THE SECOND HOMOLOGY GROUP OF CURRENT LIE ALGEBRAS

PAUL ZUSMANOVICH

0. Introduction

It is a well known fact that the current Lie algebra $\mathcal{G} \otimes \mathbb{C}[[t,t^{-1}]]$ associated to a simple finite-dimensional Lie $\mathbb{C}$-algebra $\mathcal{G}$ has a central extension leading to the affine non-twisted Kac-Moody algebra $\mathcal{G} \otimes \mathbb{C}[[t,t^{-1}]] \oplus \mathbb{C}z$ with bracket

$$\{x \otimes f, y \otimes g\} = [x,y] \otimes fg + (x,y) Res \frac{df}{dt} g\, z$$

where $(\cdot\, , \cdot)$ is the Killing form on $\mathcal{G}$ (cf. [Kac]).

In view of the known relationship between central extensions and the second (co)homology group with coefficients in the trivial module, one of the main results of this paper can be considered as a generalization of this fact for general current Lie algebras, i.e., Lie algebras of the form $L \otimes A$, where L is a Lie algebra and A is associative commutative algebra, equipped with bracket

$$[x \otimes a, y \otimes b] = [x,y] \otimes ab.$$

Theorem 0.1. *Let L be an arbitrary Lie algebra over a field K of characteristic $p \neq 2$ and A an associative commutative algebra with unit over K. Then there is an isomorphism of K-vector spaces:*

$$(0.1) \quad H_2(L \otimes A) \simeq H_2(L) \otimes A \,\oplus\, B(L) \otimes HC_1(A) \\ \oplus\, \wedge^2(L/[L,L]) \otimes Ker(S^2(A) \to A) \,\oplus\, S^2(L/[L,L]) \otimes T(A)$$

where the mapping $S^2(A) \to A$ induced by multiplication in A and $T(A) = \langle ab \wedge c + ca \wedge b + bc \wedge a \mid a,b,c \in A \rangle$.

Here $B(L)$ is the space of coinvariants of the L-action on $S^2(L)$, $HC_1(A)$ is the first-order cyclic homology group of A, and $\wedge^2$ and S^2 denote the skew and symmetric products, respectively. Notice that in the case $L = [L,L]$, the third and fourth terms in the right-hand side of (0.1) vanish.

Many particular cases of this theorem were proved by different authors previously. An exhaustive description of all previous works on this theme may be found in [H] and [S].

For the first time, a cohomology formula of the type (0.1) has appeared in [S], where Theorem 0.1 was proved assuming that L is 1-generated over an augmentation ideal of its enveloping algebra. A. Haddi [H] obtained a result similar to Theorem 0.1 in the case where K is a field of characteristic zero (however, it seems that his arguments work over any field of characteristic $p \neq 2,3$).

Our method of proof differs from all previous ones and is based on the Hopf formula expressing $H_2(L)$ in terms of a presentation $0 \to I \to \mathcal{L}(X) \to L \to 0$, where $\mathcal{L} = \mathcal{L}(X)$ is the free Lie algebra over K freely generated by the set X:

$$(0.2) \qquad H_2(L) \simeq ([\mathcal{L},\mathcal{L}] \cap I)/[\mathcal{L},I]$$

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K-Theory (ed. C. Kassel et al.), Astérisque **226** (1994), 435–452.

(see, for example, [KS]).

The contents of the paper are as follows. §1 is devoted to some technical preliminary results. In §2 we determine the presentation of a current Lie algebra $L \otimes A$. In §3 Theorem 0.1 is proved. As it corollary we get in §4 a description of the space $B(L \otimes A)$. In §5 a "noncommutative version" of Theorem 0.1 is proved (Theorem 5.1). Namely, we derive the formula for the second homology group of the Lie algebra $(A \otimes B)^{(-)}$, where A, B are associative (noncommutative) algebras with unit, and $(-)$ in superscript denotes passing to the associated Lie algebra. The technique used here is no longer based on the Hopf formula, but on more or less direct computations in some factorspaces of cycles. However, arguments used in proof, resemble, to a great extent, the previous ones. Getting a particular case $B = M_n(K)$, we recover, after a slight modification, an isomorphism $H_2(sl_n(A)) \simeq HC_1(A)$ obtained in [KL].

The following notational convention will be used: the letters $a, b, c, \dots$, possibly with sub- and superscripts, denote elements of algebra A, while letters $u, v, w, \dots$ denote elements of the free Lie algebra $\mathcal{L}(X)$ with the set of generators $X = \{x_i\}$, if the otherwise is not stated. $\mathcal{L}^n(X)$ denotes the nth term in the derived series of $\mathcal{L}(X)$. The arrows $\rightarrowtail$ and $\twoheadrightarrow$ denote injection and surjection, respectively.

All other undefined notions and notation are standard, and may be found, for example, in [F] for Lie algebra (co)homology, and in [LQ] for cyclic homology. In some places we use diagram chasing and 3×3-Lemma without explicitly mentioning it.

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1. Preliminaries

Looking at formula (0.1), one can distinguish between the first two "principal" terms and other two "non-principal" ones. In order to simplify calculations, we will obtain a variant of the Hopf formula leading to the appearance of "principal" terms only, and then the general case will be derived.

Each nonperfect Lie algebra L, i.e., not coinciding with its commutant $[L, L]$, possesses a "trivial" homology classes of 2-cycles with coefficients in the module K, namely, classes whose representatives do not lie in $L \wedge [L, L]$. More precisely, consider a natural homomorphism $\psi : H_2(L) \to H_2(L/[L, L]) \simeq \wedge^2(L/[L, L])$ and denote $H_2^{ess}(L) = Ker\psi$, the homology classes of "essential" cycles.

