SPECIAL AND EXCEPTIONAL MOCK-LIE ALGEBRAS

PASHA ZUSMANOVICH

ABSTRACT. We observe several facts and make conjectures about commutative algebras satisfying the Jacobi identity. The central question is which of those algebras admit a faithful representation (i.e., in Lie parlance, satisfy the Ado theorem, or, in Jordan parlance, are special).

INTRODUCTION

A while ago, a new class of algebras emerged in the literature – the so-called mock-Lie algebras. These are commutative algebras satisfying the Jacobi identity. These algebras are locally nilpotent, so there are no nontrivial simple objects. Nevertheless, they seem to have an interesting structure theory which gives rise to interesting questions. And, after all, it is always curios to play with a classical notion by modifying it here and there and see what will happen – in this case, to replace in Lie algebras anti-commutativity by commutativity.

In [AM], it was asked whether a finite-dimensional mock-Lie algebra admits a finite-dimensional faithful representation. In fact, an example providing a negative answer to this question was given a while ago – hidden in a somewhat obscure place, [HJS]. We provide an independent computer verification of that and similar examples, and examine several constructions and arguments for Lie algebras to see what works and what breaks in the mock-Lie case.

1. DEFINITIONS. PRELIMINARY FACTS AND OBSERVATIONS

The standing assumption throughout the paper is that the base field K is of characteristic $\neq 2, 3$. An algebra L over K, with multiplication denoted by $\circ$, is called *mock-Lie* if it is commutative:

$$x \circ y = y \circ x,$$

and satisfies the Jacobi identity:

$$(x \circ y) \circ z + (z \circ x) \circ y + (y \circ z) \circ x = 0$$

for any $x, y, z \in L$.

A substitution $x = y = z$ into the Jacobi identity yields $x^{\circ 3} = (x \circ x) \circ x = 0$. Conversely, linearizing the latter identity, we get back the Jacobi identity. Moreover, it is easy to see that assuming commutativity, the Jacobi identity is equivalent to the Jordan identity $(x^{\circ 2} \circ y) \circ x = x^{\circ 2} \circ (y \circ x)$ (see, for example, [BF, Lemma 2.2]). (On the other hand, commutative Leibniz and commutative Zinbiel algebras form a narrower class of mock-Lie algebras, namely, commutative algebras of nilpotency index 3: $(x \circ y) \circ z = 0$).

Thus, mock-Lie algebras can be characterized at least in the following four equivalent ways:

$$\frac{\text{commutative}}{\text{Jordan}} \text{ algebras } \frac{\text{satisfying the Jacobi identity}}{\text{of nil index 3}}.$$

This class of algebras appeared in the literature under different names, reflecting, perhaps, the fact that it was considered from different viewpoints by different communities, sometimes not aware of each other's results. Apparently, for the first time these algebras appeared in [Zhev, §5], where an example of infinite-dimensional solvable but not nilpotent mock-Lie algebra was given (reproduced in [ZSSS, §4.1, Example 1]); further examples can be found in [ZSSS, §4.1, Example 2 and §5.4, Exercise 4] and [W, §2.5]. In this and other Jordan-algebraic literature these algebras are called just "Jordan algebras of nil index 3". In [OK] they are called "Lie-Jordan algebras" (superalgebras are also considered there), and, finally, in the recent papers [BF] and [AM] the term "Jacobi–Jordan algebras" was used. The term "mock-Lie" comes from [GK, §3.9], where the corresponding operad appears in the list of quadratic cyclic operads with one generator. Despite that Getzler and Kapranov downplayed this class of algebras by (unjustly, in our opinion) calling them "pathological" (basing on the fact that the mock-Lie operad is not Koszul), we prefer to stick to the term "mock-Lie". These algebras live a dual life: as members of a very particular class of Jordan algebras, and as strange cousins of Lie algebras.

Finitely generated mock-Lie algebras are nilpotent and finite-dimensional (see remark at the very end of [Zhev]; in the class of finite-dimensional algebras, or, more generally, algebras with maximal condition for subalgebras, this follows from the more general result about nilpotency of such Jordan nil-algebras, [ZSSS, §4.4]). The nilpotency index of finite-dimensional mock-Lie algebras can be arbitrarily large (take, for example, finitely generated subalgebras of either of the algebras presented in [ZSSS, §4.1, Examples 1,2]), and is bounded by 3 for 1-generated algebras, by 5 for 2-generated algebras, by 9 for 3-generated algebras, and by $n+5$ for n-generated algebras, $n \geq 4$ ([GG, Main Theorem]). Any (not necessarily finite-dimensional) mock-Lie algebra L satisfies $(((L^2 \circ L^2) \circ L^2) \circ L^2) \circ L = 0$: in [HJPS, Lemma 3] this is established via heavy computer calculations, and in [Sv, Theorem 5] – via heavy manipulations by hand with nonassociative identities. The same identity is also established in [Jacobs, Theorem 1] via a straightforward computation using a computer algebra system Albert, though, strictly speaking, the latter cannot be counted as a rigorous (computer-assisted) proof, as the identity is being verified there only in the case of characteristic $p = 251$ of the base field. (Albert computations can be, however, turned into a rigorous proof by performing the same computation for different p's and using a variant of the Chinese remainder theorem; also, nowadays one can compute in characteristic zero directly, see §4 below). Mock-Lie algebras also satisfy the 3-Engel identity $((x \circ y) \circ y) \circ y = 0$ (see, for example, [HJPS, Eq. (**) at p. 3268]).

Most of the basic definitions from the Lie structure theory are carried over verbatim to the mock-Lie case. The center of a mock-Lie algebra L is an ideal consisting of elements z such that $z \circ x = 0$ for any $x \in L$. The members of the lower central series are defined inductively as $L^1 = L$, and $L^n = L^{n-1} \circ L$ for $n > 1$. The Jacobi identity implies that any monomial of length n, with arbitrary bracketing, lies in L^n. In particular, $L^k \circ L^{n-k} \subseteq L^n$ for any $0 < k < n$. Thus, a mock-Lie algebra L is nilpotent in any conceivable sense (general nonassociative, Jordan, etc.) if and only if $L^n = 0$ for some n.

