1)\lambda_1$. So it is infinitesimally irreducible only if $a = 1$. For the sake of reference we record this:

(9.1.3) Let $\tau : SL_2(p^a) \longrightarrow Alt_{p^a+1}$, $p \neq 2$, be the permutation representation of $SL_2(p^a)$ on $\mathbb{P}^1(\mathbb{F}_{p^a})$ and $\omega : Alt_{p^a+1} \longrightarrow GL_{p^a}(\overline{\mathbb{F}}_p)$ the reduction modulo p of the non-trivial component of a doubly-transitive permutation representation of Alt_{p^a+1} on p^a+1 letters. Then $\omega \circ \tau$ is irreducible; it is infinitesimally irreducible if and only if $a = 1$.

Note that when (and only when) $p^a = 5$, Alt_{p^a+1} has 2 different doubly-transitive permutation representations of degree p^a+1.

We also have an analogue of (9.1.3) for $SU_3(p^a)(\simeq {}^2A_2(p^{2a}))$.

(9.1.4) Let $\tau : SU_3(p^a) \longrightarrow Alt_{p^{3a}+1}$, $p^a > 2$, be the permutation representation of $SU_3(p^a)$ on the points of the absolute in $\mathbb{P}^2(\mathbb{F}_{p^{2a}})$ of the corresponding Hermitian form. Let $\omega : Alt_{p^{3a}+1} \longrightarrow GL_{p^{3a}}(\bar{\mathbb{F}}_p)$ be the reduction modulo p of the non-trivial component of a doubly-transitive representation of $Alt_{p^{3a}+1}$ on $p^{3a}+1$ letters. Then $\omega\circ\tau$ is irreducible, it is infinitesimally irreducible if and only if $a = 1$.

(9.2) A construction of a $\mathfrak{R}_3$ can be based directly on the estimates of V. Landazuri and G.M. Seitz [, p. 419], or on the estimates in my paper [NAS]. But it seems preferable to use our [ ] (which is based on [ ]). Set

$$(*) \quad f(x) := (2x+1)^{2\log_3(2x+1)+1}.$$

Then

(9.2.1) Proposition. There exists a finite list $\mathfrak{R}_3'$ of universal groups of Lie r-type, r a prime, which contains isomorphism classes of all non-centrally simple such groups such that if G is a universal group of Lie p-type, not isomorphic to a group from $\mathfrak{R}_3'$ and $\bar{\omega} : G \longrightarrow PGL_m(\bar{\mathbb{F}}_\ell)$, $m > 1$, an irreducible projective representation with ℓ a prime, $\ell \neq p$, then $|G| \leq f(m)$.

Proof. This follows from [, (4.4.2) and (4.4.3)(a)].

(9.2.2) Lemma. Let M be a universal group of Lie r-type, r a prime. Then M has a irreducible representation over $\overline{\mathbb{F}}_r$ of dimension $\leq d$ where

type of M:	A_n	B_n	C_n	D_n	E_6	E_7	E_8	F_4	G_2
d:	n+1	2n+1	2n	2n	27	56	248	26	7

By the type of M we understand the type of the corresponding algebraic group $\mathfrak{M}$.

Proof. Let L be a simple Lie algebra over $\mathbb{C}$ of the same type as M. Then L has an irreducible representation ψ over $\mathbb{C}$ of dimension d (see, e.g., E.B. Dynkin [, Table 30]). The "reduction" of this representation modulo p (as in R. Steinberg [, §12] or in A. Borel [, 5.11]) has as a subquotient an irreducible representation of $\mathfrak{M}$ (and, therefore, one of M) with the same highest weight as ψ. Its dimesnion is, of course, $\leq d$ (and can be, for special p, $< d$).

(9.2.3) Let $\mathfrak{K}_3''$ denote the set of isomorphism classes of centrally simple universal groups of Lie r-type, r a prime, which have non-trivial central extensions (see [, (4.3.3)] for an explicit list). For a group M of Lie type we denote by $d(M)$ the number d given in (9.2.2) for the corresponding universal group.

(9.2.4) Corollary. Let (G,M,τ,ω) be an admissible quadruple with G not isomorphic to a group from $\mathfrak{K}_3' \cup \mathfrak{K}_3''$. Then for $d = d(M)$ we have

$$|G| \leq f(d).$$

Proof. Since G is not from $\mathfrak{K}_3''$, G is its own universal cover and therefore the homomorphism $\tau : G \longrightarrow M$ lifts to a homomorphism $\tilde{\tau} : G \longrightarrow \tilde{M}$

where $\tilde{M}$ is the universal group of Lie r-type corresponding to M. Let $\eta : \tilde{M} \longrightarrow GL_s(\bar{\mathbb{F}}_r)$ be an irreducible representation with $1 < s \le d$ as in (9.2.2). Then not all compositions factors of $\eta\circ\tilde{\tau}$ are trivial (for otherwise $\eta\circ\tilde{\tau}(G)$ would be a unipotent group). Let $\eta' : G \longrightarrow GL_t(\bar{\mathbb{F}}_r)$ be a non-trivial composition factor of $\eta\circ\tilde{\tau}$. Then $1 < t \le s \le d$ and by (9.2.1) we have the inequalities claimed with t instead of d. Since the functions involved are monotonous and $t \le d$ the claims follow.