Lemma 1.1. *One has an exact sequence*

$$0 \to H_2^{ess}(L) \to H_2(L) \xrightarrow{\psi} \wedge^2(L/[L, L]) \xrightarrow{\pi} [L, L]/[[L, L], L] \to 0$$

where π is induced by multiplication in L.

Proof. This is just an obvious consequence of a 5-term exact sequence derived from the Hochschild-Serre spectral sequence $H_n(L/[L, L], H_m([L, L])) \Rightarrow H_{n+m}(L)$. $\square$

Further, we need a version of Hopf formula for $H_2^{ess}(L)$.

Lemma 1.2. *Given a presentation* $0 \to I \to \mathcal{L} \to L \to 0$ *of a Lie algebra* L*, one has*

$$H_2^{ess}(L) \simeq \frac{\mathcal{L}^3 \cap I}{\mathcal{L}^3 \cap [\mathcal{L}, I]}. \tag{1.1}$$

Proof. Since $L/[L,L] \simeq \mathcal{L}/(\mathcal{L}^2 + I)$, the Hopf formula (0.2) being applied to the algebra $L/[L,L]$ gives $H_2(L/[L,L]) \simeq \mathcal{L}^2/[\mathcal{L}, \mathcal{L}^2 + I]$, and

$$Ker\psi = Ker\left(\frac{\mathcal{L}^2 \cap I}{[\mathcal{L}, I]} \to \frac{\mathcal{L}^2}{[\mathcal{L}, \mathcal{L}^2 + I]}\right) \simeq \frac{\mathcal{L}^2 \cap I \cap [\mathcal{L}, \mathcal{L}^2 + I]}{[\mathcal{L}, I]} \simeq \frac{\mathcal{L}^3 \cap I}{\mathcal{L}^3 \cap [\mathcal{L}, I]}.$$

□

Now consider an action of a Lie algebra L on $S^2(L)$ via

$$[z, x \vee y] = [z, x] \vee y + x \vee [z, y].$$

Let $B(L) = S^2(L)/[L, S^2(L)]$ be the space of coinvariants of this action. The dual $B(L)^*$ is the space of symmetric bilinear invariant forms on L.

Let I, J be ideals of L. Define $B(I, J)$ to be the space of coinvariants of the action of L on $I \vee J$. One has a natural embedding $B(I, J) \to B(L)$. The natural map $L \vee J \to (L/I) \vee ((I + J)/I)$ defines a surjection $B(L, J) \to B(L/I, (I + J)/I)$.

Lemma 1.3. *The short sequence*

$$0 \to B(L, I \cap J) + B(I, J) \to B(L, J) \to B(L/I, (I + J)/I) \to 0 \tag{1.2}$$

is exact.

Proof. Since $Ker(L \vee J \to L/I \vee (I + J)/I) = L \vee (I \cap J) + I \vee J$, the factorization through $[L, S^2(L)]$ yields

$$\begin{aligned} &Ker(B(L, J) \to B(L/I, (I + J)/I)) \\ &\qquad = (L \vee (I \cap J) + I \vee J + [L, S^2(L)])/[L, S^2(L)] \simeq B(L, I \cap J) + B(I, J). \end{aligned}$$

□

Remark. Actually we need the following two cases of this Lemma:

(1) $J = [L, L]$. Since $I \vee [L, L]$ and $[I, L] \vee L$ are congruent modulo $[L, S^2(L)]$ and $[I, L] \subseteq I \cap [L, L]$, then $B(I, [L, L]) \subseteq B(L, I \cap [L, L])$ and we get a short exact sequence

$$0 \to B(L, I \cap [L, L]) \to B(L, [L, L]) \to B(L/I, [L/I, L/I]) \to 0. \tag{1.3}$$

(2) $I = [L, L]$ and $J = L$. Then taking into account that for an abelian Lie algebra M, $B(M) \simeq S^2(M)$, the short exact sequence (1.2) becomes

$$0 \to B(L, [L, L]) \to B(L) \to S^2(L/[L, L]) \to 0. \tag{1.4}$$

2. Presentation of $L \otimes A$

In this section starting from a presentation of L we construct a presentation of $L \otimes A$.

Let $0 \to I \to \mathcal{L}(X) \xrightarrow{p} L \to 0$ be a presentation of the Lie algebra L. Tensoring by A, we get a short exact sequence

$$0 \to I \otimes A \to \mathcal{L}(X) \otimes A \xrightarrow{p \otimes 1} L \otimes A \to 0. \tag{2.1}$$

Let $X(A)$ be a set of symbols $x(a), x \in X, a \in A$. Define a homomorphism $\phi : \mathcal{L}(X(A)) \to \mathcal{L}(X) \otimes A$ by

$$\phi : u(x_1(a_1), \dots, x_n(a_n)) \mapsto u(x_1, \dots, x_n) \otimes a_1 \dots a_n.$$

Obviously this mapping is surjective, and taking into account (2.1), gives rise to the following exact sequence:

$$0 \to \phi^{-1}(I \otimes A) \to \mathcal{L}(X(A)) \overset{(p\otimes 1)\circ\phi}{\longrightarrow} L \otimes A \to 0 \tag{2.2}$$

which gives the presentation of $L \otimes A$.

In order to determine the structure of $\phi^{-1}(I \otimes A)$, let us introduce one notation. For each homogeneous element $u = u(x_1, \dots, x_n)$ of $\mathcal{L}(X)$, define $u(a)$ to be $u(x_1(a), x_2(1), \dots, x_n(1))$. Now having an arbitrary element $u \in \mathcal{L}(X)$, define $u(a)$ as $u_1(a) + \dots + u_k(a)$, where $u = u_1 + \dots + u_k$ is decomposition of u into the sum of homogeneous components.