Representations of mock-Lie algebras may be approached in a universal way which goes back to Eilenberg and is described nicely in [Jacobson, Chapter II, §5]: for an algebra A from a given variety, a vector space V is declared to carry a structure of A-bimodule, if the corresponding semidirect sum $A \ltimes V$ (with V carrying a trivial multiplication) belongs to the same variety. As mock-Lie algebras are commutative, the left and right actions of an algebra coincide, so we can speak about just modules. According to this approach, a vector space V is a module over a mock-Lie algebra L, if there is a linear map (a representation) $\rho : L \to \mathrm{End}(V)$ such that

$$(1) \qquad \rho(x \circ y)(v) = -\rho(x)(\rho(y)v) - \rho(y)(\rho(x)v)$$

for any $x, y \in L$ and $v \in V$. Representations of mock-Lie algebras are, essentially, the same as homomorphisms to associative algebras: if ρ is a representation, then the map $-2\rho : L \to \mathrm{End}(V)^{(+)}$ is a homomorphism of Jordan algebras, where the superscript $^{(+)}$ denotes, as usual, the passage from the underlying associative algebra to the Jordan algebra structure defined on the same vector space via

$$(2) \qquad a \circ b = \frac{1}{2}(ab + ba).$$

As any associative algebra A can be embedded into the algebra of all endomorphisms of a vector space, any homomorphism of Jordan algebras $L \to A^{(+)}$ can be prolonged to a homomorphism $L \to \mathrm{End}(V)^{(+)}$. Conversely, any such homomorphism, multiplied by $-\frac{1}{2}$, yields a map satisfying (1). As in the Lie case, a representation is called *faithful*, if its kernel is zero.

As finite-dimensional mock-Lie algebras are nilpotent, their irreducible representations are one-dimensional trivial. However, it seems to be interesting to try to study finite-dimensional indecomposable representations, and to try to classify finite-dimensional mock-Lie algebras into finite, tame, and wild representation types (such study for general Jordan algebras was initiated in [KOS]).

An entertaining fact (though not related to what follows): algebras over the operad Koszul dual to the mock-Lie operad can be characterized in three equivalent ways:

- anticommutative antiassociative algebras;
- anticommutative 2-Engel algebras;
- anticommutative alternative algebras.

Here by antiassociative algebras we mean, following [OK] and [MR], algebras satisfying the identity $(xy)z = -x(yz)$, and the 2-Engel identity is $(xy)y = 0$.

Another entertaining fact (noted, for example, in [OK]) is that mock-Lie algebras can be produced from antiassociative algebras the same way as they are produced from associative ones. Namely, given an antiassociative algebra A, the new algebra $A^{(+)}$ with multiplication given by "anticommutator" (2), is a mock-Lie algebra.

As the tensor product of algebras over Koszul dual operads form a Lie algebra under the usual commutator bracket, the following question seems to be natural.

Question 1. *Which "interesting" Lie algebras can be represented as the tensor product of a mock-Lie algebra and an anticommutative antiassociative algebra?*

As was noted, for example, in [GK, §3.9(d)], the mock-Lie operad is not Koszul, and hence does not admit a standard (co)homology theory (like, for example, Lie or associative algebras). This however, does not preclude that cohomology can be constructed in some ad-hoc, nonstandard, manner.

In the class of mock-Lie algebras, when extending an algebra L by a linear map $D: L \to L$ to a semidirect sum $L \ltimes KD$, a role analogous to derivations in the Lie case is played by antiderivations. Indeed, a necessary condition for such semidirect sum to be a mock-Lie algebra, is D to be an antiderivation of L. (This condition is not sufficient: in addition, we should impose conditions following from the Jacobi identity involving two and three D's; in particular, since every mock-Lie algebra is 3-Engel, we have $D^3 = 0$). Recall that an *antiderivation* of a mock-Lie algebra L is a linear map $D: L \to L$ such that $D(x \circ y) = -D(x) \circ y - x \circ D(y)$ for any $x, y \in L$. Antiderivations, and, more generally, δ-derivations (where -1 is replaced by an arbitrary $\delta \in K$) for various classes of algebras were studied in a number of papers (see, for example, [K] with a transitive closure of references therein).

More generally, given a linear map $D: L \to V$ from a mock-Lie algebra L into an L-module V (with the corresponding representation being denoted by ρ), the vector space $V \dotplus KD$ (here and below the symbol $\dotplus$ is reserved for the direct sum of vector spaces, while $\oplus$ denotes the direct sum of algebras) carries an L-module structure if and only if D is an antiderivation of L with values in V, i.e.,

$$D(x \circ y) = -\rho(y)D(x) - \rho(x)D(y)$$

for any $x, y \in L$. The space of all such antiderivations is denoted by $\mathrm{Der}_{-1}(L, V)$. The *inner* antiderivations are antiderivations of the form $x \mapsto \rho(x)v$ for a fixed $v \in V$.

Interpreting in the standard way outer antiderivations (i.e., the quotient of antiderivations by inner antiderivations) as the 1st degree cohomology, and writing deformations of modules over mock Lie algebras as power series (à la Gerstenhaber), what also gives us the idea how the 1st degree cohomology should look like, we get what might be the beginning of a complex responsible for cohomology of a mock-Lie algebra L with coefficients in an L-module V:

$$0 \to S^0(L, V) \xrightarrow{\mathrm{d}} S^1(L, V) \xrightarrow{\mathrm{d}} S^2(L, V),$$

where $S^n(L, V)$ is the vector space of symmetric multilinear n-ary maps $L \times \cdots \times L \to V$, and the formulas for the differential are:

(3)
$$\begin{aligned} \mathrm{d}\,\varphi(x) &= \rho(x)\varphi \\ \mathrm{d}\,\varphi(x, y) &= \varphi(xy) + \rho(x)\varphi(y) + \rho(y)\varphi(x). \end{aligned}$$