(9.3) Let now $\mathfrak{K}_3''$ be the set of isomorphism classes of universal groups G of Lie r-type, r a prime, which can be subgroups of a group $M \simeq A_1(4)$, $A_1(9)$, $A_2(4)$, $B_2(2)$, $B_3(3)$, $D_4(2)$, $F_4(2)$, ${}^2A_3(9)$, ${}^2B_2(8)$, ${}^2E_6(4)$, ${}^2B_2(2)$, ${}^2F_4(2)$, ${}^2G_2(3)$ with r different from the characteristic of the argument (such groups satisfy, of course, a simpler condition $|G| \le |{}^2E_6(4)| < 3.1\cdot 10^{23}$).

(9.3.1) Lemma. If (G,M,τ,ω) is an admissible pair with G not isomorphic to a group from $\mathfrak{K}_3''$ and with a classical M then $n \ge 2^{(d-3)/2} - 0.5$ where $d = d(M)$.

Proof. Let s be the rank of the algebraic group $\mathfrak{M}$ corresponding to M. Then one sees (from (9.2.2) where $n = s$ and the type of M is A_s, B_s, C_s, D_s) that $s \ge (d-1)/2$. On the other hand from [, (4.4.2)] applied to M (the cases excluded there are excluded here by the assumption that G is not in $\mathfrak{K}_3''$) we have $n \ge (m^b-1)/2$ where b is the number given in [, Table T4.4] and where M is centrally isomorphic to ${}^cX_s(m^c)$ in the notation of [, (4.1)]. By [, Table T4.4] we have further that $b \le s$.

Still further, our assumptions excluded cases when M is centrally isomorphic to ${}^2B_2(2)$, ${}^2F_4(2)$, ${}^2G_2(3)$ (where $m = \sqrt{2}, \sqrt{2}, \sqrt{3}$ respectively). Thus we have that $m \geq 2$. Thus we have (under our assumptions) that $2n + 1 \geq m^b \geq 2^b \geq 2^s \geq 2^{(d-1)/2}$ or $n \geq 2^{(d-3)/2} - 0.5$ as claimed.

(9.3.2) Remark. One may find it more handy to verify the inequality $n \geq (2^s-1)/2$ directly from V. Landazuri and G.M. Seitz [, p.419].

(9.3.3) Corollary. Under the assumptions of (9.3.1) if G is not isomorphic to a group from $\mathcal{R}_3' \cup \mathcal{R}_3'' \cup \mathcal{R}_3'''$ and if $\omega\circ\tau$ is infinitesimally irreducible then $d \leq 78$ and so $|G| \leq 3\cdot 10^{22}$.

Proof. Let t be the number of positive roots of $\mathfrak{G}$. Then the dimension of any infinitesimally irreducible representation of $\mathfrak{G}$ is $\leq p^t$. Thus $n \leq p^t$. But $|G| \geq p^{2t}$. Thus $n \leq |G|^{1/2}$, whence by (9.2.4) $n \leq (f(d))^{1/2}$. Thus by (9.3.1) $2^{(d-3)/2} - 0.5 \leq (f(d))^{1/2}$. One checks that this inequality is violated for $d > 78$ whence our claim.

(9.3.4) Corollary. Under the assumptions of (9.1), if G is not isomorphic to a group from $\mathcal{R}_3' \cup \mathcal{R}_3'' \cup \mathcal{R}_3'''$, then either $|G| \leq 3\cdot 10^{22}$ or M is of type E_8 and $|G| < 1.5\cdot 10^{33}$.

Proof. We know by (9.3.3) that if M is classical then $d \leq 78$. But $d \leq 78$ also for all exceptional types but E_8, whence the claim in view of (9.2.4).

(9.3.5) Corollary. (9.1) holds with $\mathcal{R}_3$ the union of $\mathcal{R}_3'$, $\mathcal{R}_3''$, $\mathcal{R}_3'''$ plus the isomorphism classes of all universal groups X of Lie r-type, r a prime, where $|X| \leq 3 \cdot 10^{22}$ or X is a subgroup of a group of type E_8 of characteristic r' and $|X| < 1.5 \cdot 10^{33}$.

