A VARIANT OF BAER'S THEOREM

PASHA ZUSMANOVICH

ABSTRACT. We provide a variant of Baer's theorem about isomorphism of endomorphism rings of vector spaces over division rings, where the full endomorphism rings are replaced by some subrings of finitary maps.

By Baer's theorem we mean the following result, formulated for the first time as Theorem 1 in [Baer, Chapter V, §4]: if V and W are two (say, right) vector spaces over division rings D and F, respectively, then an isomorphism of their endomorphism rings $\mathrm{End}_D(V)$ and $\mathrm{End}_F(W)$ implies a semilinear isomorphism of V and W. Baer's original proof uses properties of idempotents in the endomorphism ring; for a streamlined and modern exposition see [KTT, Theorem 4.1]. Another proof offered in [W, Theorem 8.1] (Ph.D. thesis under Baer) uses Jacobson's density theorem. In [R] and [Bal] Baer's theorem is extended to the cases of super vector spaces and graded vector spaces over super division rings and graded division rings, respectively. Later on Baer's theorem was also extended to modules over large classes of abelian groups (the so-called Baer–Kaplansky theorem); see [KTT] for a survey.

If V is infinite-dimensional over D, the ring $\mathrm{End}_D(V)$ is huge, and one may wonder whether in the formulation of Baer's theorem it can be replaced by something smaller. Somewhat anachronistically, this was done, with small variations, already before Baer, in the 1940s, in the works of Dieudonné, Jacobson, and others; see [J2, Chapter IV, §11] or [J1, Chapter IX, §11, Theorem 7]: the condition involves isomorphism of rings of *finitary* linear maps, and the proof uses again Jacobson's density theorem. The origin of these works seems to be in similar results established earlier in the analytic setting, for rings of bounded operators on Banach or Hilbert spaces, or rings of continuous operators on normed spaces, by Eidelheit, Mackey, and others; see historical references at [J2, p. 94].

Here we offer another variation on this topic. We retain a relatively narrow context of vector spaces over division rings, and – similarly to Jacobson and others – instead of the full endomorphism ring, consider its subrings of finitary linear maps. However, the rings we consider are, in the infinite-dimensional case, rather "small", significantly "smaller" even than the ring of all finitary maps, so this can be viewed as a substantial extension of the original Baer theorem. Our rings are, generally, not dense, thus all the previous methods of proofs of Baer's theorem and its variants, based on consideration of idempotents or other structural gadgets from ring theory, or on Jacobson's density theorem, do not work. Instead, we use an elementary linear-algebraic technique of "decomposing" conditions imposed on linear maps on tensor products, and consideration of traces.

All rings in this note are assumed to be associative. For the standard linear algebra over a (noncommutative) division ring, we refer to [Baer] or [J1]. Some notation and terminology is also borrowed from [CO] (where things are treated in the Lie-algebraic context).

Let D be a division ring with unit 1, and V a right vector space over D; then the dual V^* is a left vector space over D. A linear map is called *finitary* if its kernel has finite codimension (or, what is equivalent, its image has finite dimension). Finitary maps have traces, defined in the usual manner. The set of all finitary linear maps forms a subring $\mathrm{FEnd}_D(V)$ of the endomorphism ring $\mathrm{End}_D(V)$, and is linearly spanned by *infinitesimal transvections* $t_{v,f} : V \to V$, defined as $t_{v,f}(u) = vf(u)$, where $v,u \in V$ and $f \in V^*$ (infinitesimal transvections are exactly linear maps whose image is one-dimensional). The trace of an infinitesimal transvection is determined by the formula $\mathrm{Tr}(t_{v,f}) = f(v)$.

The ring $\mathrm{FEnd}_D(V)$ is isomorphic to the ring $V \otimes_D V^*$, with multiplication given by

$$(v \otimes f) \cdot (u \otimes g) = vf(u) \otimes g, \tag{1}$$

where $v,u \in V$ and $f,g \in V^*$. The isomorphism is given by sending the infinitesimal transvection $t_{v,f}$ to the decomposable tensor $v \otimes f$.

