ON SIMPLE 15-DIMENSIONAL LIE ALGEBRAS IN CHARACTERISTIC 2

ALEXANDER GRISHKOV, HENRIQUE GUZZO JR., MARINA RASSKAZOVA, AND PASHA ZUSMANOVICH

ABSTRACT. Motivated by the recent progress towards classification of simple finite-dimensional Lie algebras over an algebraically closed field of characteristic 2, we investigate such 15-dimensional Skryabin algebras.

INTRODUCTION

The classification of finite-dimensional simple Lie algebras over an algebraically closed field has a long and interesting history. The characteristic zero case, due to Killing, Cartan, and Dynkin, is nowadays a classic. The case of characteristic $p > 3$ was accomplished relatively recently, due to the efforts of dozens of people, spread over more than 50 years and hundreds of papers, and culminated in the three volumes by Strade, [St]. The cases of characteristics 2 and 3 remain open, although a lot of efforts were done recently to augment the classification program by small characteristics specifics, in particular, to put Lie superalgebras into play (see, for example, the monumental treatises [BGLLS] and [BLLS], and references therein).

In [GZ], we did a small step towards the classification of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic 2: it was proved that any such algebra of absolute toral rank 2, and having a Cartan subalgebra of toral rank one, is 3-dimensional. In this, we relied on the paper by Skryabin, [Sk], where it was proved, among other things, that in characteristic 2 there are no simple Lie algebras of absolute toral rank 1, and simple Lie algebras having a Cartan subalgebra of toral rank 1 were characterized as certain filtered deformations of semisimple Lie algebras with the socle of the form $S \otimes \mathcal{O}$, where S is a simple Lie algebra either of Zassenhaus or Hamiltonian type, and $\mathcal{O}$ is the algebra of truncated polynomials.

In the process of the proof of the main result of [GZ], we have constructed a 2-parameter family $\mathscr{L}(\beta,\delta)$ of 15-dimensional simple Lie algebras ([GZ, §6]). The algebra $\mathscr{L}(0,0)$ in this family coincides with the smallest algebra in the series constructed by Skryabin ([Sk, Example at pp. 691–692]).

The purpose of this paper is to study the family $\mathscr{L}(\beta,\delta)$. Among other things, we prove that all algebras within the family are isomorphic to the same algebra $\mathscr{L}$ (§1), and we determine the absolute toral rank (§3), and the automorphism group of $\mathscr{L}$ (§4). In passing, we introduce the notion of a thin decomposition of a simple Lie algebra with respect to a torus (§3.4) which, we suggest, should play a role in classification efforts (see open questions in §7).

Throughout the paper, the ground field K is assumed to be perfect of characteristic 2, unless stated otherwise. Although the initial problem assumes the ground field is algebraically closed, for our purposes it will be enough to assume that square roots exist in K. This allows us to include in our consideration the case $K = \mathsf{GF}(2)$; this can be useful in some circumstances (cf. §3 and §6).

Our terminology and notation is mostly standard: $C_L(X)$ denotes the centralizer of a set X in a Lie algebra L; the linear span of a set X is denoted either by KX or by $\langle X \rangle$; the 2-envelope of a Lie algebra L is denoted by L_2; id_L denotes the identity map on L. Other notation is explained as soon as it is introduced in the text.

Date: First written March 26, 2021; last revised November 22, 2021.

2020 *Mathematics Subject Classification.* 17B20; 17B40; 17B50; 17-04.

Key words and phrases. Simple Lie algebra; characteristic 2; tori; automorphisms; thin decomposition; GAP.

1. The 2-parameter family $\mathscr{L}(\beta,\delta)$

1.1. Definition of the 2-parameter family. Recall the definition of the family $\mathscr{L}(\beta,\delta)$, depending of two parameters $\beta,\delta \in K$, of 15-dimensional simple Lie algebras constructed in [GZ, §6]. These algebras are filtered deformations of the semisimple Lie algebra of the form

$$\mathsf{S} \otimes \mathcal{O}_1(2) + g \otimes \langle 1, x\rangle + \partial, \tag{1}$$

where S is the 3-dimensional simple Lie algebra with the basis $\{e,f,h\}$ and multiplication table

$$[e,h]=e, \quad [f,h]=f, \quad [e,f]=h, \tag{2}$$

$g=(\mathrm{ad}\, f)^2$ is an outer derivation of S, $\mathcal{O}_1(2)$ is the 4-dimensional divided power algebra with the basis $\{1,x,x^{(2)},x^{(3)}\}$, with the multiplication given by

$$x^{(i)}x^{(j)} = \binom{i+j}{i} x^{(i+j)},$$

and $\partial : x^{(i)} \mapsto x^{(i-1)}$ is the special derivation of $\mathcal{O}_1(2)$.

For our purpose, it will be convenient to relabel the basis elements of the algebras $\mathscr{L}(\beta,\delta)$ as follows:

$$\begin{array}{ll} b_1 = e\otimes 1, & c_1 = e \otimes x^{(3)}, \\ b_2 = f\otimes 1, & c_2 = f \otimes x^{(3)}, \\ b_3 = h\otimes 1, & c_3 = h \otimes x^{(3)}, \\ b_4 = e\otimes x, & c_4 = g \otimes 1, \\ b_5 = f\otimes x, & c_5 = g \otimes x, \\ b_6 = h\otimes x, & d = \partial, \\ b_7 = e\otimes x^{(2)}, & \\ b_8 = f\otimes x^{(2)}, & \\ b_9 = h\otimes x^{(2)}. & \end{array} \tag{3}$$

In terms of this basis, the multiplication table of $\mathscr{L}(\beta,\delta)$ (see [GZ, (5.4)]) reads:

(4)

	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9	c_1	c_2	c_3	c_4	c_5	d
b_1	b_3	b_1	βc_5	b_6	b_4	δc_4	b_9	b_7	$\delta c_5 + d$	c_3	c_1	b_2	b_5	βc_2
b_2		b_2	b_6	0	b_5	b_9	0	b_8	c_3	0	c_2	0	0	0
b_3			b_4	b_5	0	b_7	b_8	0	c_1	c_2	0	0	0	0
b_4				0	0	$\delta c_5 + d$	c_3	c_1	b_3	c_4	b_2	b_5	0	b_1
b_5					0	c_3	0	c_2	c_4	0	0	0	0	b_2
b_6						c_1	c_2	0	0	0	c_4	0	0	b_3
b_7							0	0	b_6	c_5	b_5	b_8	c_2	b_4
b_8								0	c_5	0	0	0	0	b_5
b_9									0	0	c_5	0	0	b_6
c_1										0	b_8	c_2	0	b_7
c_2											0	0	0	b_8
c_3												0	0	b_9
c_4													0	0
c_5														c_4

1.2. Isomorphisms within the family.

Lemma 1.1. *For any $\beta,\delta \in K$, we have $\mathscr{L}(\beta,\delta) \simeq \mathscr{L}(\beta,0)$.*

Proof. Let us consider a new basis

$$\{b_1', b_2, b_3', b_4', b_5, b_6, b_7', b_8, b_9', c_1', c_2, c_3, c_4, c_5, d'\}$$

of the algebra $\mathscr{L}(\beta,\delta)$, where

$$\begin{aligned}
b_1' &= b_1 + \sqrt{\delta} b_6 + \delta b_8,\\
b_3' &= b_3 + \sqrt{\delta} b_5,\\
b_4' &= b_4 + \delta c_2,\\
b_7' &= b_7 + \sqrt{\delta} c_3,\\
b_9' &= b_9 + \sqrt{\delta} c_2,\\
c_1' &= c_1 + \sqrt{\delta} c_4,\\
d' &= d + \sqrt{\delta} b_2.
\end{aligned}$$

It is straightforward to check that in this basis the multiplication table coincides with the multiplication table (4) with $\delta = 0$. □

Lemma 1.2. *For any $\beta \in K$, we have $\mathscr{L}(\beta,0) \simeq \mathscr{L}(0,0)$.*

Proof. Consider a new basis

$$\{b_1', b_2, b_3', b_4', b_5, b_6', b_7, b_8, b_9, c_1', c_2, c_3, c_4, c_5, d'\}$$

of the algebra $\mathscr{L}(\beta,0)$, where

$$\begin{aligned}
b_1' &= b_1 + \beta^2 b_9,\\
b_3' &= b_3 + \beta^2 b_8,\\
b_4' &= b_4 + \beta^2 c_3,\\
b_6' &= b_6 + \beta^2 c_2,\\
c_1' &= b_1 + \beta^2 c_5,\\
d' &= d + \beta^2 b_5.
\end{aligned}$$

It is straightforward to check that in this basis the multiplication table coincides with the multiplication table (4) with $\beta = \delta = 0$. □

An immediate corollary of these two lemmas is

Theorem 1.3. *For any $\beta,\delta \in K$, we have $\mathscr{L}(\beta,\delta) \simeq \mathscr{L}(0,0)$.*

Since the algebra $\mathscr{L} = \mathscr{L}(0,0)$ is isomorphic to the smallest 15-dimensional algebra in the family of simple Lie algebras constructed by Skryabin in [Sk, pp.691–692], we will refer to it in the rest of the paper as the *Skryabin algebra*. The rest of the paper is devoted to elucidation of some properties, and computation of some invariants of the Skryabin algebra.