(9.3.6) Final comment. In our proof of (9.1) we tried to exhibit not only that a finite $\mathfrak{R}$ can be found, but also that this $\mathfrak{R}$ is not outrageous in size and that there are lines of analysis which clearly restrict $\mathfrak{R}$ further. Among further constrictions we mention

(i) the relation of the maximal degree of infinitesimally irreducible representations of G to the order of G was used only tangentially,

(ii) a very powerful tool completely left out is the T.A. Springer and R. Steinberg result [, I.5.17]; for classical groups V. Landazuri and G. Seitz [, p. 416] is probaly stronger,

(iii) the condition : $|G/\text{center}|$ divides $|M/\text{center}|$ is certain to eliminate many cases.

A final analysis seems certain to involve a computer.

10. Tight embeddings of groups of Lie p-type into groups of Lie p-type

(10.1) We consider quadruples (G,M,τ,ω) consisting of a universal group G of Lie p-type, a group M of Lie p-type, a homomorphism $\tau : G \longrightarrow M$, and a representation $\omega : M \longrightarrow GL_n(\overline{\mathbb{F}}_p)$, $n > 1$. We assume above and throughout this section that G and M are associated to absolutely almost simple algebraic groups $\mathfrak{G}$ and $\mathfrak{M}$. We call a quadruple as above admissible (in this section). We let $\varphi := \omega \circ \tau$ and let $\tilde{\varphi}$ and $\tilde{\omega}$ be represetnations of $\mathfrak{G}$ and $\mathfrak{M}$ corresponding to φ and ω as in (4.1.2).

(10.2) Theorem. There exists an integer $m_0(\leq 3888)$ and a finite list $\mathfrak{R}_4$ (independent of p) of isomorphism classes of universal finite groups of Lie p-type such that the following holds: If (G,M,τ,ω) is an admissible quadruple, $\mathfrak{G},\mathfrak{M},\tilde{\varphi},\tilde{\omega}$ etc are as above and

(a) $\tilde{\varphi}$ is $\otimes$-indecomposable,

(b) $p \neq 2$

(c) $\mathbb{F}_p(\varphi) = \mathbb{F}_{p^m}$, $m \geq m_0$ $(= 12)$

(d) the isomorphism class of G is not in $\mathfrak{R}_4$,

then $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

(10.2.1) Comment (10.2)(c) is used only for exceptional $\mathfrak{M}$ via (2.3.3). It can, no doubt, be replaced by $p^m \geq q_0$, thus including (c) in (d).

(10.3) Proposition. Suppose that

(a) $\mathfrak{M}$ is classical,

(b) $\mathfrak{M}$ is not of type D_n or B_n if $p=2$,

(c) G is not isomorphic to a group from $\mathcal{R}_1 \cup \mathcal{R}_2$,

(d) (G,M,τ,ω) is an admissible quadruple,

(e) $\omega \circ \tau$ is $\otimes$-indecomposable.

Then $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

Proof. Let $\eta : \mathfrak{M} \to GL_m$ denote a non-trivial representation of $\mathfrak{M}$ of the smallest dimension. Then

(10.3.1) If G is not isomorphic to a group from $\mathcal{R}_1 \cup \mathcal{R}_2$ then either $\eta\circ\tau$ is irreducible or $\eta(\mathfrak{M}) = SO_m$ and $\eta\circ\tau$ stabilizes a non-singular point.

Indeed, let $V = \overline{\mathbb{F}}_p^m$ and suppose that $\eta\circ\tau$ is reducible by $V_1 \subseteq V$. Let $\mathcal{H}$ denote the stabilizer of V_1 in $\mathfrak{M}$. It is an algebraic group containing G. If $\omega|\mathcal{H}$ is reducible then so is G. Thus $\omega|\mathcal{H}$ must be irreducible. If it is imprimitive, $\otimes$-decomposable or $\otimes$-imprimitive then so is G. This is again impossible by (7.1), assumptions, and (8.1) respectively.

Since $\omega|\mathcal{H}$ is irreducible $\mathcal{H}$ must be reductive. Since it is not imprimitive, or $\otimes$-decomposable, or $\otimes$-imprimitive, the semi-simple part of the connected component of $\mathcal{H}$ must be simple. One sees easily (say by E.B. Dynkin [, Theorems 4.2, 5.2 and Lemma 6.1] or M. Aschbacher [, p. 472]) that this can happen only when $\mathfrak{M}$ is an orthogonal group SO_m, $\dim V_1 \leq 2$ and V_1 is non-singular. However, if $\dim V_1 = 2$ then the connected component of $\mathcal{H}$ has non-trivial center C. Since C is not the center of $\mathfrak{M}$ and since G is centrally simple, it follows that $\tilde{\omega}|C$ is not scalar and centralizes G. This contradicts the irreducibility of $\omega\circ\tau$.