Similarly, writing a deformation of a mock-Lie algebra itself as a formal power series, we get the part of the complex responsible for the second cohomology:

$$S^1(L, V) \xrightarrow{\mathrm{d}} S^2(L, V) \xrightarrow{\mathrm{d}} S^3(L, V),$$

where

(4)
$$\begin{aligned} \mathrm{d}\,\varphi(x, y) &= \varphi(xy) - \rho(x)\varphi(y) - \rho(y)\varphi(x) \\ \mathrm{d}\,\varphi(x, y, z) &= \varphi(xy, z) + \varphi(zy, x) + \varphi(xz, y) + \rho(z)\varphi(x, y) + \rho(x)\varphi(z, y) + \rho(y)\varphi(x, z). \end{aligned}$$

However, a (naive) idea to modify the Chevalley–Eilenberg complex for Lie algebra cohomology by injecting appropriately some signs does not work: it is not clear how to mangle the two definitions above, for the 1st and 2nd degree cohomology, together (note the difference in signs in formulas for $\mathrm{d}\,\varphi(x, y)$ in (3) and (4)). We may try to pursue a more modest goal and to construct just cohomology with trivial coefficients; then all terms containing ρ in the formulas (3)–(4) vanish, and together they give cohomology up to degree 3; nevertheless, it is still not clear how to proceed in higher degrees. (The only sensible

cohomology can be constructed in this way in characteristic 2, what is an entirely different story and will be treated in a separate paper).

Note also that according to [GK], an analog of cyclic (co)homology of mock-Lie algebras may be constructed, but it is not clear what its utility might be.

Question 2. *Do mock-Lie algebras admit a "good" cohomology theory?*

Another natural question is whether a mock-Lie algebra L admits not a merely homomorphism to an associative algebra, but an embedding, or, what is equivalent, whether L admits a faithful representation. (It is well-known that not every Jordan algebra admits such an embedding; algebras, which do, are called *special*; algebras, which do not, are called *exceptional*). Note that since an analog of the universal enveloping algebra for finite-dimensional Jordan algebras is finite-dimensional, for finite-dimensional algebras this is equivalent to existence of a *finite-dimensional* faithful representation, i.e., the validity of the analog of the Ado theorem for Lie algebras.

A similar question about embedding of any mock-Lie algebra into an algebra of the form $A^{(+)}$ for an *anti*associative algebra A admits, as noted in [OK], a trivial negative answer. Indeed, it is easy to see that in any antiassociative algebra A, and hence in any mock-Lie algebra of the form $A^{(+)}$, a product of any 4 elements (under arbitrary bracketing) vanishes. In particular, any mock-Lie algebra which is embeddable into an algebra of the form $A^{(+)}$ for an antiassociative algebra A, is necessarily nilpotent of index 4.
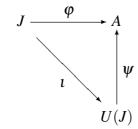
We discuss properties of universal enveloping algebras of mock-Lie algebras in §2; in particular, we see how and why the arguments used for Lie algebras (notably, the Poincaré–Birkhoff–Witt theorem) fail, and establish speciality of mock-Lie algebras of small dimension. In §3 we see how and why another chain of arguments, used in [Zu] to establish the Ado theorem not utilizing universal enveloping algebras, fails in the mock-Lie case. An example showing that, in general, mock-Lie algebras are not special, was given in [HJS]. That and similar examples are discussed in §4.

2. Universal enveloping algebras

For a Jordan algebra J, a natural analog $U(J)$ of the universal enveloping algebra is defined as the quotient of the tensor algebra $T(J)$ (= free associative algebra freely generated by a basis of J) by the ideal generated by elements of the form

$$(5) \qquad \frac{1}{2}(x \otimes y + y \otimes x) - x \circ y$$

for $x, y \in J$. The restriction to J of the natural homomorphism $T(J) \to U(J)$ yields a homomorphism of Jordan algebras $\iota : J \to U(J)^{(+)}$. The standard argument shows then that indeed, $U(J)$ is a universal object in the category of such homomorphisms, i.e., for any associative algebra A and homomorphism of Jordan algebras $\varphi : J \to A^{(+)}$, there is a unique homomorphism of associative algebras $\psi : U(J) \to A$ such that the diagram

$$
\begin{array}{ccc}
J & \xrightarrow{\ \varphi\ } & A \\
& {}_{\iota}\searrow & \ \uparrow{\scriptstyle \psi} \\
& & U(J)
\end{array}
$$

commutes. See [Jacobson, Chapter II, §1] for further details.

Specializing this construction to the mock-Lie case, one might expect that it will play a role similar to the universal enveloping algebra in the Lie case. For example, if L is *abelian*, i.e., with trivial multiplication, then $U(L) = \bigwedge(L)$, the exterior algebra on the vector space L. However, the major property of Lie-algebraic universal enveloping algebras – the Poincaré–Birkhoff–Witt theorem – fails miserably in the mock-Lie case. The shortest and, arguably, the most elegant proof of the Poincaré–Birkhoff–Witt theorem uses the Gröbner bases technique: one proves that the set of defining relations of the universal enveloping algebra of a Lie algebra is closed with respect to compositions and hence forms a Gröbner base, and then the result follows from the composition lemma (see, for example, [U, §2.6, Example 4]; we also follow the terminology adopted there). It is edifying to look at this proof, and see how it breaks in the mock-Lie case.