Lemma 2.1.

(1) *$Ker\phi$ is linearly generated by elements of the form*

$$\sum_j u(x_{i_1}(a_1^j), \dots, x_{i_n}(a_n^j)) \tag{2.3}$$

where $u(x_{i_1}, \dots, x_{i_n})$ is homogeneous element of $\mathcal{L}(X)$ and $\sum_j a_1^j \dots a_n^j = 0$.

(2) *$\phi^{-1}(I \otimes A)$ is linearly generated modulo $Ker\phi$ by elements of the form $u(a)$, where $u \in I$.*

Proof. (1) Evidently each element of $\mathcal{L}(X(A))$ may be expressed as a sum of elements of the form $u(a)$ and elements of the form (2.3), the latter lying in $Ker\phi$. To prove that they exhaust all $Ker\phi$, take a nonzero element $\sum_i \sum_j u_i(a_{ij})$ belonging to $Ker\phi$, where u_i's are linearly independent, and obtain $\sum_i \sum_j u_i \otimes a_{ij} = 0$, which implies $\sum_j a_{ij} = 0$ for each i.

(2) The factorspace $\phi^{-1}(I \otimes A)/Ker\phi$, consisting from cosets $u(a) + Ker\phi$, maps onto $I \otimes A$, whence the conclusion. □

We also need the following technical result.

Lemma 2.2. *For any $u, v, w \in \mathcal{L}(X)$ and $a, b, c \in A$, the elements*

$$[[w,u](a), v(b)] - [[w,u](b), v(a)] + [[w,v](a), u(b)) - [[w,v](b), u(a)]$$

and

$$\begin{aligned} &[[u,v](ab), w(c)] - [[u,v)(c), w(ab)] \\ +&[[u,v](ca), w(b)] - [[u,v](b), w(ca)] \\ +&[[u,v](bc), w(a)] - [[u,v](a), w(bc)] \end{aligned}$$

belong to $[\mathcal{L}(X(A)), Ker\phi]$.

Proof. Consider the first case only, the second one is analogous. We have modulo $[\mathcal{L}(X(A)), Ker\phi]$:

$$\begin{aligned} &[[w,u](a), v(b)] - [[w,u](b), v(a)] + [[w,v)(a), u(b)] - [[w,v](b), u(a)] \\ &\qquad \equiv [[w(1), u(a)], v(b)] + [[v(b), w(1)], u(a)] \\ &\qquad + [[w(1), v(a)], u(b)] + [[u(b), w(1)], v(a)] \\ &\qquad\qquad \equiv -[[u(a), v(b)], w(1)] + [[u(b), v(a)], w(1)] \equiv 0 \end{aligned}$$

□

3. The second homology of $L \otimes A$

The aim of this section is to prove Theorem 0.1.

Consider the following commutative diagram with exact rows and columns, where ϕ^{-1} stands for $\phi^{-1}(I \otimes A)$ (we will use this notation in some places further):

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}] \cap Ker\phi \arrow[r] \arrow[d] & \mathcal{L}^3(X(A)) \cap Ker\phi \arrow[d] & & \\
0 \arrow[r] & \mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}] \arrow[r] \arrow[d, "\phi"'] & \mathcal{L}^3(X(A)) \cap \phi^{-1} \arrow[r] \arrow[d, "\phi"'] & H_2^{ess}(L \otimes A) \arrow[r] & 0 \\
0 \arrow[r] & (\mathcal{L}^3(X) \otimes A) \cap [\mathcal{L}(X) \otimes A, I \otimes A] \arrow[r] \arrow[d] & (\mathcal{L}^3(X) \otimes A) \cap (I \otimes A) \arrow[d] & & \\
 & 0 & 0 & &
\end{tikzcd}
$$

The middle row follows from the Lemma 1.2 applied to the presentation (2.2).

Completing this diagram to the third column, we get a short exact sequence

$$
\begin{aligned}
(3.1)\quad 0 \to \frac{\mathcal{L}^3(X(A)) \cap Ker\phi}{\mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}(I \otimes A)] \cap Ker\phi} &\to H_2^{ess}(L \otimes A) \\
&\to \frac{\mathcal{L}^3(X) \cap I}{\mathcal{L}^3(X) \cap [\mathcal{L}(X), I]} \otimes A \to 0.
\end{aligned}
$$

According to Lemma 1.2, the right term here is nothing but $H_2^{ess}(L) \otimes A$. Let us compute the left term.

Let $\mathcal{F}(Y)$ be a free skewcommutative algebra on an alphabet Y with nonassociative product denoted by $[\cdot,\cdot]$. Define a mapping $\alpha : \mathcal{F}^2(X(A)) \to S^2(\mathcal{F}(X)) \otimes (A \wedge A)$ by

$$
\begin{aligned}
(3.2)\quad \alpha : [u(x_1(a_1), \dots, x_n(a_n)), v(x_1(b_1), \dots, x_m(b_m))] &\mapsto \\
&(u(x_1, \dots, x_n) \vee v(x_1, \dots, x_m)) \otimes (a_1 \dots a_n \wedge b_1 \dots b_m).
\end{aligned}
$$

(recall that $\mathcal{F}^2(Y)$ is just $[\mathcal{F}(Y), \mathcal{F}(Y)]$).

It is easy to see that this mapping is well defined and surjective.

Let $J(Y)$ be an ideal of $\mathcal{F}(Y)$ generated by elements of the form $[[u,v],w] + [[w,u],v] + [[v,w],u], u,v,w \in \mathcal{F}(Y)$ such that $\mathcal{F}(Y)/J(Y) \simeq \mathcal{L}(Y)$.