1.3. **Note on cohomology and deformations.** As explained in [GZ, §5], the algebras $\mathscr{L}(\beta,\delta)$ are obtained as a linear deformation of the algebra (1) by a linear combination of three 2-cocycles; one of them has to enter the linear combination with nonzero coefficient (to ensure simplicity of the deformed algebra) which can be normalized to be equal to 1, and the other two enter with coefficients β and δ. In terms of the basis (3), these 2-cocycles are:

$$\begin{aligned}
b_1 \wedge b_4 &\mapsto c_5,\\
b_1 \wedge d &\mapsto c_2
\end{aligned} \tag{5}$$

and

$$\begin{aligned} b_1 \wedge b_7 &\mapsto c_4, \\ b_4 \wedge b_7 &\mapsto c_5, \\ b_1 \wedge c_1 &\mapsto c_5 \end{aligned} \tag{6}$$

(only nonzero values on all possible pairs of the basis elements are given).

Either direct calculations, or simple reasonings based on the relationship between cohomology of the graded algebra (1) and its filtered deformation $\mathscr{L}$ (the Massey bracket between all the involved 2-cocycles vanishes) show that the cocycles (5) and (6) are also cohomologically independent 2-cocycles of the algebra $\mathscr{L}$ (note in passing that the computer calculation in GAP shows that the dimension of the whole cohomology $\mathrm{H}^2(\mathscr{L},\mathscr{L})$ is equal to 13). Thus, each $\mathscr{L}(\beta,\delta)$ is a deformation of the algebra $\mathscr{L}$ by a nontrivial 2-cocycle, which turns out to be isomorphic to the original algebra $\mathscr{L}$ via ad-hoc isomorphism. Such deformations are dubbed in [BLLS] (see also [BGLLS, §3.1.2], and references therein) as *semitrivial*. According to Lemmas 1.1 and 1.2, the isomorphism between $\mathscr{L}(\beta,\delta)$ and $\mathscr{L}$ is polynomial (actually, quadratic) in β and $\sqrt{\delta}$; hence by [BLLS, Lemma 2.2], each of the cocycles (5) and (6) is cohomologous to a cocycle of the form $x \wedge y \mapsto [D(x),D(y)]$ for some derivation D of $\mathscr{L}$.

2. 2-ENVELOPE, DERIVATIONS, SANDWICH SUBALGEBRA

We continue to employ the basis $\{b_1,\dots,b_9,c_1,\dots,c_5,d\}$ of the Skryabin algebra $\mathscr{L}$ as given by (3) (referred as the *standard basis* in what follows). The multiplication table is given by (4), where $\beta=\delta=0$.

The Skryabin algebra is not a 2-algebra; its 2-envelope $\mathscr{L}_2$ has dimension 19, with additional basis elements $b_1^{[2]}, b_4^{[2]}, b_7^{[2]}, c_3^{[2]}$, the 2-map:

$$\begin{gathered} b_2^{[2]} = c_4, \quad b_3^{[2]} = b_3, \quad c_1^{[2]} = b_9, \quad d^{[2]} = b_4^{[2]}, \\ b_5^{[2]} = b_6^{[2]} = b_8^{[2]} = b_9^{[2]} = c_2^{[2]} = c_4^{[2]} = c_5^{[2]} = 0, \\ b_7^{[4]} = b_7^{[2]}, \quad b_1^{[4]} = b_4^{[4]} = c_3^{[4]} = 0, \end{gathered}$$

and the multiplication:

	$b_1^{[2]}$	$b_4^{[2]}$	$b_7^{[2]}$	$c_3^{[2]}$
b_1	0	0	0	b_8
b_2	b_1	0	0	0
b_3	0	0	0	0
b_4	0	0	b_4	c_2
b_5	b_4	0	b_5	0
b_6	0	0	b_6	0
b_7	0	b_1	0	0
b_8	b_7	b_2	0	0
b_9	0	b_3	0	0
c_1	0	b_4	c_1	0
c_2	c_1	b_5	c_2	0
c_3	d	b_6	c_3	0
c_4	b_3	0	0	0
c_5	b_6	0	c_5	0
d	0	0	d	c_5

	$b_4^{[2]}$	$b_7^{[2]}$	$c_3^{[2]}$
$b_1^{[2]}$	0	0	b_9
$b_4^{[2]}$		0	c_4
$b_7^{[2]}$			0

In what follows, we will employ the following standard notation for arbitrary elements of $\mathscr{L}$:

$$\lambda_1 b_1 + \cdots + \lambda_9 b_9 + \mu_1 c_1 + \cdots + \mu_5 c_5 + \eta d,$$

and of $\mathscr{L}_2$:

$$\lambda_1 b_1 + \cdots + \lambda_9 b_9 + \mu_1 c_1 + \cdots + \mu_5 c_5 + \eta d + \xi_1 b_1^{[2]} + \xi_4 b_4^{[2]} + \xi_7 b_7^{[2]} + \xi_3 c_3^{[2]}, \tag{7}$$

where $\lambda_1, \dots, \lambda_9, \mu_1, \dots, \mu_5, \eta, \xi_1, \xi_4, \xi_7, \xi_3 \in K$.

Proposition 2.1. *The derivation algebra of the Skryabin algebra coincides with its* 2*-envelope.*

Proof. This is amenable to straightforward, though tedious, calculations. A short computer-assisted proof can be utilized instead: we can check in GAP that the derivation algebra of the Skryabin algebra over GF(2) is 19-dimensional. Since derivation algebra does not change under field extensions, the same is true for the Skryabin algebra over an arbitrary field. For any Lie algebra, its 2-envelope is contained in the derivation algebra. But the 2-envelope of the Skryabin algebra is 19-dimensional, as specified above. Hence, the derivation algebra coincides with the 2-envelope.

Yet another, more conceptual, proof will be given below in §3.4; it utilizes a different realization of the Skryabin algebra. □

Recall that an element x of a Lie algebra L is called a *sandwich* if $(\mathrm{ad}\,x)^2 = 0$ and $[[L,x],[L,x]] = 0$. (It is well-known – and easy to see – that if the characteristic of the ground field is different from 2, then the second condition follows from the first one, but in characteristic 2 this is not true). The set of all sandwiches is multiplicatively closed; this implies that the *sandwich subalgebra*, i.e., the subalgebra of L generated by all sandwiches, is just the linear span of sandwiches.

It follows from the result first proved by Kostrikin and Zelmanov, that a Lie algebra generated by sandwiches is nilpotent. See [V-L1, §3.2] for details and further references.

More generally, we will call a derivation D of a Lie algebra L a *sandwich derivation*, if $D^2 = 0$ and $[D(L), D(L)] = 0$.

Lemma 2.2. *The set of elements $x \in \mathscr{L}_2$ such that*

$$[[\mathscr{L}, x], [\mathscr{L}, x]] = 0 \tag{8}$$

coincides with the linear span of elements c_2, c_4, c_5, $c_3^{[2]}$.

Proof. It is a matter of straightforward verification that any linear span of elements c_2, c_4, c_5, $c_3^{[2]}$ satisfies the condition (8).

Conversely, let x be an arbitrary element (7) of $\mathscr{L}_2$ satisfying this condition. We perform the following calculations:

- Collecting in the equality $[[x,b_8],[x,c_5]] = 0$ the terms containing c_1 and c_2, we get, respectively, $\xi_1^2 = 0$ and $\lambda_1^2 + \xi_1\xi_7 = 0$, whence $\xi_1 = 0$ and $\lambda_1 = 0$.
- Collecting in the equality $[[x,b_6],[x,c_4]] = 0$ the terms containing c_5, we get $\lambda_7^2 = 0$, whence $\lambda_7 = 0$.
- Collecting in the equality $[[x,b_1],[x,c_4]] = 0$ the terms containing b_8, we get $\mu_1^2 = 0$, whence $\mu_1 = 0$.
- Collecting in the equality $[[x,b_6],[x,c_2]] = 0$ the terms containing b_8, we get $\eta^2 = 0$, whence $\eta = 0$.
- Collecting in the equality $[[x,b_8],[x,c_3]] = 0$ the terms containing b_5 and c_2, we get, respectively, $\xi_4^2 = 0$ and $\lambda_4^2 + \lambda_3\xi_4 = 0$, whence $\xi_4 = 0$ and $\lambda_4 = 0$.
- Collecting in the equality $[[x,b_1],[x,b_8]] = 0$ the terms containing b_9 and c_4, we get, respectively, $\lambda_3^2 = 0$ and $\lambda_6^2 = 0$, whence $\lambda_3 = 0$ and $\lambda_6 = 0$.
- Collecting in the equality $[[x,b_4],[x,c_2]] = 0$ the terms containing c_4, we get $\xi_7^2 = 0$, whence $\xi_7 = 0$.
- Collecting in the equality $[[x,b_1],[x,b_6]] = 0$ the terms containing b_5 and c_2, we get, respectively, $\lambda_2^2 = 0$ and $\mu_3^2 = 0$, whence $\lambda_2 = 0$ and $\mu_3 = 0$.
- Collecting in the equality $[[x,b_1],[x,c_3]] = 0$ the terms containing c_2, we get $\lambda_9^2 = 0$, whence $\lambda_9 = 0$.