If L is a mock-Lie algebra with a basis $\{x_1, \dots, x_n\}$, $U(L)$ is an associative algebra with generators $\{x_i\}$ and relations

(6)
$$x_i x_j + x_j x_i - 2x_i \circ x_j, \quad i < j$$
$$x_i^2 - x_i \circ x_i.$$

If one assumes the lexicographic order on monomials induced by $x_1 > x_2 > \dots$, and tries to compute compositions between these relations, one arrives, after a series of simple compositions and reductions as in the Lie case, to expressions of the form

(7)
$$x_i(x_j \circ x_k) + x_j(x_i \circ x_k) + x_k(x_i \circ x_j).$$

Depending on the multiplication table of L, further reduction of this expression with respect to the defining relations may be possible. In general, however, there is no reason why the expression (7) should be further reduced to zero, and thus, unlike in the Lie case, the relations defining the universal enveloping algebra are not closed with respect to composition. Of course, this still does not prove that the monomials

$$x_{i_1} \dots x_{i_k}, \quad 1 \le i_1 < \dots < i_k \le n$$

cannot form a basis of $U(L)$, but this is a strong indication of the failure of the Poincaré–Birkhoff–Witt theorem.

Conjecture 1. *The mock-Lie algebras for which the Poincaré–Birkhoff–Witt theorem holds, are exactly the abelian algebras.*

For finite-dimensional algebras, the conjecture holds true.

Theorem 1 (I. Shestakov)**.** *The finite-dimensional mock-Lie algebras for which the Poincaré–Birkhoff–Witt theorem holds, are exactly abelian algebras.*

The validity of the Poincaré–Birkhoff–Witt theorem for a mock-Lie algebra L implies that $\dim U(L) = \dim \bigwedge(L) = 2^{\dim L}$. We will establish Theorem 1 by providing two different proofs of the fact that for a finite-dimensional nonabelian mock-Lie algebra L, $\dim U(L) < 2^{\dim L}$.

For the first proof, we need two simple lemmas.

Lemma 1. *Let L be a mock-Lie algebra, and I its ideal. Then $\dim U(L) \le \dim U(L/I) \cdot \dim U(I)$.*

Proof. Every element in $U(L)$ can be represented as the sum of products XY, where X is a monomial of the form $x_{i_1} \dots x_{i_k}$, x's belong to representatives of the cosets in L/I, and Y is a monomial of the form $y_{j_1} \dots y_{j_\ell}$, y's belong to I. The elements X are subject to the same relations (6) modulo I, and elements Y are subject to relations (6), with x's replaced by y's (of course, there could be additional relations in $U(L)$ between x's and y's, but we do not care here about them). Thus all X's linearly span $U(L/I)$, all Y's linearly span $U(I)$, and the statement of the lemma follows. $\square$

Lemma 2. *A nonabelian mock-Lie algebra such that all its proper ideals and all its proper quotients are abelian, is isomorphic to one of the following algebras:*
  (i) *2-dimensional algebra $\langle a, b \mid a^2 = b \rangle$;*
  (ii) *3-dimensional algebra $\langle a, b, c \mid ab = c \rangle$;*
 (iii) *3-dimensional algebra $\langle a, b, c \mid a^2 = c; ab = c \rangle$.*

In the multiplication tables between basic elements of algebras, we specify the nonzero products only. Mock-Lie algebras of low dimension over an algebraically closed field were described in [BF]. Algebra from heading (i) is denoted as A_{12} in this classification, and is the only nonabelian 2-dimensional algebra. Algebra from heading (ii), the "commutative Heisenberg algebra", is isomorphic to the algebra A_{13} (see [BF, Remark 3.3]), and algebra from heading (iii) is isomorphic to $A_{12} \oplus A_{01}$, the direct sum of algebra from heading (i) and one-dimensional abelian algebra.

Proof of Lemma 2. Let L be a mock-Lie algebra as specified in the condition of the lemma. Since L^2 is a proper ideal of L, it can be enlarged to an ideal I of codimension one, which is abelian. Write $L = I \dotplus Ka$ for some element $a \notin I$, and $a \circ x = f(x)$ for $x \in I$, where $f : I \to I$ is a linear map. The condition $a^{\circ 3} = 0$ implies $a^{\circ 2} \in \operatorname{Ker} f$, and the Jacobi identity for triple (a, a, x), where $x \in I$, implies $f^2 = 0$.

If V is a proper f-invariant subspace of I, then it is a proper ideal of L, and the quotient $L/V = I/V \dotplus Ka$ is abelian, i.e., $\operatorname{Im} f \subseteq V$ and $a^{\circ 2} \in V$. So f is a nilpotent of index ≤ 2 linear map such that its image lies

in any proper invariant subspace, what implies that either $f = 0$, or f is a map on a 2-dimensional vector space, represented by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the canonical basis. Moreover, $a^{\circ 2}$ lies in any proper f-invariant subspace, what implies that either $a^{\circ 2} = 0$, or f is the zero map on an one-dimensional vector space, or, again, f is a map on a 2-dimensional vector space, represented by the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ in the canonical basis. Combining all these possibilities, we get the claimed list of algebras. $\qquad\square$

First proof of Theorem 1. Let us prove by induction on dimension of L, that for a nonabelian mock-Lie algebra L, $\dim U(L) < 2^{\dim L}$. The only one-dimensional mock-Lie algebra is abelian, so the statement is vacuously true for $\dim L = 1$. The induction step: if L has either a nonabelian ideal, or nonabelian quotient, we are done by Lemma 1, otherwise L is one of the algebras described in Lemma 2, which can be dealt with directly (see also computer calculations below). $\qquad\square$

Now we turn to the second proof.

Lemma 3. *For any commutative algebra A, one of the following holds:*

  (i) *A is algebra with trivial multiplication;*
 (ii) *A is of the form $V \dotplus Ka$, where V is a vector space with trivial multiplication, $a \circ x = x$ for any $x \in V$, and $a^{\circ 2} = 2a$;*
(iii) *A contains an element x such that x and $x^{\circ 2}$ are linearly independent.*

Note that the algebra A here is not assumed to be mock-Lie, or Jordan, or to satisfy any other distinguished identity beyond commutativity. We continue to denote the binary multiplication by $\circ$.