Now let Π be a D-subspace of V^*. Then $V \otimes_D \Pi$ is still closed with respect to the multiplication (1), and hence forms a subring of $V \otimes_D V^*$. Going back to $\mathrm{End}_D(V)$ and its subrings, the ring $V \otimes_D \Pi$ is isomorphic

Date: First written September 15, 2022; last minor revision April 10, 2024.
2020 *Mathematics Subject Classification.* 16S50.
Key words and phrases. Baer's theorem; endomorphism ring; finitary linear map.
Rocky Mountain J. Math., to appear.

to the subring $\mathrm{FEnd}_D(V,\Pi)$ of $\mathrm{FEnd}_D(V)$ generated by all infinitesimal transvections $t_{v,f}$ with $v \in V$ and $f \in \Pi$.

Note that all the just mentioned rings, $\mathrm{End}_D(V)$, $\mathrm{FEnd}_D(V)$, and $\mathrm{FEnd}_D(V,\Pi)$, are also right vector spaces over D, and hence have a structure of a right D-algebra.

Theorem. *Let V,W be right vector spaces over a division ring D, Π a nonzero finite-dimensional subspace of V^*, Γ a finite-dimensional subspace of W^*, and $\Phi : \mathrm{FEnd}_D(V,\Pi) \to \mathrm{FEnd}_D(W,\Gamma)$ an isomorphism of D-algebras. Then there is an isomorphism of D-vector spaces $\alpha : V \to W$ such that*

$$\Phi(f) = \alpha \circ f \circ \alpha^{-1} \tag{2}$$

for any $f \in \mathrm{FEnd}_D(V,\Pi)$.

Proof. Write the rings $\mathrm{FEnd}_D(V,\Pi)$ and $\mathrm{FEnd}_D(W,\Gamma)$ in the isomorphic form as the tensor products $V \otimes_D \Pi$ and $W \otimes_D \Gamma$ as above, and, by abuse of notation, denote by the same symbol Φ the isomorphism of D-algebras $\Phi : V \otimes_D \Pi \to W \otimes_D \Gamma$. Due to finite-dimensionality of Π, we have isomorphism of vector spaces over D:

$$\mathrm{Hom}_D(V \otimes_D \Pi, W \otimes_D \Gamma) \simeq \mathrm{Hom}_D(V,W) \otimes_D \mathrm{Hom}_D(\Pi,\Gamma),$$

and hence we can write Φ in the form

$$\Phi(v \otimes f) = \sum_{i \in I} \alpha_i(v) \otimes \beta_i(f), \tag{3}$$

where $\alpha_i : V \to W$ and $\beta_i : \Pi \to \Gamma$ are two linearly independent families of D-linear maps, indexed by a finite set I. The condition that Φ is a homomorphism, written for a pair of decomposable tensors $v \otimes f$ and $u \otimes g$, where $v,u \in V$ and $f,g \in \Pi$, is equivalent to

$$\sum_{i \in I} \Big(\alpha_i(v) f(u) - \sum_{j \in I} \alpha_j(v) \beta_j(f)(\alpha_i(u)) \Big) \otimes \beta_i(g) = 0.$$

Since the family $\{\beta_i\}_{i \in I}$ is linearly independent over D, each first tensor factor in the external sum vanishes, i.e.,

$$\alpha_i(v) f(u) - \sum_{j \in I} \alpha_j(v) \beta_j(f)(\alpha_i(u)) = 0$$

for any $i \in I$, $v,u \in V$, and $f \in \Pi$. This can be rewritten as

$$\alpha_i(v) \Big(f(u) - \beta_i(f)(\alpha_i(u)) \Big) - \sum_{\substack{j \in I \\ j \neq i}} \alpha_j(v) \beta_j(f)(\alpha_i(u)) = 0.$$

Since the family $\{\alpha_i\}_{i \in I}$ is linearly independent over D, each coefficient from D in the last sum vanishes; in particular,

$$\beta_i(f)(\alpha_i(u)) = f(u) \tag{4}$$

for any $u \in V, f \in \Pi$, and $i \in I$.