- Collecting in the equality $[[x,b_1],[x,b_4]]=0$ the terms containing c_5, we get $\lambda_8^2=0$, whence $\lambda_8=0$.
- Collecting in the equality $[[x,b_1],[x,b_7]]=0$ the terms containing c_4, we get $\lambda_5^2=0$, whence $\lambda_5=0$.

We are left with a linear combination of c_2, c_4, c_5, $c_3^{[2]}$. □

As an immediate corollary of this lemma, we have

Proposition 2.3.

(i) *The sandwich subalgebra of* $\mathscr{L}$ *is* 3*-dimensional abelian, linearly spanned by elements* c_2, c_4, c_5.

(ii) *The set of sandwich derivations of* $\mathscr{L}$ *forms a* 4*-dimensional abelian subalgebra of* $\mathscr{L}_2$ *spanned by the inner derivations* $\operatorname{ad} c_2$, $\operatorname{ad} c_4$, $\operatorname{ad} c_5$, *and by the outer derivation* $\operatorname{ad} c_3^{[2]}$.

Proof. (i) By Lemma 2.2, the sandwich subalgebra of $\mathscr{L}$ lies in the linear span of c_2, c_4, c_5. Since these elements pairwise commute, they span the 3-dimensional abelian subalgebra of $\mathscr{L}$, and

$$(\mu_1 c_2+\mu_4 c_4+\mu_5 c_5)^{[2]}=\mu_1^2 c_2^{[2]}+\mu_4^2 c_4^{[2]}+\mu_5^2 c_5^{[2]}=0,$$

so any element x in this 3-dimensional subalgebra satisfies the condition $(\operatorname{ad} x)^2=0$.

(ii) By Proposition 2.1, any derivation is an element of $\mathscr{L}_2$. Apply Lemma 2.2 and reason as above. □

Note that from Lemma 2.2 and Proposition 2.3 it follows that in the Skryabin algebra $\mathscr{L}$, the condition $[[\mathscr{L},x],[\mathscr{L},x]]=0$ implies $[[\mathscr{L},x],x]=0$. In general, this is, of course, not true: take, for example, any metabelian non-nilpotent Lie algebra.

3. Computations over GF(2), the absolute toral rank, thin decomposition

3.1. **Some numerology.** In this section we report on computations over GF(2), performed on the computer in GAP, [G]†. A brute-force search on the computer shows that in the 2-envelope of the Skryabin algebra $\mathscr{L}$ over GF(2) there are 384 toral elements, 6144 2-dimensional tori, 21504 3-dimensional tori, 26880 4-dimensional tori, and no 5-dimensional tori (no attempt was made to determine their conjugacy classes with respect to the automorphism group).

The centralizer in $\mathscr{L}$ of each of the 384 toral elements is 7-dimensional, and for 240 toral elements the centralizer is (central) simple. Among these 240 simple algebras, 48 have absolute toral rank 2, and 192 have absolute toral rank 3. As proved in [V-L2, Theorem 1] (and confirmed by computations in [E]), over GF(2) there exist two simple 7-dimensional Lie algebras. These algebras are identified as (forms) of the Zassenhaus algebra $W_1'(3)$, and the Hamiltonian algebra $H_2''\big((2,1),(1+x_1^{(3)}x_2)\,dx_1\wedge dx_2\big)$, denoted by us here simply as W and H, respectively (see [GG] and [GGA] for further info, including explicit multiplication tables of these algebras, and their identification with some other simple 7-dimensional Lie algebras from the literature). Both algebras have absolute toral rank 3 over an algebraically closed field, but over GF(2), the algebra W has absolute toral rank 2, while the algebra H has absolute toral rank 3; thus the absolute toral rank can be used to distinguish them as subalgebras of the Skryabin algebra in our computations.

3.2. **Some examples.** Let us exhibit explicitly one of the maximal tori, and one of the 7-dimensional simple subalgebras mentioned in the previous subsection.

Here is just one of the 4-dimensional tori, linearly spanned by the toral elements

$$b_1+b_3+b_1^{[2]},\quad b_2+b_3+c_4+b_7^{[2]},\quad b_4+b_6+b_4^{[2]}+b_7^{[2]},\quad b_8+b_9+c_1+c_3+b_7^{[2]}+c_3^{[2]}. \tag{9}$$

Now take the first toral element in this torus, $h=b_1+b_3+b_1^{[2]}$. Its centralizer $C_{\mathscr{L}}(h)$ has the basis

$$b_1+b_3,\quad b_2+c_3+c_4,\quad b_4+b_6,\quad b_5+b_6+c_5,\quad b_7+b_9,\quad c_1+c_3,\quad d.$$

†The GAP code is available at `https://web.osu.cz/~Zusmanovich/papers/15dim/`.

It is straightforward to check that in this basis the multiplication table of $C_{\mathscr{L}}(h)$ coincides with the multiplication table of the simple 7-dimensional Hamiltonian algebra H (see [GG, §3] or [GGA, §1]) via the identification

$$V_0 \leftrightarrow b_5 + b_6 + c_5, \quad V_1 \leftrightarrow c_1 + c_3, \quad E_1 \leftrightarrow b_7 + b_9, \quad E_0 \leftrightarrow b_2 + c_3 + c_4,$$
$$F_1 \leftrightarrow b_4 + b_6, \quad F_0 \leftrightarrow d, \quad G \leftrightarrow b_1 + b_3.$$

3.3. **The absolute toral rank.** The computer calculation in §3.1 shows that the absolute toral rank of the Skryabin algebra over GF(2) is equal to 4. However, a bit of additional work allows to establish this result over an arbitrary field.

Theorem 3.1. *The absolute toral rank of the Skryabin algebra is equal to* 4.

Proof. As we want to establish this result over an arbitrary field K, we will distinguish the Skryabin algebra $\overline{\mathscr{L}} = \mathscr{L} \otimes_{\mathsf{GF}(2)} K$ over K, and its GF(2)-form $\mathscr{L}$.

A direct computer verification shows that each of the 26880 4-dimensional tori T in $\mathscr{L}_2$ coincides with its normalizer, i.e., is a Cartan subalgebra of $\mathscr{L}_2$. Consequently, $\overline{T} = T \otimes_{\mathsf{GF}(2)} K$ is a Cartan subalgebra of $\overline{\mathscr{L}_2} = (\overline{\mathscr{L}})_2$. By [P, Theorem 2(ii)], $\overline{T}$ is a torus of the maximal possible dimension in $\overline{\mathscr{L}_2}$, and hence the absolute toral rank of $\overline{\mathscr{L}}$ is equal to 4.

Alternatively, a direct simple proof free from reference to the computer can be provided by just looking at one of the 4-dimensional tori, for example, at (9). Indeed, a direct calculation shows that (9) coincides with its normalizer in the 2-envelope of the Skryabin algebra $\overline{\mathscr{L}}$ over an arbitrary ground field K, and hence by the same reference to [P] is a torus of the maximal possible dimension. ☐

3.4. **Thin decomposition.** Let us consider the following situation. Assume that a Lie algebra L has dimension $2^n - 1$, the absolute toral rank of L is n, T is a torus in the 2-envelope of L of (the maximal) dimension n, and $T \cap L = 0$. Assume further that the roots of the action of T on L are exactly nonzero tuples in $\mathsf{GF}(2)^n$ (in particular, the centralizer of T in L is zero), and each root space is one-dimensional. Thus, the root space decomposition is of the form

$$L = \bigoplus_{\substack{\alpha \in \mathsf{GF}(2)^n \\ \alpha \neq (0,\dots,0)}} K e_\alpha \tag{10}$$

for some elements $e_\alpha \in L$. In particular, the multiplication table of L in the basis $\{e_\alpha \mid \alpha \in \mathsf{GF}(2)^n \backslash (0,\dots,0)\}$ has the following form: either $[e_\alpha, e_\beta] = e_{\alpha+\beta}$, or $[e_\alpha, e_\beta] = 0$.

In such a situation, we will call the decomposition (10) *thin*. We suggest that this is an important property of simple Lie algebras which should be taken into account in the classification efforts.

A direct computer verification shows that the Skryabin algebra admits a thin decomposition with respect to *each* of the 26880 4-dimensional tori in its 2-envelope. We will explicitly provide one of them, corresponding to the torus (9).