Proof. Suppose that for any element x of the algebra A, x and $x^{\circ 2}$ are linearly dependent, i.e., for any $x \in A$, there is $\lambda(x) \in K$ such that

$$(8) \qquad\qquad x^{\circ 2} = \lambda(x)x.$$

Linearizing this equality, and using bilinearity and commutativity of $\circ$, we get that there exists a linear map $\alpha : A \to K$ such that $x \circ y = \alpha(y)x + \alpha(x)y$ for any $x, y \in A$. If α is the zero map, the multiplication in A is trivial. Otherwise, set $V = \operatorname{Ker}\alpha$. Since V is of codimension one in A, we may write $A = V \dotplus Ka$, and normalize α by assuming $\alpha(a) = 1$. The rest is obvious. $\qquad\square$

Algebras with the condition that any element satisfies a cubic polynomial, or, in other words, algebras of rank ≤ 3 (of which the condition to be of nil index 3 is the particular case) were studied in a number of papers – see, for example [W] and references therein. The condition (8) is the quadratic particular case, i.e., it defines algebras of rank ≤ 2 (without the free term in the defining polynomial equation), and headings (i) and (ii) of Lemma 3 give an easy description of such commutative algebras. Note that algebra in (ii) is Jordan, but not mock-Lie.

Second proof of Theorem 1. Obviously, for abelian mock-Lie algebras the Poincaré–Birkhoff–Witt theorem holds. Conversely, suppose L is a nonabelian mock-Lie algebra of dimension n. By Lemma 3, there is $x_2 \in L$ such that x_2 and $x_1 = x_2^{\circ 2}$ are linearly independent. Complete x_1, x_2 to a basis $\{x_1, x_2, x_3, \ldots, x_n\}$ of L. Since in $U(L)$ holds $x_1 = x_2^2$, $U(L)$ is generated, as an algebra with unit, by $n - 1$ elements $\{x_2, x_3, \ldots, x_n\}$, and hence is linearly spanned by monomials of the form $x_2^k x_{i_1} \ldots x_{i_\ell}$, where $0 \leq k \leq 2$, $3 \leq i_1 < \cdots < i_\ell \leq n$; consequently, $\dim U(L) \leq 3 \cdot 2^{n-2} < 2^n$. $\qquad\square$

In fact, the second proof "almost" establishes Theorem 1 in the broader class of all Jordan algebras: the finite-dimensional Jordan algebras for which the Poincaré–Birkhoff-Witt theorem holds, are exactly algebras with trivial multiplication, and algebras specified in Lemma 3(ii).

Direct calculations, performed with the aid of GAP [GA] and GAP package GBNP [GB], show that, as a rule, the dimension of $U(L)$ is much smaller than $2^{\dim L}$. In the table below, the left column indicates the name of a mock-Lie algebra, and the right column the dimension of its universal enveloping algebra. The table contains all nonabelian algebras of dimension ≤ 5, and all nonassociative algebras of dimension 6. We follow the nomenclature of [BF], with the exception of $\mathfrak{L}$ which denotes the unique, over an algebraically closed field, 5-dimensional nonassociative mock-Lie algebra (see [BF, Proposition 4.1]). In the family of 6-dimensional algebras $A_{26}(\beta, 0)$ and $A_{26}(\beta, 1)$, the parameter β assumes values $0, 1$.

dimension 2	
A_{12}	3

dimension 3	
$A_{12} \oplus A_{01}$	5
A_{13}	5

dimension 4	
A_{14}	7
A_{24}	9
$A_{12} \oplus A_{01} \oplus A_{01}$	9
$A_{13} \oplus A_{01}$	9
$A_{12} \oplus A_{12}$	6

dimension 5	
A_{15}	9
A_{25}	10
A_{35}	10
A_{45}	10
A_{55}	10
A_{65}	11
A_{75}	17
$A_{12} \oplus A_{01} \oplus A_{01} \oplus A_{01}$	17
$A_{13} \oplus A_{01} \oplus A_{01}$	17
$A_{24} \oplus A_{01}$	17
$A_{12} \oplus A_{12} \oplus A_{01}$	10
$A_{12} \oplus A_{13}$	10
$A_{14} \oplus A_{01}$	11
$\mathfrak{L}$	9

dimension 6	
A_{16}	11
$A_{26}(\beta, 0)$	13
$A_{26}(\beta, 1)$	12

Note also that, unlike in the Lie case, $U(L_1 \oplus L_2)$ is not isomorphic to (in fact, in most of the cases much smaller than) $U(L_1) \otimes U(L_2)$.

Theorem 2. *For any mock-Lie algebra L, the kernel of the map $\imath : L \to U(L)$ lies in L^4.*

Proof. Suppose that $z \in \operatorname{Ker} \imath$, i.e., z belongs to the ideal of $T(L)$ generated by elements of the form (5). We may write

$$(9) \qquad z = \sum_{i \geq 0} \sum_{j \geq 0} \sum_{k \in \mathbb{I}_{ij}} a_i^{(k)} \otimes \left(\frac{1}{2}(x_{ij}^{(k)} \otimes y_{ij}^{(k)} + y_{ij}^{(k)} \otimes x_{ij}^{(k)}) - x_{ij}^{(k)} \circ y_{ij}^{(k)} \right) \otimes b_j^{(k)},$$

where $a_i^{(k)}, b_i^{(k)}$ are homogeneous elements of $T(L)$ of degree i, and $x_{ij}^{(k)}, y_{ij}^{(k)} \in L$ (we identify elements $\lambda \otimes x$, $x \otimes \lambda$, and λx for $\lambda \in T^0(L) = K$ and $x \in T(L)$). By modifying x's and y's appropriately, we may normalize the degree 0 elements $a_0^{(k)}$ and $b_0^{(k)}$ by equating them to 1. Moreover, by expanding the terms in further sums, if necessary, we may assume that a's and b's are monomials, i.e., $a_i^{(k)} = a_i^{(k1)} \otimes \cdots \otimes a_i^{(ki)}$, where $a_i^{(k*)} \in L$, and similarly for b's.