This proves (10.3.1).

(10.3.2) If in (10.3.1) $\eta\circ\tau$ is irreducible, then $\tilde{\varphi}(\mathbb{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

Indeed, by (2.1) $\eta\circ\tau$ extends to $\tilde{\eta}:\mathbb{G} \longrightarrow \mathfrak{M}$. We have, of course, that $\tilde{\omega}\circ\tilde{\eta}$ agrees with $\tilde{\varphi}$ on G. By the unicity of the extension to an algebraic group (see (2.1.1)) we have, therefore, $\tilde{\varphi} = \tilde{\omega}\circ\tilde{\eta}$.

(10.3.3) If in (10.3.1) $\eta\circ\tau$ is reducible, then $\tilde{\varphi}(\mathbb{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

Indeed then $\mathfrak{M}$ is SO_m and $\eta\circ\tau(G) \subseteq \mathfrak{H} \simeq SO_{m-1}$. Since by (10.3.1) $\eta\circ\tau(G)$ can not preserve a 2-dimensional subspace, we see $\eta\circ\tau$ is irreducible in the natural representation of $\mathfrak{H}$. Thus by (10.3.2), $\tilde{\varphi}(\mathbb{G}) \subseteq \tilde{\omega}(\mathfrak{H})$, whence our claim.

(10.3.4) Possible example. It appears that in characteristic 2 every irreducible representation of SO_{2n+1} is irreducible when restricted to SO_{2n}.

(10.4) Proposition. In the notation of (10.1) assume that

(a) $\mathfrak{M}$ is exceptional,

(b) $|\mathbb{F}_p(\varphi)| = p^m$, $m \geq m_0 := 122$,

(c) $\omega\circ\tau$ is $\otimes$-indecomposable.

Then $\tilde{\varphi}(\mathbb{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

(10.4.1) Lemma. Let $G \simeq {}^cX_a(\tilde{q}^c)$ and let q be as in (2.3.2). Let t be the number of positive roots of $\mathfrak{G}$. If $p \neq 2$ and $q > 2p^t$ then $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

Proof. Suppose first that G is not ${}^2G_2(3^{2s+1})$. Then after an appropriate twist by the Frobenius we can assume that the highest weight λ of $\tilde{\omega}$ is of the form $\lambda = \sum a_i\lambda_i$ where $0 \le a_i < p$ and the λ_i are fundamental weights. From Weyl's formula we see that $d_0(\lambda) \le d_0((p-1)\rho) = p^t$ whence our claim by (2.3.3).

If $G \simeq {}^2G_2(3^{2s+1})$ then (up to a twist by the Frobenius) G has only 3 $\otimes$-indecomposable representations (with highest weights $0 \cdot \lambda_1$, λ_1, $2\lambda_1$). We have $d_0(2\lambda_1) = 27 < 2 \cdot 3^6$ whence our claim by (2.3.3).

It remains now to obtain a bound on t from (10.4.1). This bound will be extracted from the lengths of certain chains of centralizers. If $g \in G$ is semi-simple (i.e. of order prime to p) then $\mathfrak{H} := Z_{\mathfrak{G}}(g)$ contains a maximal torus $\mathfrak{T}$ of $\mathfrak{G}$. The root system Σ_1 of $\mathfrak{H}$ with respect to $\mathfrak{T}$ is a subsystem of the root system Σ of $\mathfrak{G}$ with respect to $\mathfrak{T}$. We consider sequences $g_0 = 1, g_1, \ldots, g_x$ of semi-simple elements of G such that

(10.4.2) (i) g_i is a p'-element for $i = 1, \ldots, x$,

(ii) $g_{i+1} \in \mathfrak{D}Z_G(g_0, \ldots, g_i)$ for $i = 0, \ldots, x-1$.

It follows from T.A. Springer and R. Steinberg [,4.3, 5.3] and a simple analysis of root subsystems of root systems of type G_2, F_4, E_6, E_7, E_8 that

(10.4.3) If $\{g_0,\dots,g_x\}$ is as in (10.4.2) and $|\tau(g_i)| > 5$ for all $i = 1,\dots,x$, then the rank of $\mathfrak{D}Z_{\mathfrak{m}}(g_0,\dots,g_{i+1})$ is strictly less than that of $\mathfrak{D}Z_{\mathfrak{m}}(g_0,\dots,g_i)$ for $i = 0,\dots,x-1$.

In particular, we have

(10.4.4) If $\{g_0,\dots,g_x\}$ is as in (10.4.2) and $|\tau(g_i)| > 5$ for $i = 1,\dots,x$, then $x \le 8$.

(10.4.5) If $G \simeq X_a(q)$, $X_a \ne A_a$, $q \ge 23$, $p \ne 2$ then $x = a$ and $\tau(g_i) \ge (q-1)/4$ in (10.4.2).