Lemma 3.1.

$$\alpha(J(X(A))) = (J(X) \vee \mathcal{F}(X) + [\mathcal{F}(X), S^2(\mathcal{F}(X))]) \otimes (A \wedge A) + (\mathcal{F}^2(X) \vee \mathcal{F}(X)) \otimes T(A).$$

Proof. Writing the generic element in $J(X(A))$, it is easy to see, by considering graded degree, that every element in $\alpha(J(X(A)))$ can be written as a sum of an element lying in $(J(X) \vee \mathcal{F}(X)) \otimes (A \wedge A)$ and an element of the form

$$(3.3)\qquad ([u,v] \vee w) \otimes (ab \wedge c) + ([w,u] \vee v) \otimes (ca \wedge b) + ([v,w] \vee u) \otimes (bc \wedge a)$$

for certain $u,v,w \in \mathcal{F}(X)$ and $a,b,c \in A$.

Substituting in (3.3) $b = c = 1$, we get an element

$$([u,v] \vee w + [w,u] \vee v - [v,w] \vee u) \otimes (1 \wedge a).$$

Now permuting the letters u, v in the last expression, one easily get

$$(\mathcal{F}^2(X) \vee \mathcal{F}(X)) \otimes (1 \wedge A) \subset \alpha(J(A(X))).$$

Substituting in (3.3) $c = 1$ and taking into account the last relation, we get

$$([w, u] \vee v + u \vee [w, v]) \otimes (A \wedge A) \subset \alpha(J(A(X))). \tag{3.4}$$

Any element in (3.3) is congruent modulo (3.4) to an element of the form

$$(\mathcal{F}^2(X) \vee \mathcal{F}(X)) \otimes (ab \wedge c + ca \wedge b + bc \wedge a)$$

proving the Lemma. □

Now factoring the surjection α through $J(X(A))$ and using Lemma 3.1, we get a mapping

$$\overline{\alpha} : \mathcal{L}^2(X(A)) \longrightarrow B(\mathcal{L}(X)) \otimes HC_1(A) + (KX \vee KX) \otimes (A \wedge A),$$

(KX denotes the space of linear terms in $\mathcal{F}(X)$ such that $\mathcal{F}(X) = KX + \mathcal{F}^2(X)$), which being restricted to $\mathcal{L}^3(X(A))$, gives rise to the surjection

$$\overline{\alpha} : \mathcal{L}^3(X(A)) \longrightarrow B(\mathcal{L}(X), \mathcal{L}^2(X)) \otimes HC_1(A),$$

where $HC_1(A) = (A \wedge A)/T(A)$ is a first order cyclic homology of A.

Further, the restriction of the mapping ϕ defined in §2 to $\mathcal{L}^3(X(A))$ leads to a surjection $\phi : \mathcal{L}^3(X(A)) \to \mathcal{L}^3(X) \otimes A$.

Lemma 3.2. $\overline{\alpha}(\mathcal{L}^3(X(A)) \cap Ker\phi) = \overline{\alpha}(\mathcal{L}^3(X(A)))$.

Proof. The Lemma follows immediately from Lemma 2.1 and equality

$$\overline{\alpha}[u(a), v(b)] = \frac{1}{2}\overline{\alpha}([u(a), v(b)] - [u(b), v(a)]),$$

where the argument in the right-hand side lies in $Ker\phi$. □

Lemma 3.3.

$$\overline{\alpha}(\mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}(I \otimes A)]) = B(\mathcal{L}(X), I \cap \mathcal{L}^2(X)) \otimes HC_1(A).$$

Proof. According to Lemma 2.1, $\overline{\alpha}(\mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}(I \otimes A)])$ consists from the linear span of the following elements:

$$\overline{u \vee v} \otimes \overline{a \wedge b}$$

where either $u \in \mathcal{L}^2(X), v \in I$ or $u \in \mathcal{L}(X), v \in I \cap \mathcal{L}^2(X)$, and

$$\sum_j \overline{u \vee v} \otimes \overline{a \wedge b_j}$$

where $\sum_j b_j = 0$. The last expression obviously vanishes.

Modulo $[\mathcal{L}(X), S^2(\mathcal{L}(X))]$ we have:

$$\mathcal{L}^2(X) \vee I \equiv \mathcal{L}(X) \vee [I, \mathcal{L}(X)] \subseteq \mathcal{L}(X) \vee (I \cap \mathcal{L}^2(X)),$$

which implies the assertion of Lemma. □

Lemma 3.3 implies that the mapping $\overline{\alpha}$, being restricted to $\mathcal{L}^3(X(A)) \cap \phi^{-1}(I \otimes A)$ and factored through $\mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}(I \otimes A)]$, gives rise to a surjection

$$\beta : \frac{\mathcal{L}^3(X(A)) \cap \phi^{-1}(I \otimes A)}{\mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}(I \otimes A)]} \to \frac{B(\mathcal{L}(X), \mathcal{L}^2(X))}{B(\mathcal{L}(X), I \cap \mathcal{L}^2(X))} \otimes HC_1(A). \tag{3.5}$$

The right-hand side here is by (1.3) isomorphic to $B(L,[L,L]) \otimes HC_1(A)$.