Isolating in (9) the homogeneous components, we get

$$(10) \qquad z = -\sum_{k \in \mathbb{I}_{00}} x_{00}^{(k)} \circ y_{00}^{(k)}$$

in degree 1 (i.e., for elements lying in L), and

$$(11) \qquad \sum_{i+j=n-2} \sum_{k \in \mathbb{I}_{ij}} a_i^{(k)} \otimes \frac{1}{2}(x_{ij}^{(k)} \otimes y_{ij}^{(k)} + y_{ij}^{(k)} \otimes x_{ij}^{(k)}) \otimes b_j^{(k)} = \sum_{i+j=n-1} \sum_{k \in \mathbb{I}_{ij}} a_i^{(k)} \otimes (x_{ij}^{(k)} \circ y_{ij}^{(k)}) \otimes b_j^{(k)}$$

in degree $n > 1$ (i.e., for elements lying in $L^{\otimes n}$). The equality (10) implies $z \in L^2$.

Applying to both sides of the equality (11) for $n = 2$ the multiplication map $\circ : L \otimes L \to L$, and using commutativity of L, we get

$$(12) \qquad \sum_{k \in \mathbb{I}_{00}} x_{00}^{(k)} \circ y_{00}^{(k)} = \sum_{k \in \mathbb{I}_{10}} (x_{10}^{(k)} \circ y_{10}^{(k)}) \circ a_1^{(k)} + \sum_{k \in \mathbb{I}_{01}} (x_{01}^{(k)} \circ y_{01}^{(k)}) \circ b_1^{(k)},$$

what, together with (10), implies $z \in L^3$.

Applying to both sides of the equality (11) for $n = 3$ the map $L \otimes L \otimes L \to L$, $a \otimes b \otimes c \mapsto (a \circ c) \circ b$, and taking into account commutativity and the Jacobi identity, we get

$$\sum_{k \in \mathbb{I}_{10}} (x_{10}^{(k)} \circ y_{10}^{(k)}) \circ a_1^{(k)} + \sum_{k \in \mathbb{I}_{01}} (x_{01}^{(k)} \circ y_{01}^{(k)}) \circ b_1^{(k)} =$$

$$-2 \left(\sum_{k \in \mathbb{I}_{02}} ((x_{02}^{(k)} \circ y_{02}^{(k)}) \circ b_2^{(k2)}) \circ b_2^{(k1)} + \sum_{k \in \mathbb{I}_{11}} (x_{11}^{(k)} \circ y_{11}^{(k)}) \circ (a_1^{(k)} \circ b_1^{(k)}) + \sum_{k \in \mathbb{I}_{20}} ((x_{20}^{(k)} \circ y_{20}^{(k)}) \circ a_2^{(k1)}) \circ a_2^{(k2)} \right),$$

what, together with (10) and (12), implies $z \in L^4$. $\qquad\square$

One may try to continue the same way, considering the equality (11) in higher degrees, and applying maps on the tensor powers of L with different bracketings and permutations. For example, in the case $n = 4$, applying to both sides of (11) the map $L \otimes L \otimes L \otimes L \to L$, $a \otimes b \otimes c \otimes d \mapsto ((a \circ b) \circ c) \circ d$, and using the Jacobi identity, we get:

$$\sum_{k \in \mathbb{I}_{02}} ((x_{02}^{(k)} \circ y_{02}^{(k)}) \circ b_2^{(k1)}) \circ b_2^{(k2)} - \frac{1}{2} \sum_{k \in \mathbb{I}_{11}} ((x_{11}^{(k)} \circ y_{11}^{(k)}) \circ a_1^{(k)}) \circ b_1^{(k)} - \frac{1}{2} \sum_{k \in \mathbb{I}_{20}} (x_{20}^{(k)} \circ y_{20}^{(k)}) \circ (a_2^{(k1)} \circ a_2^{(k2)})$$

$$= \sum_{k \in \mathbb{I}_{03}} (((x_{03}^{(k)} \circ y_{03}^{(k)}) \circ b_3^{(k1)}) \circ b_3^{(k2)}) \circ b_3^{(k3)} + \sum_{k \in \mathbb{I}_{12}} ((a_1^{(k)} \circ (x_{12}^{(k)} \circ y_{12}^{(k)})) \circ b_2^{(k1)}) \circ b_2^{(k2)}$$

$$+ \sum_{k \in \mathbb{I}_{21}} ((a_2^{(k1)} \circ a_2^{(k2)}) \circ (x_{21}^{(k)} \circ y_{21}^{(k)})) \circ b_1^{(k)} + \sum_{k \in \mathbb{I}_{30}} ((a_3^{(k1)} \circ a_3^{(k2)}) \circ a_3^{(k3)}) \circ (x_{30}^{(k)} \circ y_{30}^{(k)}).$$

But the complexity of such manipulations explodes and it is not clear how exactly to combine them to proceed. Moreover, since there exist finite-dimensional (and hence nilpotent) exceptional mock-Lie algebras (§4), it is impossible that $z \in L^n$ for any n, so this process should stop somewhere (actually, at $n \le 8$, as there is an exceptional mock-Lie algebra whose nilpotency index is 9).

Proposition. *Any mock-Lie algebra of dimension ≤ 6 is special.*

First proof. According to [BF, Propositions 4.1 and 5.1], any nonassociative mock-Lie algebra L of dimension ≤ 6 is isomorphic to one of the following algebras: $\mathfrak{L}$ (of dimension 5), A_{16}, or $A_{26}(\beta, \delta)$ (of dimension 6). Using the same GAP/GBNP computations, we may check that Gröbner basis of the universal enveloping algebra of these algebras does not contain elements of degree 1 (i.e., elements of L), hence the defining ideal of $U(L)$ does not contain such elements, and L embeds into $U(L)$. $\qquad\square$

Second proof. For any nonassociative mock-Lie algebra L of dimension ≤ 6, we have $L^4 = 0$, hence according to Theorem 2, L embeds into $U(L)$. $\qquad\square$

Third proof. Follows from the result of Slin'ko [Sl, Theorem 2] that a nilpotent Jordan algebra of nilpotency index ≤ 5 is special. $\qquad\square$

3. No alternative route to Ado

In [Zu], an alternative proof of the Ado theorem for nilpotent Lie algebras was given, not utilizing universal enveloping algebras, but working entirely inside the category of finite-dimensional Lie algebras. Our initial attempt was to modify this proof for the mock-Lie case. This approach, however, meets several obstacles, some of them seems difficult to surmount (and was doomed to failure anyway due to existence of exceptional mock-Lie algebras, see the next §4). We think it is instructive to examine these obstacles, in order to better understand the peculiarities of the mock-Lie case.