Indeed, the center of G is elementary abelian of period at most 3 (for E_6) if $X_a \ne A_a$. One then chooses in the Dynkin diagram a sequence of length a of embedded diagrams such that no member of the sequence is a subdiagram of type A_b, $b \ge 3$. Then one can choose elements $g_1,\dots,g_a$ in the split torus of G such that $|g_i| = q - 1$, and then $|\tau(g_i)| \ge (q-1)/4$.
For example in E_6 take subsystems $D_5 \supset D_4 \supset A_3 \supset A_2 \supset A_1$.

Combining (10.4.4) and (10.4.5) we have

(10.4.6) If $G \simeq X_a(q)$, $X_a \ne A_a$, $q \ge 23$, $p \ne 2$ then $a \le 8$.

(10.4.7) If $G \simeq A_a(q)$, $q \ge 61$, $p \ne 2$, then $a \le 8$.

Indeed, take in the Dynkin diagram of type A_a the sequence $A_{a-1} \supset A_{a-2} \supset \dots$ of embedded diagrams. Taking for $\tilde{g}_i$, $i = 1,\dots,a$, the generator of $Z_{A_{a-i+1}(q)}(A_{a-i}(q))(\simeq \mathbb{Z}/(q-1))$ we see that the image of $\tilde{g}_i$ in G/center is of the same order as $\tilde{g}_i$ if $i > 1$, and it is of order $(q-1)/(a+1,q-1)$ if $i = 1$. since $q \geq 61$ this gives us a sequence of g_i as in (10.4.2) such that $\tau(g_i) \geq 5$ for at least the 9 last g_i. Therefore by (10.4.5) $a \leq 8$.

To deal wit $^2A_a(q^2)$ we consider it as $SU_{a+1}(q)$ and view in it a sequence of naturally embedded subgroups $SU_{a+1}(q) \supset U_a(q) \supset U_{a-1}(q) \supset \dots$. As in the case of $A_a(q)$ this gives (one has, though, to replace in the proof q-1 by q+1):

(10.4.8) If $G \simeq {}^2A_a(q)$, $a \geq 2$, $q \geq 61$, $p \neq 2$ then $a \leq 8$.

In $^2D_a(q^2)$ we consider the sequence $^2D_a(q^2) \supset {}^2D_{a-1}(q^2) \supset \dots \supset {}^2D_3(q^2) \simeq {}^2A_3(q^2) \supset {}^2A_2(q^2) \supset {}^2A_1(q^2) \simeq A_1(q)$ where in $^2A_3(q)$ we consider the sequence of subgroups as for $^2A_a(q^2)$ above. This gives:

(10.4.9) If $G \simeq {}^2D_a(q^2)$, $a \geq 3$, $q \geq 23$, $p \neq 2$, then $a \leq 8$.

Combining (10.4.6)-(10.4.9) and verifying directly for $^2E_6(q^2)$, $^3D_4(q^3)$, $^2G_2(3^{2s+1})$, we see that

(10.4.10) If $G \simeq {}^cX_a(q^c)$, $q \geq 61$, $p \neq 2$, then the number t of positive roots of $\mathfrak{G}$ satisfies $ct \leq 120$ and for $G \simeq {}^2G_2(3^{2s+1})$, $2\sqrt{3}\ t = 12\sqrt{3} \leq 120$.

[Indeed, 120 is achieved for E_8 when $c = 1$, $t = 120$. The next worst case is ${}^2D_8(q^2)$ when $c = 2$, $t = 56$.]

(10.4.11) Proposition. Let $\mathfrak{M}$ be exceptional, $G \simeq {}^cX_a(q^c)$, $q \geq 61$, $p \neq 2$ and φ $\otimes$-indecomposable. If $q^c \geq p^{122}$, then $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$.

Indeed, by (10.4.1) (with change of notation) we have that $\tilde{\varphi}(\mathfrak{G}) \subseteq \tilde{\omega}(\mathfrak{M})$ if $q^c > 2^c p^{ct}$. But $ct \leq 120$ by (10.4.11) and $2^c < p^c$ evidently. Thus $q^c \geq p^{122} > 2^c p^{ct}$ whence our claim.

(10.4.12) We can now complete our proof of (10.4). We have that $\mathbb{F}_p(\varphi) = \mathbb{F}_q$ or $\mathbb{F}_{q^c}$ if $G \simeq {}^cX_a(q^c)$. Thus if $\mathbb{F}_p(\varphi) = p^m$, $m \geq 122$ implies that $q^c \geq p^{122}$ whence (10.4) follows directly from (10.4.11).