Further, according to Lemma 3.2, β can be restricted to a surjection

$$\beta : \frac{\mathcal{L}^3(X(A)) \cap Ker\phi}{\mathcal{L}^3(X(A)) \cap [\mathcal{L}(X(A)), \phi^{-1}(I \otimes A)] \cap Ker\phi} \to B(L,[L,L]) \otimes HC_1(A). \tag{3.6}$$

Lemma 3.4. *β in (3.6) is injective.*

Proof. Denoting the left-hand side in (3.5) as $Frac$, consider the following diagram:

$$\begin{array}{ccccc}
Ker([L,L] \otimes A \wedge L \otimes A \to L \otimes A) & \xrightarrow{h} & H_2^{ess}(L \otimes A) & \xrightarrow{j} & Frac \\
{\scriptstyle i}\uparrow & & & & \downarrow{\scriptstyle \beta} \\
L \vee [L,L] \otimes A \wedge A & & \xrightarrow{n} & & B(L,[L,L]) \otimes HC_1(A)
\end{array}$$

where h is the obvious factorization, j is the isomorphism following from Lemma 1.2 applied to presentation (2.2), $n = l \otimes s$, where $l : L \vee [L,L] \to B(L,[L,L])$ and $s : A \wedge A \to HC_1(A)$ are obvious factorizations, and i is defined as

$$i : (x \vee y) \otimes (a \wedge b) \mapsto \frac{1}{2}(x \otimes a \wedge y \otimes b - x \otimes b \wedge y \otimes a) \tag{3.7}$$

for $x \in [L,L], y \in L$.

The following calculation verifies the commutativity of this diagram:

$$\begin{aligned}
&\beta \circ j \circ h \circ i((x \vee y) \otimes (a \wedge b)) \\
&= \frac{1}{2}\beta \circ j \circ h(x \otimes a \wedge y \otimes b - x \otimes b \wedge y \otimes a) \\
&= \frac{1}{2}\beta \circ j(\overline{x \otimes a \wedge y \otimes b - x \otimes b \wedge y \otimes a}) \\
&= \frac{1}{2}\beta \circ j(\overline{(u(a) + \phi^{-1}) \wedge (v(b) + \phi^{-1}) - (u(b) + \phi^{-1}) \wedge (v(a) + \phi^{-1})}) \\
&= \frac{1}{2}\beta(\overline{[u(a), v(b)] - [u(b), v(a)]}) \\
&= \frac{1}{2}(\overline{(x \vee y) \otimes (a \wedge b) - (x \vee y) \otimes (b \wedge a)}) \\
&= \overline{x \vee y} \otimes \overline{a \wedge b} \\
&= n((x \vee y) \otimes (a \wedge b))
\end{aligned}$$

where the overlined elements denote cosets in the corresponding factorspaces, and $x = u + I, y = v + I$.

It is also clear from the previous calculation and Lemmas 2.1 and 3.2 that the image of $j \circ h \circ i$ coincides with the left-hand side of (3.6).

Thus the kernel of the mapping (3.6) can be evaluated as

$$
\begin{aligned}
Ker\beta &= j\circ h\circ i(Ker\, n)\\
&= j\circ h\circ i(\langle [z,x]\vee y + [z,y]\vee x\rangle \otimes \langle a\wedge b\rangle\\
&+ \langle [x,y]\vee z\rangle \otimes \langle ab\wedge c + ca\wedge b + bc\wedge a\rangle)\\
&= j(\langle \overline{[z,x]\otimes a\wedge y\otimes b - [z,x]\otimes b\wedge y\otimes a}\\
&+ \overline{[z,y]\otimes a\wedge x\otimes b - [z,y]\otimes b\wedge x\otimes a}\rangle\\
&+ \langle \overline{[x,y]\otimes ab\wedge z\otimes c - [x,y]\otimes c\wedge z\otimes ab}\\
&+ \overline{[x,y]\otimes ca\wedge z\otimes b - [x,y]\otimes b\wedge z\otimes ca}\\
&+ \overline{[x,y]\otimes bc\wedge z\otimes a - [x,y]\otimes a\wedge z\otimes bc}\rangle)\\
&= \langle \overline{[[w,u](a),v(b)] - [[w,u](b),v(a)]}\\
&+ \overline{[[w,v](a),u(b)] - [[w,v](b),u(a)]}\rangle\\
&+ \langle \overline{[[u,v](ab),w(c)] - [[u,v](c),w(ab)]}\\
&+ \overline{[[u,v](ca),w(b)] - [[u,v](b),w(ca)]}\\
&+ \overline{[[u,v](bc),w(a)] - [[u,v](a),w(bc)]}\rangle
\end{aligned}
$$

(here $u = x + I, v = y + I, w = z + I$). The latter expression vanishes thanks to Lemma 2.2. $\square$

Putting together (3.1), (3.6) and Lemma 3.4, we get

Proposition 3.5. $H_2^{ess}(L\otimes A) \simeq H_2^{ess}(L)\otimes A \,\oplus\, B(L,[L,L])\otimes HC_1(A)$.

By Lemma 1.1 we have an exact sequence

$$
0 \to H_2^{ess}(L\otimes A) \to H_2(L\otimes A) \to \wedge^2(L/[L,L]\otimes A) \xrightarrow{\pi_A} [L,L]/[[L,L],L]\otimes A \to 0. \tag{3.8}
$$

Lemma 3.6.

$$
\begin{aligned}
Ker\pi_A \simeq Ker(\wedge^2(L/[L,L]) &\xrightarrow{\pi} [L,L]/[[L,L],L]) \otimes A\\
&\oplus\, \wedge^2(L/[L,L])\otimes Ker(S^2(A)\to A) \,\oplus\, S^2(L/[L,L])\otimes \wedge^2(A).
\end{aligned}
$$