The corresponding generators of the one-dimensional root spaces are:

$$\begin{aligned}
e_{0001} &= b_2 + b_3 + b_4 + b_6 + c_4\\
e_{0010} &= b_2 + b_3 + c_1 + c_3 + c_4\\
e_{0011} &= b_2 + b_3 + b_4 + b_6 + c_1 + c_3 + c_4\\
e_{0100} &= b_1 + b_3 + b_7 + b_9 + d\\
e_{0101} &= b_7 + b_9 + d\\
e_{0110} &= b_1 + b_3 + b_5 + b_6 + c_5 + d\\
e_{0111} &= b_5 + b_6 + c_5\\
e_{1000} &= b_2 + b_3 + b_8 + b_9 + c_1 + c_2 + c_3\\
e_{1001} &= b_2 + b_3 + b_4 + b_5 + b_6\\
e_{1010} &= b_2 + b_3 + c_1 + c_2 + c_3\\
e_{1011} &= b_2 + b_3 + b_4 + b_5 + b_6 + c_1 + c_2 + c_3\\
e_{1100} &= b_1 + b_2 + b_3 + b_7 + b_8 + b_9 + c_2 + c_3 + d\\
e_{1101} &= b_5 + b_6 + b_7 + b_8 + b_9 + c_2 + c_3 + d\\
e_{1110} &= b_1 + b_2 + b_3 + b_5 + b_6 + c_2 + c_3 + d\\
e_{1111} &= b_5 + b_6
\end{aligned}$$

and the corresponding multiplication table of $\mathscr{L}$ reads:

	e_{0010}	e_{0011}	e_{0100}	e_{0101}	e_{0110}	e_{0111}	e_{1000}	e_{1001}	e_{1010}	e_{1011}	e_{1100}	e_{1101}	e_{1110}	e_{1111}
e_{0001}	e_{0011}	e_{0010}	e_{0101}	e_{0100}	0	0	e_{1001}	0	e_{1011}	e_{1010}	e_{1101}	e_{1100}	e_{1111}	0
e_{0010}		e_{0001}	e_{0110}	e_{0111}	e_{0100}	0	0	e_{1011}	0	e_{1001}	e_{1110}	e_{1111}	e_{1100}	0
e_{0011}			e_{0111}	e_{0110}	e_{0101}	0	e_{1011}	e_{1010}	e_{1001}	0	0	e_{1110}	e_{1101}	0
e_{0100}				0	e_{0010}	e_{0011}	e_{1100}	e_{1101}	e_{1110}	0	e_{1000}	e_{1001}	0	e_{1011}
e_{0101}					e_{0011}	e_{0010}	0	e_{1100}	e_{1111}	e_{1110}	0	e_{1000}	e_{1011}	e_{1010}
e_{0110}						e_{0001}	e_{1110}	e_{1111}	e_{1100}	e_{1101}	0	0	e_{1000}	e_{1001}
e_{0111}							0	0	0	0	e_{1011}	e_{1010}	e_{1001}	0
e_{1000}								e_{0001}	0	e_{0011}	e_{0100}	0	e_{0110}	0
e_{1001}									e_{0011}	e_{0010}	e_{0101}	e_{0100}	0	0
e_{1010}										e_{0001}	e_{0110}	e_{0111}	e_{0100}	0
e_{1011}											e_{0111}	e_{0110}	e_{0101}	0
e_{1100}												e_{0001}	e_{0010}	e_{0011}
e_{1101}													0	e_{0010}
e_{1110}														e_{0001}

Using this realization of the Skryabin algebra, it is possible to provide an alternative proof of the fact that its derivation algebra coincides with its 2-envelope, not utilizing computer or tedious calculations.

Another proof of Proposition 2.1. Consider the torus T, and the corresponding thin decomposition of $\mathscr{L}$ as above. We shall prove that the derivation algebra of $\mathscr{L}$ is isomorphic to the semidirect sum $T \ltimes \mathscr{L}$.

As follows, for example, from [F, Proposition 1.2], any derivation of $\mathscr{L}$ can be represented as a sum of a derivation D preserving the thin decomposition, and an inner derivation. We have $D(e_\alpha) = \lambda_\alpha e_\alpha$ for any nonzero $\alpha \in \mathsf{GF}(2)^4$, and some $\lambda_\alpha \in K$. Writing the derivation condition for each pair of basic elements e_α, e_β, we have

$$\lambda_{\alpha+\beta} = \lambda_\alpha + \lambda_\beta \quad \text{if} \quad [e_\alpha, e_\beta] \neq 0. \tag{11}$$

Let us denote by h_1, h_2, h_3, h_4 the respective basis elements of T (listed in (9)). Modifying D by the action of the toral element $\operatorname{ad}\left(\lambda_{0001}h_1 + \lambda_{0010}h_2 + \lambda_{0100}h_3 + \lambda_{1000}h_4\right)$, we may assume that

$$\lambda_{0001} = \lambda_{0010} = \lambda_{0100} = \lambda_{1000} = 0. \tag{12}$$

Now, looking at the multiplication table above, and using the conditions (11) and (12), it is easy to see that $\lambda_\alpha = 0$ for any nonzero $\alpha \in \mathsf{GF}(2)^4$, i.e., $D = 0$. □

Other examples of simple Lie algebras admitting a thin decomposition include the 7-dimensional algebra H (as shown in [GG]; but not the other 7-dimensional algebra $W = W_1(3)'$, see below), and certain 15-dimensional Hamiltonian algebras and algebras constructed by Eick, see §6.

Recall that the well-known Zassenhaus algebra $W_1(n)'$ of dimension $2^n - 1$ has a realization in the basis $\{ e_\alpha \mid \alpha \in \mathsf{GF}(2^n)^\times \}$ with multiplication $[e_\alpha, e_\beta] = (\alpha+\beta)e_{\alpha+\beta}$. The grading $W_1(n)' = \bigoplus_{\alpha \in GF(2^n)^\times} K e_\alpha$ has most of the properties of a thin decomposition: it is the root space decomposition with respect to the maximal n-dimensional torus $\langle e_0, e_0^{[2]}, \dots, e_0^{[2^{n-1}]} \rangle$, and the root spaces are one-dimensional. However, it fails to be a thin decomposition: the roots lie in $\mathsf{GF}(2^n)$, and not in $\mathsf{GF}(2)^n$.

4. The automorphism group

The goal of this section is to determine the automorphism group of the Skryabin algebra. First, we define three types of automorphisms – exponential automorphisms, certain explicitly defined three one-parameter families, and diagonal automorphisms, determine the group generated by them, and then prove that they exhaust the whole automorphism group.

4.1. **Exponential automorphisms.** If D is a sandwich derivation of a Lie algebra L, then $\exp(D) = 1 + D$ is an automorphism of L, called an *exponential automorphism*. Since $\exp(D)^2 = \exp(2D) = 1$, exponential automorphisms are of order 2, and generate a unipotent subgroup of exponent 2, denoted by $\mathrm{Exp}(L)$, of the automorphism group $\mathrm{Aut}(L)$. Note also that if $D_1, D_2 \in L$ are two commuting sandwich derivations, then the corresponding automorphisms also commute: $\exp(D_1) \circ \exp(D_2) = \exp(D_2) \circ \exp(D_1)$.

According to Proposition 2.3(ii), the group $\mathrm{Exp}(\mathscr{L})$ is 4-dimensional abelian, isomorphic to the additive group $K^4 = K \oplus K \oplus K \oplus K$. Let us write down its one-parameter generators explicitly (here and below we indicate only those basis elements on which the automorphism acts non-identically):

$$\exp(\mathrm{ad}\,\alpha c_2): \begin{array}{l} b_1 \mapsto b_1 + \alpha c_3 \\ b_3 \mapsto b_3 + \alpha c_2 \\ b_4 \mapsto b_4 + \alpha c_4 \\ b_7 \mapsto b_7 + \alpha c_5 \\ d \;\mapsto d + \alpha b_8 \end{array}$$

$$\exp(\mathrm{ad}\,\alpha c_4): \begin{array}{l} b_1 \mapsto b_1 + \alpha b_2 \\ b_4 \mapsto b_4 + \alpha b_5 \\ b_7 \mapsto b_7 + \alpha b_8 \\ c_1 \mapsto c_1 + \alpha c_2 \end{array}$$

$$\exp(\mathrm{ad}\,\alpha c_5): \begin{array}{l} b_1 \mapsto b_1 + \alpha b_5 \\ b_7 \mapsto b_7 + \alpha c_2 \\ d \;\mapsto d + \alpha c_4 \end{array}$$

$$\exp(\mathrm{ad}\,\alpha c_3^{[2]}): \begin{array}{l} b_1 \mapsto b_1 + \alpha b_8 \\ b_4 \mapsto b_4 + \alpha c_2 \\ d \;\mapsto d + \alpha c_5. \end{array}$$

Here $\alpha \in K$ is a parameter.

4.2. **On** $\exp(\operatorname{ad}\alpha c_4)$**.** For a moment, let us return to the realization of $\mathscr{L}$ as a filtered deformation of the semisimple Lie algebra (1). The exponential automorphisms $\exp(\operatorname{ad}\alpha c_4)$ are the only automorphisms of $\mathscr{L}$ which are "lifted" from automorphisms of the algebra (1). Using the results about automorphisms of the tensor product of a simple Lie algebra and a divided power algebra (see, for example, [W, §2.2]), it is not difficult to describe the automorphism group of (1) (roughly, those are automorphisms of $\mathsf{S}\otimes\mathcal{O}_1(2)$ invariant under the action of g and of ∂). Automorphisms which are preserved by the cocycles defining the deformation, are also automorphisms of $\mathscr{L}$. Among the automorphisms of (1), the only automorphisms satisfying this condition, are automorphisms acting on $\mathsf{S}\otimes\mathcal{O}_1(2)$ as $\phi(\alpha)\otimes\mathrm{id}_{\mathcal{O}_1(2)}$, where $\phi(\alpha)$ is an automorphism of S of the form

$$e\mapsto e+\alpha f,\quad f\mapsto f,\quad h\mapsto h,$$

and leaving $g\otimes\langle 1,x\rangle$ and ∂ invariant. In terms of the standard basis of $\mathscr{L}$, this is exactly $\exp(\operatorname{ad}\alpha c_4)$. In general, the automorphism group of the algebra (1) is much smaller than the automorphism group of its deformation $\mathscr{L}$, which shows that, generally, there is no strong relationship between automorphisms of a Lie algebra and of its deformation.