(10.5) Now we can complete the proof of (10.2). Let $\mathfrak{K}_4$ consist of $\mathfrak{K}_1$ and $\mathfrak{K}_2$. Then (10.3) holds and it remains to consider exceptional $\mathfrak{M}$. Then $q \geq 61$ is implied by (10.2)(c) whence (10.4) holds.

11. Sufficient conditions for maximality of a subgroup of a Lie p-type of a classical finite group of characteristic p

(11.1) We let H_0 be a finite simple classical group and $H_0 \subseteq H \subseteq \mathrm{Aut}\, H$. If H_0 is $D_4(q)$ we assume that H does not contain the triality automorphism of H_0. We denote by $\tilde{H_0}$, as in (6.1), a lifting of H_0 to its natural projective representation on a vector space V over a finite field K, $K \supseteq \mathbb{F}_p$, with a form f, so that, as in (6.1), $\tilde{H_0} = \Omega(V,f)$. We denote by $\mathcal{H}$ the algebraic linear group associated to (V,f).

(11.1.1) We further consider a universal group G of Lie p-type which is associated to an absolutely almost simple simply connected algebraic $\overline{\mathbb{F}}_p$-group $\mathfrak{G}$. We consider a homomorphism $\varphi : G \longrightarrow \tilde{H_0}$. By (2.1), except when $p = 2$ and f is quadratic, φ extends to a homorphism $\tilde{\varphi} : \mathfrak{G} \longrightarrow \mathcal{H}$ of algebraic groups.

(11.1.2) Let $\overline{G}$ be the image of $\varphi(G)$ in H and $N := N_H(\overline{G})$. Our second main result is

(11.2) Theorem. Suppose that

(i) $p \neq 2$

(ii) $|K| \geq p^{122}$,

(iii) G is not isomorphic to a group from a finite list $\mathcal{R}$ of groups,

(iv) H contains no algebraic automorphism of H_0 if $H_0 \simeq D_4(q)/\text{center}$ which do not preserve the given representation,

(v) $H = NH_0$,

(vi) $\mathbb{F}_p(\varphi) = K$,

(vii) φ is absolutely irreducible and absolutely $\otimes$-indecomposable,

(viii) $\tilde{\varphi}(\mathbb{G})$ is maximal among connected algebraic subgroups of $\mathbb{G}$ ($\tilde{\varphi}$ is defined over K).

Then N is a maximal subgroup of H.

(11.2.1) Discussion.

(i) is probably not needed but at many points the proof would become much more complicated.

(ii) is needed at two points to apply (10.4) (see (11.) below) and, in a weaker form $|K| > p^2$ to exclude the possibility of an embedding $\varphi(G) \to Alt_d$ (see (11.3.3)). Since we believe that (10.4) holds without the assumption on $\log_p \mathbb{F}_p(\varphi)$, (ii) can be, probably, replaced by $|K| > p^2$. This condition still excludes infinitely many G as examples (9.1.3) and (9.1.4) show.

(iii) excludes finitely many isomorphism classes of G. The set $\mathfrak{R}$ is, probably, much smaller than we construct.

(iv) is dependent on our use of M. Aschbacher's results [ ].

(v) excludes "uninteresting" maximal subgroups.

(vi) and (vii) are justified by M. Aschbacher's result and our (6.2); they are necessary if we exclude groups from $\mathcal{EC}_H$.

(viii) is, actually, raison d'etre of the whole paper; it reduces the question on maximal subgroups for finite groups to that for algebraic groups. This latter problem might be solvable using the ideas of E.B. Dynkin [ ] and [ ]. We discuss in Section 12 below some cases when (vii) is also necessary. The reason that it may fail to be necessary is that the Galois group part of Out H_0 contained in H may not preserve the conjugacy class of $\overline{G}$ in H_0.

(11.3) We assume here that G and φ are as in (11.1) but N is not maximal. Let R be a maximal subgroup of H such that $R \supseteq N$. If R belongs to the family C_1 (see (5.3)) then $\varphi(G))$ is reducible (contradicts (11.2 (vii)). If R belongs to C_2 or C_3 then $\varphi(G)$ is (absolutely) imprimitive in contradiction with (7.1). If R belongs to C_4, C_7 or C_9 (see (5.4)) then $\varphi(G)$ is absolutely $\otimes$-decomposable (in contradiction with (11.2)(vii)) or absolutely $\otimes$-imprimitive (in contradiction with (11.2)(iii) in view of (8.1)). R can not belong to C_5 by (11.2)(vi). In the case of the family C_8 only the case when R is orthogonal in dimension 4 is excluded by (11.2)(vii).

The remaining cases will be excluded in view of (11.2)(viii) below in ( ) if $R \neq N$. This leaves the family C_6.