Proof. The following commutative diagram with exact rows and columns

$$
\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & S^2(L/[L,L])\otimes\wedge^2(A) & \xrightarrow{=} & S^2(L/[L,L])\otimes\wedge^2(A) & \xrightarrow{\pi} & 0 & & \\
 & & & & {\scriptstyle i}\downarrow & & \downarrow & & \\
0 & \to & Ker\pi_A & \to & \wedge^2(L/[L,L]\otimes A) & \xrightarrow{\pi_A} & [L,L]/[[L,L],L]\otimes A & \to & 0\\
 & & & & {\scriptstyle k}\downarrow & & \| & & \\
0 & \to & Ker(\pi\otimes m) & \to & \wedge^2(L/[L,L])\otimes S^2(A) & \xrightarrow{\pi\otimes m} & [L,L]/[[L,L],L]\otimes A & \to & 0\\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}
$$

where i is defined in (3.7), and

$$
\begin{aligned}
k &: x\otimes a\wedge y\otimes b \mapsto (x\wedge y)\otimes(a\vee b)\\
m &: a\vee b \mapsto ab
\end{aligned}
$$

for $x, y \in L/[L,L], a, b \in A$, implies

$$Ker\pi_A \simeq Ker(\pi \otimes m) \ \oplus \ S^2(L/[L,L]) \otimes \wedge^2(A). \tag{3.9}$$

Considering the commutative diagram with exact rows and columns

$$\begin{tikzcd}
 & & 0 \arrow[d] & \\
 & \wedge^2(L/[L,L]) \otimes Ker\, m \arrow[r, "="] & \wedge^2(L/[L,L]) \otimes Ker\, m \arrow[r, "\pi"] \arrow[d] & 0 \arrow[d] \\
0 \arrow[r] & Ker(\pi \otimes m) \arrow[r] & \wedge^2(L/[L,L]) \otimes S^2(A) \arrow[r, "\pi\otimes m"] \arrow[d, "1\otimes m"'] & {[L,L]/[[L,L],L] \otimes A} \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & Ker(\pi \otimes 1) \arrow[r] & \wedge^2(L/[L,L]) \otimes A \arrow[r, "\pi\otimes 1"] \arrow[d] & {[L,L]/[[L,L],L] \otimes A} \arrow[r] & 0 \\
 & & 0 & &
\end{tikzcd}$$

we get

$$\begin{aligned} Ker(\pi \otimes m) \simeq \wedge^2(L/[L,L]) \otimes Ker(S^2(A) \to A) & \\ \oplus \ Ker(\wedge^2(L/[L,L]) \xrightarrow{\pi} [L,L]/[[L,L],L]) \otimes A. \end{aligned} \tag{3.10}$$

Putting (3.9) and (3.10) together proves the Lemma. □

Combining Proposition 3.5, (3.8) and Lemma 3.6, we get

$$\begin{aligned} H_2(L \otimes A) \simeq H_2^{ess}(L) \otimes A \ \oplus \ Ker(\wedge^2(L/[L,L]) \to [L,L]/[L,[L,L]]) \otimes A \\ \oplus \ B(L,[L,L]) \otimes HC_1(A) \ \oplus \ S^2(L/[L,L]) \otimes \wedge^2(A) \\ \oplus \ \wedge^2(L/[L,L]) \otimes Ker(S^2(A) \to A). \end{aligned}$$

By Lemma 1.1 the first two terms here give $H_2(L) \otimes A$. Using a (noncanonical) splitting $\wedge^2(A) = HC_1(A) \oplus T(A)$ and the exact sequence (1.4), the third and fourth terms give $B(L) \otimes HC_1(A) \oplus S^2(L/[L,L]) \otimes T(A)$. Combining these identifications gives Theorem 0.1.

Remark. It is interesting to compare Theorem 0.1 with the two-dimensional case of the homological operation

$$H_n(L \otimes A) \to \bigoplus_{i+j=n-1} HC_i(U(L)) \otimes HC_j(A)$$

defined in [FT] ($U(L)$ is the universal enveloping algebra of L and the ground field assumed to be of characteristic zero). Taking $n = 2$, we obtain a mapping

$$H_2(L \otimes A) \to HC_1(U(L)) \otimes HC_0(A) \ \oplus \ HC_0(U(L)) \otimes HC_1(A). \tag{3.11}$$

Cyclic homology of universal enveloping algebras was studied in [FT] and [Kas2]. Using their results, we may observe that if $S(L)$ denotes the whole symmetric algebra over L, then

$$HC_0(U(L)) = H_0(L, S(L)) = S(L)/[L, S(L)]$$

and $HC_1(U(L))$ is a certain factorspace of $H_1(L, S(L))$ containing $H_2(L)$. This implies that in general (3.11) is neither injection, nor surjection. However, if $L = [L, L]$, then (3.11) is an injection.

4. Computation of $B(L \otimes A)$

Theorem 0.1 allows us to compute $B(L\otimes A)$ in terms of L and A (of course, an alternative but longer proof may be given by means of direct computations).

Theorem 4.1. *$B(L \otimes A) \simeq B(L, [L, L]) \otimes A \,\oplus\, S^2(L/[L, L] \otimes A)$.*

Proof. It is more convenient to use Proposition 3.5 rather then Theorem 0.1 to obtain a formula for $B(L \otimes A, [L, L] \otimes A)$ and then to derive the general case.