4.3. **Automorphisms** Φ**,** Ψ**, and** Θ**.** Consider the following three one-parameter families of linear maps on $\mathscr{L}$, depending on the parameter $\alpha\in K$:

$$\Phi(\alpha):\ \begin{array}{l} b_1\mapsto b_1+\alpha b_4\\ b_2\mapsto b_2+\alpha b_5\\ b_7\mapsto b_7+\alpha^2 b_2+\alpha^3 b_5+\alpha c_1\\ b_8\mapsto b_8+\alpha c_2\\ b_9\mapsto b_9+\alpha^2 c_4\\ c_1\mapsto c_1+\alpha^2 b_5\\ c_3\mapsto c_3+\alpha c_4\\ d\ \mapsto d+\alpha b_3+\alpha^2 b_6 \end{array}$$

$$\Psi(\alpha):\ \begin{array}{l} b_1\mapsto b_1+\alpha b_6+\alpha c_1+\alpha^2 c_4\\ b_2\mapsto b_2+\alpha c_2\\ b_3\mapsto b_3+\alpha b_5\\ b_4\mapsto b_4+\alpha b_2+\alpha^2 c_2\\ b_6\mapsto b_6+\alpha c_4\\ b_7\mapsto b_7+\alpha b_5+\alpha c_3\\ b_9\mapsto b_9+\alpha c_2+\alpha c_5\\ c_1\mapsto c_1+\alpha b_8+\alpha c_4\\ d\ \mapsto d+\alpha b_2+\alpha b_9+\alpha^2 c_2+\alpha^2 c_5 \end{array}$$

$$\Theta(\alpha):\ \begin{array}{l} b_1\mapsto b_1+\alpha b_7+\alpha^3 b_8+\alpha^2 b_9\\ b_2\mapsto b_2+\alpha b_8\\ b_3\mapsto b_3+\alpha^2 b_8\\ b_4\mapsto b_4+\alpha^2 b_5+\alpha c_1+\alpha^2 c_3+\alpha^3 c_5\\ b_5\mapsto b_5+\alpha c_2\\ b_6\mapsto b_6+\alpha^2 c_2+\alpha^2 c_5\\ c_1\mapsto c_1+\alpha^2 c_2+\alpha^2 c_5\\ c_3\mapsto c_3+\alpha c_5\\ d\ \mapsto d+\alpha^2 b_5+\alpha b_6+\alpha^3 c_2+\alpha^2 c_3 \end{array}$$

Direct calculations show that all of them are automorphisms of $\mathscr{L}$, and

$$\begin{aligned} \Phi(\alpha)\circ\Phi(\alpha') &= \Phi(\alpha+\alpha')\\ \Psi(\alpha)\circ\Psi(\alpha') &= \Psi(\alpha+\alpha')\circ\exp\left(\operatorname{ad}\alpha\alpha' c_3^{[2]}\right)\\ \Theta(\alpha)\circ\Theta(\alpha') &= \Theta(\alpha+\alpha')\circ\exp\left(\operatorname{ad}(\alpha^2\alpha'+\alpha\alpha'^2)c_3^{[2]}\right) \end{aligned} \tag{13}$$

for any $\alpha, \alpha' \in K$. In particular, the automorphisms $\Phi(\alpha)$ and $\Theta(\alpha)$ are of order 2, and the automorphisms $\Psi(\alpha)$ are of order 4.

4.4. **Diagonal automorphisms.** Let L be a Lie algebra with a basis B. We will call an automorphism of L *diagonal with respect to B* (or just *diagonal* if it is clear which basis B is meant), if it leaves invariant each one-dimensional subspace Kx, where $x \in B$.

Lemma 4.1. *Each diagonal automorphism of $\mathscr{L}$ with respect to the standard basis is of the form*

$$\Delta(\lambda):\quad \begin{array}{ll} b_1 \mapsto \lambda^{-2} b_1 & c_1 \mapsto \lambda c_1 \\ b_2 \mapsto \lambda^2 b_2 & c_2 \mapsto \lambda^5 c_2 \\ b_3 \mapsto b_3 & c_3 \mapsto \lambda^3 c_3 \\ b_4 \mapsto \lambda^{-1} b_4 & c_4 \mapsto \lambda^4 c_4 \\ b_5 \mapsto \lambda^3 b_5 & c_5 \mapsto \lambda^5 c_5 \\ b_6 \mapsto \lambda b_6 & d \mapsto \lambda^{-1} d \\ b_7 \mapsto b_7 & \\ b_8 \mapsto \lambda^4 b_8 & \\ b_9 \mapsto \lambda^2 b_9 & \end{array} \tag{14}$$

where $\lambda \in K^\times$.

Proof. Let $x \mapsto \alpha(x)x$, where x is an element in the standard basis, be a diagonal automorphism of $\mathscr{L}$. Denote $\alpha(b_4) = \lambda^{-1}$.

We perform the following calculations:

- $\alpha(b_3) = \alpha(b_1)\alpha(b_2)$, $\alpha(b_1) = \alpha(b_1)\alpha(b_3)$, and $\alpha(b_2) = \alpha(b_2)\alpha(b_3)$ imply $\alpha(b_2) = \alpha(b_1)^{-1}$ and $\alpha(b_3) = 1$.
- $\alpha(b_1) = \alpha(b_4)\alpha(d)$ and $\alpha(d) = \alpha(b_1)\alpha(c_1)$ imply $\alpha(c_1) = \lambda$.
- $\alpha(b_7) = \alpha(c_1)\alpha(d) = \alpha(b_1)\lambda^2$.
- $b_7^{[4]} = b_7^{[2]}$ implies $\alpha(b_7)^4 = \alpha(b_7)^2$, thus $\alpha(b_7)^2 = 1$ and $\alpha(b_7) = 1$, $\alpha(b_1) = \lambda^{-2}$, $\alpha(b_2) = \lambda^2$, and $\alpha(d) = \lambda^{-1}$.
- $\alpha(c_3) = \alpha(b_1)\alpha(c_2) = \alpha(c_2)\lambda^{-2}$.
- $\alpha(c_1) = \alpha(b_1)\alpha(c_3)$ implies $\lambda = \alpha(c_2)\lambda^{-4}$, thus $\alpha(c_2) = \lambda^5$, and $\alpha(c_3) = \lambda^3$.
- $\alpha(b_2) = \alpha(b_1)\alpha(c_4)$ implies $\alpha(c_4) = \lambda^4$.
- $\alpha(b_9) = \alpha(b_2)\alpha(b_7) = \lambda^2$.
- $\alpha(b_8) = \alpha(b_2)\alpha(b_9) = \lambda^4$.
- $\alpha(c_2) = \alpha(b_7)\alpha(c_5)$ implies $\alpha(c_5) = \lambda^5$.
- $\alpha(b_5) = \alpha(b_4)\alpha(c_4) = \lambda^3$.
- $\alpha(b_6) = \alpha(b_1)\alpha(b_5) = \lambda$.

□

Conversely, it is straightforward to verify that each map of the form (14) is an automorphism of $\mathscr{L}$. Therefore, the group of diagonal automorphisms $\mathrm{Diag}(\mathscr{L})$ is isomorphic to $K^\times$, the multiplicative group of K.

4.5. **Putting all these automorphisms together.** Denote by $\mathrm{Aut}_0(\mathscr{L})$ the group generated by all automorphisms of $\mathscr{L}$ we have defined so far: exponential, Φ, Ψ, Θ, and diagonal.

Direct calculations show that for each $\alpha, \gamma \in K$, any of $\Phi(\alpha)$, $\Psi(\alpha)$, and $\Theta(\alpha)$ commutes with any of $\exp(\mathrm{ad}\,\gamma c_2)$, $\exp(\mathrm{ad}\,\gamma c_4)$, $\exp(\mathrm{ad}\,\gamma c_5)$, and $\exp(\mathrm{ad}\,\gamma c_3^{[2]})$. Additionally,

$$\begin{aligned} \Psi(\gamma) \circ \Phi(\alpha) &= \Phi(\alpha) \circ \Psi(\gamma) \circ \exp\left(\mathrm{ad}\,\alpha\gamma c_4\right) \\ \Theta(\gamma) \circ \Phi(\alpha) &= \Phi(\alpha) \circ \Theta(\gamma) \circ \exp\left(\mathrm{ad}\,\alpha\gamma^2 c_2\right) \circ \exp\left(\mathrm{ad}\,\alpha^2\gamma c_4\right) \circ \exp\left(\mathrm{ad}\,\alpha\gamma^2 c_5\right) \circ \exp\left(\mathrm{ad}\,\alpha^2\gamma^2 c_3^{[2]}\right) \\ \Theta(\gamma) \circ \Psi(\alpha) &= \Psi(\alpha) \circ \Theta(\gamma) \circ \exp\left(\mathrm{ad}\,\alpha\gamma c_2\right) \circ \exp\left(\mathrm{ad}\,\alpha\gamma c_5\right). \end{aligned}$$

Together with (13) this implies that the group $\mathcal{N}$ generated by the exponential automorphisms, and automorphisms Φ, Ψ and Θ, is a 7-dimensional unipotent algebraic group. Further, taking into account that $\Delta(\lambda)^{-1} = \Delta(\lambda^{-1})$, we have:

$$
\begin{aligned}
\exp(\operatorname{ad} \alpha c_2)^{\Delta(\lambda)} &= \exp(\operatorname{ad} \lambda^{-5} \alpha c_2) \\
\exp(\operatorname{ad} \alpha c_4)^{\Delta(\lambda)} &= \exp(\operatorname{ad} \lambda^{-4} \alpha c_4) \\
\exp(\operatorname{ad} \alpha c_5)^{\Delta(\lambda)} &= \exp(\operatorname{ad} \lambda^{-5} \alpha c_5) \\
\exp(\operatorname{ad} \alpha c_3^{[2]})^{\Delta(\lambda)} &= \exp(\operatorname{ad} \lambda^{-6} \alpha c_3^{[2]}) \\
\Phi(\alpha)^{\Delta(\lambda)} &= \Phi(\lambda^{-1}\alpha) \\
\Psi(\alpha)^{\Delta(\lambda)} &= \Psi(\lambda^{-3}\alpha) \\
\Theta(\alpha)^{\Delta(\lambda)} &= \Theta(\lambda^{-2}\alpha).
\end{aligned}
\tag{15}
$$

(here the superscript denotes the group conjugation: $\varphi^{\psi} = \psi^{-1} \circ \varphi \circ \psi$).