In this case $\varphi(G)$ is contained (see M. Aschbacher [, p. 472]) in the normalizer of an extraspecial group S of order r^{2m+1} and $\dim_K V = r^m$, r a prime, $r \neq p$. A projection of this normalizer modulo S gives a homomorphism $\bar{\tau} : \varphi(G) \longrightarrow PSp_{2m}(r)$. By our conditions the projective representation $\bar{\omega} : PSp(r) \longrightarrow PGL(V)$ lifts to a linear representation $\omega : Sp_{2m} \longrightarrow GL(V)$ (contained in the normalizer of S) and $\bar{\tau}$ to an embedding $\tau : \varphi(G) \longrightarrow Sp_{2m}(r)$. Thus $(G, Sp_{2m}(r), \tau\circ\varphi, \omega)$ is an admissible pair as in Section 9 and is thus excluded by the assumption (11.2(iii)) in view of (9.1). Thus we have, by M. Aschbacher [, p. 469] and since the groups in C_8 not excluded so far can be seen to satisfy the conclusions below in view of the condition (11.2)(iii), (11.3.1) The socle M of R is simple.

Our major step is

(11.3.2) Proposition. M is of Lie p-type.

In the proof we consider the other possibilities

(i) M is of Lie p'-type,

(ii) $M \simeq Alt_m$ for some m,

(iii) M is sporadic.

The case (iii) is excluded in an ad hoc manner by creating a finite set $\mathfrak{R}_5'$ of the isomorphism classes of finite universal groups of Lie type which have central isomorphisms with subgroups of sporadic groups and then including $\mathfrak{R}_5'$ into $\mathfrak{R}$.

In the case (i) we have that $Aut\ M/M$ is solvable, whence R/M is solvable. Since the groups G which are not centrally simple are rejected by (11.2)(ii) (family $\mathfrak{R}_3'$) it follows that $\overline{G} \subseteq M$. Lifting M to $\tilde{H}_0$ we obtain an admissible pair of Section 9 satisfying the assumptions of (9.1). Thus this is impossible by (11.2)(iii) and (9.1). [Note that we lifted G to $\tilde{H}_0$ using universality of G and the fact that G has no sporadic extensions, consult family $\mathfrak{R}_3''$.]

(11.3.3) In the case $M \simeq Alt_m$ we consider first the case when m lifts in $\tilde{H}_0$ to a group isomorphic to Alt_m. We know by G.D. James [, 11.5] that for every irreducible representation ω of Sym_m over $\overline{\mathbb{F}}_p$ we have $\mathbb{F}_p(\omega) =$

$\mathbb{F}_p$. Since Alt_m has index 2 in Sym_d the restriction ω' of ω to Alt_m is either irreducible, and then $\mathbb{F}_p(\omega') = \mathbb{F}_p$, or it has two irreducible components, say ω_1' and ω_2'. For $\gamma \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ we, therefore, have that either $\gamma \circ \omega_1' \simeq \omega_1'$ or $\gamma \circ \omega_1' \simeq \omega_2'$. If the first case happens for all $\gamma \in \mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ then $\mathbb{F}_p(\omega_1') = \mathbb{F}_p(\omega_2') = \mathbb{F}_p$. Otherwise the stabilizer of (the equivalence class of) ω_1' is of index 2 in $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ and then $\mathbb{F}_p(\omega_1') = \mathbb{F}_p(\omega_2') = \mathbb{F}_{p^2}$.

In view of (11.2)(ii) and (vi) we can not, therefore, have that $\varphi(G)$ is isomorphic to a subgroup of Alt_m.

(11.3.4) When $M \simeq \mathrm{Alt}_d$ and M lifts in $\tilde{H}_0$ to a group which is a non-trivial central extension of Alt_d, the corresponding (projective) representation of a cover of Sym_d may require a quadratic extension of $\mathbb{F}_p$ for accomodating values of its characters (see I. Schur [, IX in §4.1]). Then Alt_d may require an extension of degree 4. We shall, however, bypass this question. First recall (I. Schur [, II in §5]) that the universal cover $\tilde{\mathrm{Alt}}_d$ of Alt_d is of degree 2 if $d > 7$.

Let $d > 7$ and let $\omega : \tilde{\mathrm{Alt}}_m \to GL_n(\mathbb{F}_p)$ be a faithful irreducible representation. We have by Weisfeiler [, (3.3)(ii)] (which follows from A. Wagner [, Theorem 1.3(ii)]) that $m \leq 5.4 + 2.2 \log_2 n$. On the other hand, as in the proof of (9.3.3) and (8.2.3), we have $n \leq p^t$ where t is the number of positive roots of $\mathcal{G}$. Thus $m \leq 5.4 + 2.2t \log_2 p$. This is completely analogous to (8.2.4). As there, the tables (8.2.7) and (8.2.9) imply that, with the exception of groups G isomorphic to a finite set $\mathfrak{R}_5''$ of groups, the above inequality fails. This concludes our proof of (11.3.2).