The proof in [Zu] goes as follows. First it is noted that an $\mathbb{N}_{<n}$-graded (i.e., $\mathbb{N}$-graded with nonzero components concentrated in degrees $1, 2, \ldots, n-1$) Lie algebra L can be embedded into its tensor product extension $L \otimes tK[t]/(t^n)$. This is carried over verbatim to the mock-Lie case.

The associative commutative algebra $tK[t]/(t^n)$ possess a nondegenerate derivation $t\frac{d}{dt}$. This allows to construct a faithful representation of the Lie algebra $L \otimes tK[t]/(t^n)$ (and hence of its subalgebra L) of the form $\left(L \otimes tK[t]/(t^n)\right) \ltimes \left(\mathrm{id}_L \otimes t\frac{d}{dt}\right)$. Completely similar with [Zu, Lemma 1.3], we have an elementary

Lemma 4. *Let L be a mock-Lie algebra, V an L-module, and D an antiderivation of L with values in V such that $\mathrm{Ker}\, D = 0$. Then L has a faithful representation.*

Proof. The required representation ρ is given by the action of L on $V \dotplus \mathrm{Der}_{-1}(L, V)$, defined naturally on the first direct summand, and via $\rho(x)(d) = d(x)$ for $x \in L$ and $d \in \mathrm{Der}_{-1}(L, V)$, on the second direct summand. $\qquad\square$

Now comes the first obstacle: unlike in the case of derivations, the algebra $tK[t]/(t^n)$ possess nondegenerate antiderivations if and only if $n \le 4$. More precisely, we have:

Lemma 5. *The space of antiderivations of the algebra $tK[t]/(t^n)$ ($n \ge 2$) is 1-dimensional for $n = 2$, 2-dimensional for $n = 3$, and 3-dimensional for $n \ge 4$. Its basis can be chosen among antiderivations of the following form (only nonzero actions on the standard basis $\{t, t^2, \ldots, t^{n-1}\}$ of $tK[t]/(t^n)$ are given):*

  (i) $t \mapsto t^{n-1}$;
  (ii) $t \mapsto -\frac{1}{2}t^{n-2}, \quad t^2 \mapsto t^{n-1}$ *(if $n \geq 3$)*;
  (iii) $t \mapsto t^{n-3}, \quad t^2 \mapsto -2t^{n-2}, \quad t^3 \mapsto t^{n-1}$ *(if $n \geq 4$)*.

Proof. Straightforward computation using induction on the degree of monomials. $\square$

Thus we have only a very limited analog of [Zu, Lemma 2.5], with $\mathbb{N}_{<4}$-gradings instead of arbitrary $\mathbb{N}$-gradings:

Lemma 6. *An $\mathbb{N}_{<4}$-graded mock-Lie algebra has a faithful representation.*

Proof. By above, an $\mathbb{N}_{<4}$-graded mock-Lie algebra L is embedded into $L \otimes tK[t]/(t^n)$, $n \leq 4$. By Lemma 5, $tK[t]/(t^n)$ has a nondegenerate antiderivation D, and hence $L \otimes tK[t]/(t^n)$ has a nondegenerate antiderivation $\mathrm{id}_L \otimes D$. Applying Lemma 4 to the adjoint module $L \otimes tK[t]/(t^n)$ and the antiderivation $\mathrm{id}_L \otimes D$, we get that $L \otimes tK[t]/(t^n)$ has a faithful representation, and hence so does its subalgebra L. $\square$

Note that, unlike in the Lie case, the semidirect sum $\left(L \otimes tK[t]/(t^n)\right) \ltimes KD$ is not a mock-Lie algebra (for example, it is not locally nilpotent). We merely consider it as an $L \otimes tK[t]/(t^n)$-module, ignoring its algebra structure.

The proof in [Zu] then proceeds by induction on $\dim I$ in the representation of an arbitrary nilpotent Lie algebra as a quotient F/I of a free nilpotent Lie algebra F. The key ingredient in this induction is an auxiliary result about possibility to distinguish elements of a Lie algebra by the kernel of a suitable representation ([Zu, Lemma 2.10]), which is proved using some combinatorics related to the tensor product of representations. And here comes another obstacle: in general, there is no notion of the tensor product of two representations of a mock-Lie algebra. Perhaps, one may try to work around it by defining an ad-hoc bialgebra structure on mock-Lie algebras in question using central elements, similarly how it is done in [Zhel, Propositions 1 and 2] in some particular cases of Jordan algebras, and get in this way that any mock-Lie algebra of nilpotency index ≤ 4 has a faithful representation, and hence is special. This result, however, would be covered by Theorem 2, or, more generally, by the above-cited Slin'ko's result about speciality of nilpotent Jordan algebras of nilpotency index ≤ 5.

4. EXCEPTIONAL MOCK-LIE ALGEBRAS

In [HJS], a 44-dimensional mock-Lie algebra was presented, on which the Glennie identity $G_8 = 0$ does not vanish (see [Mc] for a nice overview of the Glennie and other special Jordan identities), and hence it is exceptional as a Jordan algebra. Though this algebra was constructed using Albert [A1], the authors meticulously define the multiplication table on the first 3 pages of their preprint, and prove that it indeed defines a mock-Lie algebra with non-vanishing Glennie identity on the next 15 pages. Here we recreate this 20-years-old effort, indicating how one can rigorously establish existence of non-special mock-Lie algebras using computer – in two, somewhat different, ways.