Therefore, $\mathcal{N}$ is a normal subgroup in $\mathrm{Aut}_0(\mathcal{L})$, and $\mathrm{Aut}_0(\mathcal{L})$ is isomorphic to the semidirect product $K^\times \ltimes \mathcal{N}$, with the action of $K^\times$ on $\mathcal{N}$ defined by (15).

4.6. **Invariant subspaces.** Now we are going to prove that the automorphisms constructed in the previous subsections exhaust all automorphisms of $\mathcal{L}$. To this aim, we determine certain invariant subspaces in the Skryabin algebra.

Proposition 4.2. *The Skryabin algebra $\mathcal{L}$ has the following* $\mathrm{Aut}(\mathcal{L})$*-invariant subspaces:*

$$\langle c_2\rangle \quad \langle c_4\rangle \quad \langle c_5\rangle$$
$$\langle b_5, c_2\rangle \quad \langle b_8, c_2\rangle$$
$$V_4 = \langle b_2, b_5, b_8, c_2\rangle$$
$$V_5 = \langle b_8, c_2, c_3, c_4, c_5\rangle$$
$$V_6 = \langle b_2, b_3, b_5, b_8, c_2, c_4\rangle \quad V_6' = \langle b_5, b_6, b_8, c_2, c_4, c_5\rangle \quad V_6'' = \langle b_5, b_8, b_9, c_2, c_4, c_5\rangle$$
$$V_7 = \langle b_5, b_6, b_8, b_9, c_2, c_4, c_5\rangle \quad V_7' = \langle b_5, b_8, b_9, c_2, c_3, c_4, c_5\rangle$$
$$V_8 = \langle b_2, b_3, b_5, b_6, b_8, c_2, c_4, c_5\rangle \quad V_8' = \langle b_2, b_5, b_6, b_8, b_9, c_2, c_4, c_5\rangle \quad V_8'' = \langle b_2, b_5, b_6, b_8, c_2, c_3, c_4, c_5\rangle$$
$$V_9 = \langle b_2, b_3, b_5, b_6, b_8, b_9, c_2, c_4, c_5\rangle \quad V_9' = \langle b_2, b_5, b_6, b_8, b_9, c_2, c_3, c_4, c_5\rangle$$
$$V_{11} = \langle b_2, b_3, b_4, b_5, b_6, b_8, c_1, \dots, c_5\rangle \quad V_{11}' = \langle b_2, b_3, b_5, b_6, b_8, b_9, c_1, \dots, c_5\rangle$$
$$V_{11}'' = \langle b_2, b_3, b_5, b_6, b_8, b_9, c_2, c_3, c_4, c_5, d\rangle$$
$$V_{12} = \langle b_2, b_3, b_4, b_5, b_6, b_8, b_9, c_1, \dots, c_5\rangle$$

Proof. By Proposition 2.3(i), the sandwich subalgebra S of $\mathcal{L}$ coincides with $\langle c_2, c_4, c_5\rangle$. Starting from this, rewrite the specified subspaces in invariant terms:

- $\langle c_5\rangle = \{x \in S \mid \dim[\mathcal{L}, x] \le 3\}$;
- $\langle c_4, c_5\rangle$ is the subspace (actually, the abelian subalgebra) linearly spanned by elements $x \in S$ such that $\dim[\mathcal{L}, x] \le 4$;
- $\langle c_4\rangle = [\mathcal{L}, c_5] \cap \langle c_4, c_5\rangle$;
- $V_4 = [\mathcal{L}, c_4]$;
- $\langle c_2\rangle = S \cap V_4$;
- $V_5 = [\mathcal{L}, c_2]$;
- $\langle b_8, c_2\rangle = V_4 \cap V_5$;
- $V_7' = [\mathcal{L}, \langle b_8, c_2\rangle]$
- $V_9' = \{x \in \mathcal{L} \mid [x, S] = 0\}$;
- $\langle b_5, c_2\rangle = [\mathcal{L}, c_5] \cap V_4 \cap [V_9', V_9']$;
- $V_8' = \{x \in V_9' \mid x^{[2]} \in \mathcal{L}\}$;
- $V_8'' = [\mathcal{L}, \langle b_5, c_2\rangle]$;

- Note that V'_8 is a subalgebra of $\mathscr{L}$ with the center S. The subspace V_7 (actually, a subalgebra) coincides with the set of elements $x \in V'_8$ such that the induced adjoint map $\mathrm{ad}\, x : V'_8/S \to V'_8/S$ has rank ≤ 1;
- $V'_6 = V_7 \cap V''_8$;
- $V''_6 = V_7 \cap V'_7$;
- $V'_{11} = \{x \in \mathscr{L} \mid [x,S] \subseteq S\}$;
- $V''_{11} = C_{\mathscr{L}}(c_4)$;
- $V_9 = [V''_{11}, V''_{11}]$;
- $V_{12} = \{x \in \mathscr{L} \mid [x, V'_{11}] \subseteq V'_{11} + Kx\}$;
- $V_{11} = [V_{12}, V_{12}]$;
- $V_8 = V_9 \cap V_{11}$;
- V_6 is a linear span of elements $x^{[2]}$, where $x \in V_8$;

All these items are verified by straightforward computations. □

Many more $\mathrm{Aut}(\mathscr{L})$-invariant subspaces of $\mathscr{L}$ can be produced in a similar fashion, here we confine ourselves only to those which will be needed in the sequel.

4.7. No other automorphisms.

Theorem 4.3. *Assume that any quadratic equation with coefficients in the ground field K has a solution in K. Then* $\mathrm{Aut}(\mathscr{L}) = \mathrm{Aut}_0(\mathscr{L})$.

Proof. Let φ be an automorphism of $\mathscr{L}$. Our strategy is to consecutively "twist" φ by taking compositions with various automorphisms from $\mathrm{Aut}_0(\mathscr{L})$, and eventually arrive at the conclusion $\varphi = \mathrm{id}_{\mathscr{L}}$.

By Proposition 4.2, $\varphi(b_3)$ lies in V_6 and does not lie in $V_4 + \langle c_4 \rangle$, i.e., is of the form

$$\varphi(b_3) = \lambda_2 b_2 + \lambda_3 b_3 + \lambda_5 b_5 + \lambda_8 b_8 + \mu_2 c_2 + \mu_4 c_4,$$

where $\lambda_3 \neq 0$. As b_3 is toral, $\varphi(b_3)$ is toral, and the equality $\varphi(b_3) = \varphi(b_3)^{[2]}$ is equivalent to the following quadratic system:

$$\begin{aligned} \lambda_2 &= \lambda_2 \lambda_3 \\ \lambda_3 &= \lambda_3^2 \\ \lambda_5 &= \lambda_3 \lambda_5 \\ \lambda_8 &= \lambda_3 \lambda_8 \\ \mu_2 &= \lambda_3 \mu_2 \\ \mu_4 &= \lambda_2^2. \end{aligned}$$

Consequently,

$$\varphi(b_3) = \lambda_2 b_2 + b_3 + \lambda_5 b_5 + \lambda_8 b_8 + \mu_2 c_2 + \lambda_2^2 c_4. \tag{16}$$

Assume that $\lambda_2 \neq 0$. Applying to both sides of the equality (16) the automorphism

$$\exp\Big(\mathrm{ad}\,(\lambda_2^{-\frac{3}{2}}\lambda_5 + \lambda_2^{-\frac{5}{2}}\mu_2)c_2\Big) \circ \Theta(\alpha) \circ \Phi\big(\lambda_2^{-\frac{3}{2}}\lambda_5\big) \circ \Delta\big(\lambda_2^{\frac{1}{2}}\big),$$

where α satisfies the quadratic equation $\alpha^2 + \alpha + \lambda_2^{-2}\lambda_8 = 0$ (this is the only place were we need the assumption that any quadratic equation with coefficients in K has a solution), we may assume that

$$\varphi(b_3) = b_2 + b_3 + c_4.$$

The automorphism φ maps the subalgebra $C_{\mathscr{L}}(b_3)$ to the subalgebra $C_{\mathscr{L}}(b_2 + b_3 + c_4)$. We have

$$C_{\mathscr{L}}(b_3) = \langle b_3, b_6, b_9, c_3, c_4, c_5, d \rangle$$

and

$$C_{\mathscr{L}}(b_2 + b_3 + c_4) = \langle b_2 + b_3 + c_4, b_5 + b_6, b_8 + b_9, c_2 + c_3, c_4, c_5, d \rangle.$$