(11.4) We can now conclude our proof of (11.1). Let, as in (11.3) $R \supseteq N$. Then by (11.3.1) and (11.3.2) the socle M of R is simple of Lie p-type. Let M' be the lifting of M to $\tilde{H}_0$ and $\mathfrak{M}$ the algebraic absolutely almost simple simply connected group corresponding to M. By (2.1) (applicable in view of (11.2)(i)) the embedding $M' \to \tilde{H}_0$ extends to a homomorphism $\tilde{\omega} : \mathfrak{M} \to \mathfrak{H}$. By (10.2) it follows that $\tilde{\varphi}(\mathbb{G}) \subseteq \tilde{\omega}(\mathfrak{M})$, whence by (11.2)(vii) either $M = H_0$ or $M = \overline{G}$ thus establishing maximality of N.

12. More necessary conditions for maximality

(12.1) Let, as before in (6.1) and (11.1), $\mathcal{H}$ be the algebraic linear group associated to a finite field K with an automorphism θ of order 1 or 2, a vector space V over k and a form f on V. Let k be the fixed field of θ. Then $\mathcal{H}$ is defined over k. Let $\mathbb{G}$ be an algebraic absolutely almost simple simply connected gruop defined over a finite field k'. Then $\mathbb{G}$ is quasi-split over k' and it is split by an extension $k'_{\mathbb{G}}$ of k' of prime degree over k' (see (4.1.2)). We set

(12.1.1) $q := |k'|$, $m := [k' : \mathbb{F}_p]$, $c := [k'_{\mathbb{G}} : k']$, r = rank of $\mathbb{G}$.

Let $\tilde{\varphi} : \mathbb{G} \to \mathcal{H}$ be an irreducible (on $V \otimes \overline{\mathbb{F}}_p$) and $\otimes$-indecomposable representation of $\mathbb{G}$ such that $\tilde{\varphi}(\mathbb{G}(k')) \subseteq \mathcal{H}(k)$. We set, as before,

(12.1.2) $G := \mathbb{G}(k')$, $\varphi := \tilde{\varphi}|G$.

Let λ be the highest weight of $\tilde{\varphi}$ and Γ_λ the stabilizer of λ in the Galois group of $\overline{\mathbb{F}}_p/k'$ acting as described in (4.1.2). Set

(12.1.3) $k'_\lambda := \overline{\mathbb{F}}_p^{\Gamma_\lambda}$.

We have by (4.3.2) (mutatis mutandi) and in view of the $\otimes$-indecompasability of $\tilde{\varphi}$, that $\mathbb{F}_p(\varphi) = k'_\lambda$. We are justified by M. Aschbacher [, p. 469] in assuming that

(12.1.4) $K = k'_\lambda$;

Otherwise $\varphi(G)$ will fall into C_H. We can also assume (after twisting $\mathcal{G}$ if necessary by a special isogeny, see A. Borel and J. Tits [, §3]) that

(12.1.5) $\ker \tilde{\varphi}$ is a central group subscheme of $\mathcal{G}$ (i.e. $d\tilde{\varphi}$ is non-zero on every root subalgebra).

We now set

(12.2.2) $H^- := \mathcal{H}^-(k)$

so that

(12.2.3) $H_0 \triangleleft H^-$.

Let further $A := \operatorname{Aut} H_0$. Set $\Gamma := \operatorname{Gal}(k/\mathbb{F}_p)$. Then it follows from R. Steinberg [, Theroems 30 and 36] that there exists an epimorphism $\sigma : A \to \Gamma$. Set $A_0 := \operatorname{Ker} \sigma$. Then $H^- \triangleleft A_0$ and

(12.2.4) $A_0/H^- \simeq \operatorname{Aut} \Delta_{\mathcal{H}}$.

Here $\Delta_{\mathcal{H}}$ is the Dynkin diagram of $\mathcal{H}$ and we assume that $|\operatorname{Aut} \Delta_{\mathcal{H}}| = 2$ if $p = 2$ and $\mathcal{H}$ is of type C_2.

(12.3) Let now $\tilde{\varphi}^- : \mathcal{G} \to \mathcal{H}^-$ be the composite of $\tilde{\varphi}$ and of the projection $\mathcal{H} \to \mathcal{H}^-$. Since $\tilde{\varphi}$ is irreducible, the center of $\tilde{\varphi}(\mathcal{G})$ consists of scalar matrices and is, therefore, contained in the center of $\mathcal{H}$. Thus

(12.3.1) $\tilde{\varphi}^{-}(\mathbb{G})$ is the adjoint group of $\mathbb{G}$.

We set $\overline{G} := \tilde{\varphi}^{-}(\mathbb{G})$.