Take any commutative unital algebra A' with $HC_1(A') \simeq K$. According to Proposition 3.5,

$$
\begin{aligned}
(4.1)\quad H_2^{ess}(L \otimes A \otimes A') &\simeq H_2^{ess}(L \otimes A) \otimes A' \,\oplus\, B(L \otimes A, [L, L] \otimes A) \\
&\simeq H_2^{ess}(L) \otimes A \otimes A' \,\oplus\, B(L, [L, L]) \otimes HC_1(A) \otimes A' \,\oplus\, B(L \otimes A, [L, L] \otimes A).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
(4.2)\quad H_2^{ess}(L \otimes A \otimes A') &\simeq H_2^{ess}(L) \otimes A \otimes A' \,\oplus\, B(L, [L, L]) \otimes HC_1(A \otimes A') \\
&\simeq H_2^{ess}(L) \otimes A \otimes A' \,\oplus\, B(L, [L, L]) \otimes HC_1(A) \otimes A' \,\oplus\, B(L, [L, L]) \otimes A.
\end{aligned}
$$

(the last isomorphism follows from the partial first-order commutative case of the Künneth formula for cyclic homology (cf. [Kas1]): $HC_1(A \otimes A') \simeq HC_1(A) \otimes A' + A \otimes HC_1(A')$).

Comparing (4.1) and (4.2), and using the naturality condition guaranteeing compatibility, one has

$$B(L \otimes A, [L, L] \otimes A) \simeq B(L, [L, L]) \otimes A.$$

Now the assertion of Theorem easily follows from the last isomorphism and the short exact sequence (1.4) applied to the algebra $L \otimes A$. □

5. The second homology of $A \otimes B$

Recall that given an associative algebra A, we may consider its associated Lie algebra $A^{(-)}$ with the same underlying space A and the bracket $[a, b] = ab - ba$, as well as a Jordan algebra $A^{(+)}$ with multiplication $a \circ b = \frac{1}{2}(ab + ba)$.

Recall that $T(A) = \langle ab \wedge c + ca \wedge b + bc \wedge a | a, b, c \in A\rangle$. For the sake of convenience we will also use the following notation:

$$
\begin{aligned}
T(A, [A, A]) &= \frac{T(A) + [A, A] \wedge A}{[A, A] \wedge A} \\
HC_1(A, [A, A]) &= \frac{A \wedge A}{[A, A] \wedge A + T(A)} \simeq \frac{\wedge^2(A/[A, A])}{T(A, [A, A])}
\end{aligned}
$$

(the second one is an analogue of $H_2^{ess}(L)$ for cyclic homology).

The aim of this section is to prove the following

Theorem 5.1. *Let A, B be associative algebras with unit over a field K of characteristic $p \neq 2$. Let $F(A, B)$ denote the direct sum of the following four vector spaces:*

(1) $A[A, A]/[A, A] \otimes HC_1(B)$
(2) $A/A[A, A] \otimes H_2(B^{(-)})$
(3) $(Ker(S^2(A) \to A/[A, A]))/[A, S^2(A)] \otimes HC_1(B, [B, B])$

(4) $Ker(S^2(A/[A,A]) \to A/A[A,A]) \otimes T(B,[B,B])$

where arrows in (3) and (4) are induced by (associative or Jordan) multiplication in A.

Then $H_2((A \otimes B)^{(-)}) \simeq F(A,B) \oplus F(B,A)$.

The proof is divided into several steps.

We employ the following short exact sequence:

$$0 \to \wedge^2 A \otimes S^2 B \xrightarrow{i} \wedge^2(A \otimes B) \xrightarrow{p} S^2 A \otimes \wedge^2 B \to 0$$

where the middle term is identified with the direct sum of two extreme ones via

$$a_1 \otimes b_1 \wedge a_2 \otimes b_2 \leftrightarrow a_1 \wedge a_2 \otimes b_1 \vee b_2 + a_1 \vee a_2 \otimes b_1 \wedge b_2,$$

and i and p are obvious imbedding and projection respectively. In what follows, this will be used without explicitly mentioning it.

The arguments are quite analogous to the ones at the beginning of §3. Here they applied to $H_2((A \otimes B)^{(-)}) \simeq Ker\, d/Im\, d$ (d is the differential in the standard homology complex of $(A \otimes B)^{(-)}$). The mapping p gives rise to the following short exact sequence:

$$0 \to \frac{Ker\, p \cap Ker\, d}{Ker\, p \cap Im\, d} \to H_2((A \otimes B)^{(-)}) \to \frac{p(Ker\, d)}{p(Im\, d)} \to 0. \tag{5.1}$$

The Lie bracket on $A \otimes B$ may be written as a sum

$$[a_1 \otimes b_1, a_2 \otimes b_2] = [a_1, a_2] \otimes b_1 \circ b_2 + a_1 \circ a_2 \otimes [b_1, b_2].$$

The proof of the following statement is quite analogous to the proof of (3.10).

Lemma 5.1.

$$p(Ker\, d) \simeq Ker(A \vee A \to A/[A,A]) \otimes B \wedge B + A \vee A \otimes Ker(B \wedge B \to B)$$

where the first arrow is induced by (associative or Jordan) multiplication in algebra A and the second one is the Lie multiplication in $B^{(-)}$.

Lemma 5.2. *$p(Im\, d)$ is a linear span of the following elements:*

(1) $[A, S^2(A)] \otimes B \wedge B$
(2) $[A,A] \vee A \otimes T(B)$
(3) $(a_1 \vee a_2 - 1 \vee a_1 \circ a_2) \otimes [B,B] \wedge B, a_i \in A$
(4) $A \vee A \otimes ([b_1,b_2] \wedge b_3 + [b_3,b_1] \wedge b_2 + [b_2,b_3] \wedge b_1), b_i \in B$.