In order to verify whether a certain identity holds in a certain variety of algebras, Albert constructs a large enough (but finite-dimensional) homomorphic image of a free algebra in a given variety (see [Jacobs] and Albert User's Guide at [A1]). Thus, in order to verify whether the Glennie identity $G_8 = 0$, where G_8 is a word of the total degree 8 in 3 variables – two of degree 3 and one in degree 2, holds in the variety of mock-Lie algebras, one constructs a quotient of the free mock-Lie algebra of rank 3, freely generated by elements, say, a, b, c, by the ideal linearly spanned by all words containing either at least 4 a's, or at least 4 b's, or at least 3 c's. This quotient, let us call it $\mathfrak{M}$, has dimension 44. Initially, Albert worked over the prime fields of characteristic ≤ 251, and we have modified it to work over the rationals ([A2]). When computing over the rationals, all nonzero coefficients in the multiplication table of $\mathfrak{M}$ belong to the set $\{\pm 2, \pm 1, \pm\frac{1}{2}\}$. This means that $\mathfrak{M}$ can be defined over any field of characteristic $\neq 2$.

For the sake of further discussion, let $\{e_1, e_2, e_3, \ldots, e_{44}\}$ be the standard basis of $\mathfrak{M}$ as produced by Albert. A quick inspection of the multiplication table of $\mathfrak{M}$, supported by simple computer calculations, reveals that $\mathfrak{M}^9 = 0$, and the center of $\mathfrak{M}$ is one-dimensional, linearly spanned by e_{44}.

After constructing the multiplication table of a suitable homomorphic image of the free algebra, Albert checks whether a given identity $f = 0$ is satisfied in that image, by computing the linearization of f on all possible combinations of the basic elements, and verifying whether the results are identically zero. Thus, we can check that $G_8 = 0$ is not an identity in the mock-Lie variety for any fixed characteristic (well, technically for any characteristic $< 2^{63}$, what is the current limitation of [A2]) including zero, but, of course, we want

to establish this for *any* characteristic $\neq 2, 3$. When the fact that some set of identities $f_1 = 0, \ldots, f_n = 0$ does not imply another identity $f = 0$ in characteristic zero, implies the same fact in (almost) any positive characteristic p? Using the standard ultraproduct argument, it is true for all $p > p_0$, for some p_0 depending on $f_1, \ldots, f_n, f$. However, this argument does not give any concrete value of p_0. One can try to employ a slightly different method – instead of verifying identity in the given variety, one can add the identity in question to the defining identities, and to compare the dimension sequences of the corresponding pieces of free algebras produced by Albert. With such approach, using a simple variant of the Chinese remainder theorem, one can give a concrete estimate on p_0 (see [DZ]), however this estimate is too big for all practical purposes.

Instead, we can perform the last Albert's step – verifying of (non-)identity – in another, general purpose computer algebra system of our choice (we prefer GAP [GA] for such sort of tasks), which gives us finer control over the whole process. Namely, we can verify that in the algebra $\mathfrak{M}$ over the rationals, $G_8(e_1, e_2, e_3) = 96e_{44}$. Obviously, this computation implies $G_8(e_1, e_2, e_3) \neq 0$ in $\mathfrak{M}$ modulo any prime $p \neq 2, 3$.

A second way to verify that a given mock-Lie algebra L is exceptional, is to compute a Gröbner basis of its universal enveloping algebra $U(L)$, and to verify whether this Gröbner basis contains elements of the first degree (i.e., elements of L). Indeed, due to universal property of $U(L)$, L is special if and only if it admits embedding into $U(L)$, what, in its turn, happens if and only if the defining ideal of $U(L)$, and hence any of the Gröbner bases of $U(L)$, does not contain elements of the first degree. Computing with the help of GAP and GBNP, as in §2, we see that the central element e_{44} belongs to a Gröbner basis of $U(\mathfrak{M})$. (By the way, $\dim U(\mathfrak{M}) = 157$).

The same procedures can be repeated for other known special Jordan identities, but as all of them are of degree > 8, the resulting algebras constructed by Albert are of higher dimensions. Among those special identities summarized in [Mc], the identities G_9 and S_9 give rise to exceptional mock-Lie algebras of dimension 52 and 177 respectively, and the rest of identities are, within the mock-Lie variety, consequences of G_8, so they do not produce a new algebra. The Medvedev special identity ([Me, §2]) gives rise to an exceptional mock-Lie algebra of dimension 144.

Question 3. *What is the minimal possible dimension of an exceptional mock-Lie algebra?*

According to Proposition in §2, this minimal possible dimension lies between 7 and 44.

The algebra $\mathfrak{M}$ has another interesting feature: since the quotient of any mock-Lie algebra by the center is special, $\mathfrak{M}$ is an one-dimensional central extension of the 43-dimensional special algebra. The analogous Lie-algebraic situation (where, alas, speciality is understood as an embedding into associative *PI* algebras) was discussed many times in the literature, with the decisive example of a special Lie algebra whose one-dimensional central extension is not special, albeit, naturally, of a very different sort ([B] and references therein).

Moreover, experimenting with GAP, we have found that all random subalgebras and random quotients of $\mathfrak{M}$ we have tried, are special, what inclines us to think that the minimal dimension in Question 3 is indeed 44.

Question 4. *What is the minimal nilpotency index of an exceptional mock-Lie algebra?*

According to the above-cited Slin'ko's theorem, for Jordan algebras this minimal nilpotency index is 6, so for mock-Lie algebras it lies between 6 and 9. Of course, the Glennie identity, being of degree 8, is satisfied in all mock-Lie algebras of nilpotency index 8, what, however, still does not guarantee that all of them are special.

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Software

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[A2]   Albert fork version 4.0M5; `http://www1.osu.cz/~zusmanovich/albert/`[†]

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Department of Mathematics, University of Ostrava, Ostrava, Czech Republic

Email address, as of April 4, 2021: `pasha.zusmanovich@gmail.com`

[†]Added April 4, 2021: currently available at `https://web.osu.cz/~Zusmanovich/soft/albert/`