By Proposition 4.2, $\varphi(b_6)$ lies in V_6', so does not contain terms with $b_2+b_3+c_4$, b_8+b_9, c_2+c_3, and d. Similarly, $\varphi(b_9)$ lies in V_6'', so does not contain terms with $b_2+b_3+c_4$, b_5+b_6, c_2+c_3, and d. Therefore, we can write

$$\begin{aligned}
\varphi(b_6) &= \alpha_6(b_5+b_6)+\alpha_4 c_4+\alpha_5 c_5\\
\varphi(b_9) &= \beta_9(b_8+b_9)+\beta_4 c_4+\beta_5 c_5\\
\varphi(d) &= \gamma_3(b_2+b_3+c_4)+\gamma_6(b_5+b_6)+\gamma_9(b_8+b_9)+\delta_3(c_2+c_3)+\delta_4 c_4+\delta_5 c_5+\gamma d
\end{aligned}$$

for certain $\alpha_i,\beta_i,\gamma_i,\delta_i,\gamma \in K$. Then, the equalities $[\varphi(b_6),\varphi(d)] = \varphi(b_3)$ and $[\varphi(b_9),\varphi(d)] = \varphi(b_6)$ are equivalent to

$$\alpha_6\gamma = 1 \tag{17}$$
$$\alpha_6\delta_3+\alpha_5\gamma = 1 \tag{18}$$

and

$$\beta_9\gamma = \alpha_6 \tag{19}$$
$$\beta_5\gamma = \alpha_4 \tag{20}$$
$$\beta_9\delta_3 = \alpha_5 \tag{21}$$

respectively. It is easy to see that the system (17)–(21) is contradictory: for example, multiplying (21) by α_6, and taking into account (18), we get $\beta_9+\alpha_5\beta_9\gamma = \alpha_5\alpha_6$, which, in its turn, together with (19) gives $\beta_9 = 0$, hence $\alpha_5 = \alpha_6 = 0$, contradicting condition (18).

Therefore, $\lambda_2 = 0$. Applying to both sides of (16) the automorphism

$$\Psi(\lambda_5)\circ\exp\Big(\mathrm{ad}\big(\lambda_5\sqrt{\lambda_8}+\mu_2\big)c_2\Big)\circ\Theta\big(\sqrt{\lambda_8}\big),$$

we may assume that $\varphi(b_3) = b_3$. Consequently, the eigenspace $\langle b_1,b_2,b_4,b_5,b_7,b_8,c_1,c_2\rangle$ corresponding to the eigenvalue 1 of $\mathrm{ad}\, b_3$, is invariant under φ, and we may write

$$\varphi(b_1) = \lambda_1 b_1+\lambda_2 b_2+\lambda_4 b_4+\lambda_5 b_5+\lambda_7 b_7+\lambda_8 b_8+\mu_1 c_1+\mu_2 c_2.$$

Applying to both sides of this equality the automorphism

$$\begin{aligned}
&\exp\Big(\mathrm{ad}\big(\lambda_1^{\frac{1}{2}}\lambda_2\lambda_4+\lambda_1^{-\frac{3}{2}}\lambda_4^3\lambda_7+\lambda_1^{-\frac{1}{2}}\lambda_4^2\mu_1+\lambda_1^{\frac{3}{2}}\lambda_5\big)c_5\Big)\\
&\circ\exp\Big(\mathrm{ad}\big(\lambda_1\lambda_2\lambda_7+\lambda_1^{-1}\lambda_4\lambda_7^2+\lambda_1^2\lambda_8\big)c_3^{[2]}\Big)\\
&\circ\exp\Big(\mathrm{ad}\big(\lambda_1\lambda_2+\lambda_1^{-1}\lambda_4\lambda_7\big)c_4\Big)\\
&\circ\Phi\Big(\lambda_1^{-\frac{1}{2}}\lambda_4\Big)\\
&\circ\Delta\big(\lambda_1^{-\frac{1}{2}}\big),
\end{aligned}$$

we may assume that $\lambda_1 = 1$ and $\lambda_2 = \lambda_4 = \lambda_5 = \lambda_8 = 0$.

Then we have

$$0 = \varphi(b_1^{[4]}) = \varphi(b_1)^{[4]} = \mu_1^2 b_4^{[2]}+\lambda_7^4 b_7^{[2]}+\mu_2^2 c_3^{[2]}+\text{ (terms lying in } \mathscr{L}),$$

which implies $\lambda_7 = \mu_1 = \mu_2 = 0$, and $\varphi(b_1) = b_1$. Further:

- $\varphi(c_4)\in\langle c_4\rangle$ by Proposition 4.2;
- $\varphi(b_2) = [\varphi(b_1),\varphi(c_4)]\in[b_1,\langle c_4\rangle] = \langle b_2\rangle$;
- $\varphi(c_2)\in\langle c_2\rangle$ by Proposition 4.2;
- $\varphi(c_3) = [\varphi(b_1),\varphi(c_2)]\in[b_1,\langle c_2\rangle] = \langle c_3\rangle$;
- $\varphi(c_1) = [\varphi(b_1),\varphi(c_3)]\in[b_1,\langle c_3\rangle] = \langle c_1\rangle$;
- $\varphi(c_5)\in\langle c_5\rangle$ by Proposition 4.2;
- $\varphi(b_5) = [\varphi(b_1),\varphi(c_5)]\in[b_1,\langle c_5\rangle] = \langle b_5\rangle$;
- $\varphi(b_6) = [\varphi(b_1),\varphi(b_5)]\in[b_1,\langle b_5\rangle] = \langle b_6\rangle$;
- $\varphi(d) = [\varphi(b_1),\varphi(c_1)]\in[b_1,\langle c_1\rangle] = \langle d\rangle$;
- $\varphi(b_7) = [\varphi(c_1),\varphi(d)]\in[\langle c_1\rangle,\langle d\rangle] = \langle b_7\rangle$;
- $\varphi(b_4) = [\varphi(b_7),\varphi(d)]\in[\langle b_7\rangle,\langle d\rangle] = \langle b_4\rangle$;

- $\varphi(b_8) = [\varphi(c_2), \varphi(d)] \in [\langle c_2 \rangle, \langle d \rangle] = \langle b_8 \rangle$;
- $\varphi(b_9) = [\varphi(c_3), \varphi(d)] \in [\langle c_3 \rangle, \langle d \rangle] = \langle b_9 \rangle$.

Hence, φ is a diagonal automorphism, and by Lemma 4.1 we have $\varphi = \Delta(\lambda)$, where $\lambda^{-2} = 1$. But then $\lambda = 1$ and $\varphi = \mathrm{id}_{\mathscr{L}}$. □

5. Gradings

Having a supply of automorphisms of $\mathscr{L}$ at hand, and using the known correspondence between automorphisms and group gradings (formulated in full generality in the language of affine group schemes – see, for example, [EK, Proposition 1.36]), we may try to construct group gradings of $\mathscr{L}$.

In practice, this is achieved by extending the ground field K to a suitable commutative ring R (not necessary a field – we are dealing with the group scheme $R \to \mathrm{Aut}_R(\mathscr{L} \otimes_K R)$), extending an automorphism φ of $\mathscr{L}$ to the automorphism $\overline{\varphi}$ of $\overline{\mathscr{L}} = \mathscr{L} \otimes_K R$ via $\overline{\varphi}(x \otimes r) = \varphi(x) \otimes r$, and considering eigenspaces $\overline{\mathscr{L}}_\lambda = \{x \in \overline{\mathscr{L}} \,|\, \overline{\varphi}(x) = \lambda x\}$. For a suitable choice of φ and R, and a suitable homomorphism χ from the group generated by all eigenvalues $\lambda \in R$ to a group G, the eigenspaces are "rational": $\overline{\mathscr{L}}_\lambda = \mathscr{L}_{\chi(\lambda)} \otimes_K R$. The ensued grading $\mathscr{L} = \bigoplus_{\chi(\lambda)} \mathscr{L}_{\chi(\lambda)}$ is a grading of $\mathscr{L}$ by G, even in the case where the eigenvalues λ do not necessarily belong to the ground field.

For example, the diagonal automorphism $\Delta(\lambda)$ for the "generic" value of λ (or, which is the same, for $\lambda \in K$ such that the order of λ in the multiplicative group $K^\times$ is > 7) produces a $\mathbb{Z}$-grading

$$\mathscr{L} = \overset{-2}{\langle b_1 \rangle} \oplus \overset{-1}{\langle b_4, d \rangle} \oplus \overset{0}{\langle b_3, b_7 \rangle} \oplus \overset{1}{\langle b_6, c_1 \rangle} \oplus \overset{2}{\langle b_2, b_9 \rangle} \oplus \overset{3}{\langle b_5, c_3 \rangle} \oplus \overset{4}{\langle b_8, c_4 \rangle} \oplus \overset{5}{\langle c_2, c_5 \rangle}. \tag{22}$$

Specializing the automorphism $\Delta(\lambda)$ to the cases $\lambda^n = 1$, $n \le 7$, we obtain a $\mathbb{Z}/n\mathbb{Z}$-grading of $\mathscr{L}$.