Proof. We adopt the notation $x \equiv 0$ denoting the fact that certain element x of $A \vee A \otimes B \wedge B$ lies in $p(Im\, d)$. The generic relation defining the quotient by $p(Im\, d)$ is

$$\begin{aligned} &[a_1,a_2] \vee a_3 \otimes (b_1 \circ b_2) \wedge b_3 + (a_1 \circ a_2) \vee a_3 \otimes [b_1,b_2] \wedge b_3 \\ &+[a_3,a_1] \vee a_2 \otimes (b_3 \circ b_1) \wedge b_2 + (a_3 \circ a_1) \vee a_2 \otimes [b_3,b_1] \wedge b_2 \\ &+[a_2,a_3] \vee a_1 \otimes (b_2 \circ b_3) \wedge b_1 + (a_2 \circ a_3) \vee a_1 \otimes [b_2,b_3] \wedge b_1 \equiv 0. \end{aligned} \tag{5.2}$$

Symmetrizing this relation with respect to a_1, a_2, we get:

$$\begin{aligned} &2(a_1 \circ a_2) \vee a_3 \otimes [b_1,b_2] \wedge b_3 \\ &\quad + ([a_3,a_1] \vee a_2 - [a_2,a_3] \vee a_1) \otimes ((b_3 \circ b_1) \wedge b_2 - (b_2 \circ b_3) \wedge b_1) \\ &\quad\quad + ((a_3 \circ a_1) \vee a_2 + (a_2 \circ a_3) \vee a_1) \otimes ([b_3,b_1] \wedge b_2 + [b_2,b_3] \wedge b_1) \equiv 0. \end{aligned} \tag{5.3}$$

Cyclic permutations of a_1, a_2, a_3 in the last relation yield:

$$((a_1 \circ a_2) \vee a_3 + (a_3 \circ a_1) \vee a_2 + (a_2 \circ a_3) \vee a_1) \otimes ([b_1, b_2] \wedge b_3 + [b_3, b_1] \wedge b_2 + [b_2, b_3] \wedge b_1) \equiv 0.$$

This relation, in its turn, evidently implies

$$A \wedge A \otimes ([b_1, b_2] \wedge b_3 + [b_3, b_1] \wedge b_2 + [b_2, b_3] \wedge b_1) \equiv 0. \tag{5.4}$$

Now rewriting (5.3) modulo (5.4) and substituting $a_3 = 1$ and $b_2 = 1$, we get, respectively:

$$(a_1 \vee a_2 - 1 \vee a_1 \circ a_2) \otimes [B, B] \wedge B \equiv 0 \tag{5.5}$$

and

$$([a_3, a_1] \vee a_2 - [a_2, a_3] \vee a_1) \otimes (b_1 \wedge b_3 + 1 \wedge b_1 \circ b_3) \equiv 0.$$

Symmetrizing the last relation with respect to b_1, b_3, one gets:

$$([a_3, a_1] \vee a_2 - [a_2, a_3] \vee a_1) \otimes B \wedge B \equiv 0. \tag{5.6}$$

Particularly, taking in (5.6) $a_2 = 1$, one gets

$$1 \vee [A, A] \otimes B \wedge B \equiv 0. \tag{5.7}$$

Now, (5.2) is equivalent modulo (5.4)–(5.6) to

$$\begin{aligned} &[a_1, a_2] \vee a_3 \otimes (b_1 b_2 \wedge b_3 + b_3 b_1 \wedge b_2 + b_2 b_3 \wedge b_1) \\ &+ 1 \vee ((a_1 \circ a_2) \circ a_3 - (a_2 \circ a_3) \circ a_1) \otimes [b_1, b_2] \wedge b_3 \\ &+ 1 \vee ((a_3 \circ a_1) \circ a_2 - (a_2 \circ a_3) \circ a_1) \otimes [b_3, b_1] \wedge b_2 \equiv 0. \end{aligned} \tag{5.8}$$

Taking into account the identity

$$(a \circ b) \circ c - (a \circ c) \circ b = \frac{1}{4}[a, [b, c]]$$

(cf. [J], p.37), and (5.7), the relation (5.8), in its turn, is equivalent to

$$[A, A] \vee A \otimes T(B) \equiv 0. \tag{5.9}$$

Putting together (5.4)–(5.6) and (5.9), we get exactly the statement of the Lemma. □

Lemma 5.3.

(1) $p(Ker\, d)/p(Im\, d) \simeq F(A, B)$.
(2) $(Ker\, p \cap Ker\, d)/(Ker\, p \cap Im\, d) \simeq F(B, A)$.

Proof. (1) is derived from Lemmas 5.1 and 5.2 after a number of routine transformations.

(2) Define a projection $p' : \wedge^2(A \otimes B) \to \wedge^2 A \otimes S^2 B$. Due to an obvious fact that p' is the identity on $Ker\, p = A \wedge A \otimes B \vee B$, we have an isomorphism

$$\frac{Ker\, d \cap Ker\, p}{Ker\, p \cap Im\, d} \simeq \frac{p'(Ker\, d \cap Ker\, p)}{p'(Ker\, p \cap Im\, d)} = \frac{p'(Ker\, d)}{p'(Im\, d)}$$

But the right-hand term here is computed as in the part (1), up to permutation of A and B. □

Now Theorem 5.1 follows immediately from (5.1) and Lemma 5.3.

Remark. Taking in Theorem 5.1 $B = M_n(K)$, we get, after a series of elementary transformations, an isomorphism

$$H_2(gl_n(A)) \simeq HC_1(A) \oplus \wedge^2(A/[A,A]).$$

Using the Hochschild-Serre spectral sequence associated with central extension

$$0 \to sl_n(A) \to gl_n(A) \to A/[A,A] \to 0$$

of Lie algebras, we derive

$$H_2(sl_n(A)) \simeq HC_1(A)$$

which is a result of C. Kassel and J.-L. Loday [KL].

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Paul Zusmanovich
Department of Mathematics
Bar-Ilan University
Ramat-Gan 52900, Israel

Email address, as of July 3, 2020: `pasha.zusmanovich@gmail.com`