As the product of any two elements from the standard basis is either zero, or again an element from the standard basis, the decomposition of $\mathscr{L}$ into one-dimensional subspaces spanned by the basis element is a grading. This is a group grading, and its universal group ([EK, §1.2]) is isomorphic to the (additive) abelian group with generators x, y (corresponding to elements b_1, b_4 respectively), and the relation $2x = 4y$, which is clearly isomorphic to the direct sum $\mathbb{Z} \oplus \mathbb{Z}/2\mathbb{Z}$. On the other hand, the thin decomposition exhibited in §3.4 is a $(\mathbb{Z}/2\mathbb{Z})^4$-grading.

In general, it seems to be a difficult task to classify all group gradings of the Skryabin algebra. Even to determine whether the automorphisms of order 2 (exponential, Φ, and Θ) lead to $\mathbb{Z}/2\mathbb{Z}$-gradings seems to be far from trivial. Perhaps, it could be approached with the method of [KL].

6. Comparison of the Skryabin algebra with algebras from the Eick list. A bit more numerology

In [E], a computer-generated list of simple Lie algebras over GF(2) of dimension ≤ 20 is presented. The Skryabin algebra (defined over GF(2)) is not isomorphic to any of the 15-dimensional algebras in the list.

One way to see this, using data from [E], is to look at automorphisms, either at the group of exponential automorphisms $\mathrm{Exp}(\mathscr{L})$, or at the whole group $\mathrm{Aut}(\mathscr{L})$. As the ground field is GF(2), $\mathrm{Exp}(\mathscr{L})$ is isomorphic to the additive group $\mathrm{GF}(2) \oplus \mathrm{GF}(2) \oplus \mathrm{GF}(2) \oplus \mathrm{GF}(2)$, thus having order 16, which is different from all the 15-dimensional algebras in the list (including the new ones, number 7 and 8, dubbed by us here as $Eick_7$ and $Eick_8$) except for the non-alternating Hamiltonian algebra $P(2,1,1)$ (number 4). The order of $\mathrm{Aut}(\mathscr{L})$ is $2^7 = 128$ (over GF(2), there are no nontrivial diagonal automorphisms), which is, again, different from all the 15-dimensional algebras in the list.

Another way to distinguish between all these algebras, is to repeat for them the same pedestrian, but informative computations concerning tori and the sandwich subalgebra, as in §3.1 and §2. The following table accumulates some information about the 15-dimensional central simple algebras from the Eick list, whose derivation algebra is 19-dimensional (and coincides in all the cases with the 2-envelope). Here TR denotes the absolute toral rank of the algebra, N_1 and N_m denote, respectively, the number of toral elements, and of tori of the maximal dimension TR in the 2-envelope, and S is the dimension of the sandwich subalgebra. We keep Eick's notation for algebras; in particular, $W(4)$ is, in a more common notation, the Zassenhaus algebra $W_1(4)'$.

algebra	TR	N_1	N_m	S
$W(4)$	2	256	1,536	10
$P(2,1,1)$	4	448	43,680	1
$P(3,1)$	3	384	10,752	5
$P(2,2)$	4	384	13,440	3
$Eick_7$	4	464	87,360	1
$Eick_8$	4	464	67,200	1

Interestingly enough, all of these algebras which are of absolute toral rank 4, i.e., $P(2,1,1)$, $P(2,2)$, $Eick_7$, and $Eick_8$, also admit thin decompositions with respect to a lot (conjecturally with respect to all) of the 4-dimensional tori.

Non-alternating Hamiltonian algebras in characteristic 2 were investigated in depth in the recent interesting preprints [KKC] and [K]. We believe that over GF(2) all 15-dimensional algebras from [K] are isomorphic to $P(3,1)$, but have not ventured into proving it.

7. Open questions

1. Is it true that two toral elements h and h' in $\mathscr{L}$ are conjugate with respect to the automorphism group, if and only if $C_{\mathscr{L}}(h) \simeq C_{\mathscr{L}}(h')$?

2. Describe all gradings of the Skryabin algebra.

3. *Conjecture*. Let L be a simple Lie algebra admitting a thin decomposition with respect to a torus T. Then the derivation algebra of L is isomorphic to the semidirect sum $T \ltimes L$.

The conjecture is true for the Skryabin algebra, as shown in §3.4. It is easy to see that the condition of simplicity is essential here: for example, the 3-dimensional nilpotent Lie algebra admits a thin decomposition, but not satisfies the conclusion of the conjecture.

4. *Conjecture*. Any simple Lie algebra of dimension > 3 over a field of characteristic 2 admitting a thin decomposition has:
a) a proper simple graded subalgebra (with respect to this decomposition);
b) a graded subalgebra isomorphic either to W or to H.

5. Classify simple finite-dimensional Lie algebras over an algebraically closed field of characteristic 2, admitting a thin decomposition.

6. Classify simple finite-dimensional $\mathbb{Z}$-graded Lie algebras over an algebraically closed field of characteristic 2, such that all homogeneous components are of dimension < 3. (Note that the Skryabin algebra belongs to this class, due to (22)).

Acknowledgements

Thanks are due to Bettina Eick, Dimitry Leites, Alexander Premet, and the anonymous referee for useful remarks. Alexander Grishkov was supported by FAPESP, grant number 2018/23690-6 and CNPq, grant number 307824/2016-0; Marina Rasskazova during the work on this paper was visiting the University of São Paulo, and was supported by FAPESP, grant number 2018/11292-6; Pasha Zusmanovich was supported by the Ministry of Education and Science of the Republic of Kazakhstan, grant number AP09259551.

References

[BGLLS] S. Bouarroudj, P. Grozman, A. Lebedev, D. Leites, I. Shchepochkina, *Simple vectorial Lie algebras in characteristic* 2 *and their superizations*, SIGMA **16** (2020), 089.

[BLLS] S. Bouarroudj, A. Lebedev, D. Leites, I. Shchepochkina, *Lie algebra deformations in characteristic* 2, Math. Res. Lett. **22** (2015), no.2, 353–402.

[E] B. Eick, *Some new simple Lie algebras in characteristic* 2, J. Symb. Comput. **45** (2010), no.9, 943–951.

[EK] A. Elduque, M. Kochetov, *Gradings on Simple Lie Algebras*, AMS, 2013.

[F] R. Farnsteiner, *Derivations and central extensions of finitely generated graded Lie algebras*, J. Algebra **118** (1988), no.1, 33–45.

[G] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.11.0, 2020; `https://www.gap-system.org/`

[GG] A. Grishkov, M. Guerreiro, *On simple Lie algebras of dimension seven over fields of characteristic* 2, São Paulo J. Math. Sci. **4** (2010), no.1, 93–107.

[GGA] A. Grishkov, M. Guerreiro, W.F. de Araujo, *On the classification of simple Lie algebras of dimension seven over fields of characteristic* 2, São Paulo J. Math. Sci. **14** (2020), no.2, 703–713.

[GZ] A. Grishkov, P. Zusmanovich, *Deformations of current Lie algebras. I. Small algebras in characteristic* 2, J. Algebra **473** (2017), 513–544.

[K] A.V. Kondrateva, *Non-alternating Hamiltonian Lie algebras in three variables*, arXiv:2101.00398.

[KKC] A.V. Kondrateva, M.I. Kuznetsov, N.G. Chebochko, *Non-alternating Hamiltonian Lie algebras in characteristic* 2. *I*, arXiv:1812.11213.

[KL] A. Krutov, A. Lebedev, *On gradings modulo 2 of simple Lie algebras in characteristic* 2, SIGMA **14** (2018), 130.

[P] A.A. Premet, *On Cartan subalgebras of Lie p-algebras*, Izv. Akad. Nauk SSSR Ser. Mat. **50** (1986), no.4, 788–800 (in Russian); Math. USSR Izvestija **29** (1987), no.1, 145–157 (English translation).

[Sk] S. Skryabin, *Toral rank one simple Lie algebras of low characteristics*, J. Algebra **200** (1998), no.2, 650–700.

[St] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic. Vol I. Structure Theory*, 2nd ed.; *Vol. II. Classifying the Absolute Toral Rank Two Case*, 2nd ed.; *Vol. III. Completion of the Classification*, De Gruyter, 2017, 2017, 2012.

[V-L1] M. Vaughan-Lee, *The Restricted Burnside Problem*, 2nd ed., Oxford Univ. Press, 1993.

[V-L2] M. Vaughan-Lee, *Simple Lie algebras of low dimension over* GF(2), LMS J. Comput. Math. **9** (2006), 174–192.

[W] B. Weisfeiler, *On the structure of the minimal ideal of some graded Lie algebras of characteristic $p > 0$*, J. Algebra **53** (1978), no.2, 344–361.

(Alexander Grishkov) University of São Paulo, Brazil and Omsk F.M. Dostoevsky State University, Russia
Email address: `shuragri@gmail.com`

(Henrique Guzzo Jr.) University of São Paulo, Brazil
Email address: `guzzo@ime.usp.br`

(Marina Rasskazova) Omsk State Technical University, Russia
Email address: `marinarasskazova@yandex.ru`

(Pasha Zusmanovich) University of Ostrava, Czech Republic
Email address: `pasha.zusmanovich@gmail.com`