This implies

$$\mathrm{Tr}\Big(\Phi(u \otimes f)\Big) = \mathrm{Tr}\Big(\sum_{i \in I} \alpha_i(u) \otimes \beta_i(f)\Big) = \sum_{i \in I} \mathrm{Tr}\Big(\alpha_i(u) \otimes \beta_i(f)\Big)$$
$$= \sum_{i \in I} \beta_i(f)(\alpha_i(u)) = \sum_{i \in I} f(u) = |I| f(u) = |I|\,\mathrm{Tr}(u \otimes f).$$

As the ring $V \otimes_D \Pi$ is linearly spanned by decomposable tensors, we have

$$\mathrm{Tr}(\Phi(\xi)) = |I|\,\mathrm{Tr}(\xi) \tag{5}$$

for any $\xi \in V \otimes_D \Pi$.

Now consider the inverse isomorphism $\Phi^{-1} : W \otimes_D \Gamma \to V \otimes_D \Pi$, with decomposition similar to (3) with the index set J. By the same reasoning as in the case of Φ, we have

$$\mathrm{Tr}(\Phi^{-1}(\eta)) = |J|\,\mathrm{Tr}(\eta) \tag{6}$$

for any $\eta \in W \otimes_D \Gamma$. Combining (5) and (6), we have

$$\mathrm{Tr}(\xi) = |I||J|\,\mathrm{Tr}(\xi) \tag{7}$$

for any $\xi \in V \otimes_D \Pi$.

Since for any nonzero $f \in V^*$ there is $v \in V$ such that $f(v) \neq 0$ (actually, we can choose $f(v)$ to be equal to 1), $\mathrm{FEnd}_D(V,\Pi)$ always contains elements of nonzero trace[†]. Picking such an element ξ in (7), we have $|I| = |J| = 1$, i.e., Φ preserves traces and can be represented as a decomposable linear map: $\Phi = \alpha \otimes \beta$ for some $\alpha : V \to W$ and $\beta : \Pi \to \Gamma$. The system of equalities (4) reduces to the single equality

$$\beta(f) \circ \alpha = f \tag{8}$$

for any $f \in \Pi$. As Φ is invertible, α and β are invertible with $\Phi^{-1} = \alpha^{-1} \otimes \beta^{-1}$, so (8) can be rewritten as $\beta(f) = f \circ \alpha^{-1}$, and hence $\Phi(v \otimes f) = \alpha(v) \otimes (f \circ \alpha^{-1})$. Rewriting the last equality in terms of $\mathrm{FEnd}_D(V,\Pi)$ for an infinitesimal transvection $t_{v,f}$, and expanding by linearity, we get (2). □

A couple of final remarks concerning possible extensions of the theorem:

(i) We require the base ring D to be a division ring in order to ensure that all D-modules are free. We can merely require that all the modules appearing in the formulation of the theorem, i.e., V, W, Π, Γ, as well as all the modules appearing in the course of the proof, are free, without imposing any conditions on D, but this will just lead to a cumbersome formulation without changing the essence of the things.

(ii) The theorem can be easily extended to the graded case (thus providing a "finitary" analog of results from [R] and [Bal]), with essentially the same proof which will keep track of the maps on each graded component separately. This is left as an exercise to the reader.

Thanks are due to the anonymous referee for indicating an erroneous remark in the previous version of the manuscript.

References

[Baer] R. Baer, *Linear Algebra and Projective Geometry*, Acad. Press, 1952.

[Bal] I.N. Balaba, *Isomorphisms of graded rings of linear transformations of graded vector spaces*, Chebyshevskii Sbornik **6** (2005), no.4, 7–24 (in Russian).

[CO] H. Cuypers, M. Oostendorp, *A geometric characterization of the finitary special linear and unitary Lie algebras*, arXiv:2105.11967.

[J1] N. Jacobson, *Lectures in Abstract Algebra. II. Linear Algebra*, Von Nostrand, 1953; reprinted by Springer-Verlag, 1975.

[J2] ———, *Structure of Rings*, 2nd ed., AMS, 1964.

[KTT] P.A. Krylov, A.A. Tuganbaev, A.V. Tsarev, *Around the Baer–Kaplansky theorem*, J. Math. Sci. **256** (2021), no.3, 278–298.

[R] M.L. Racine, *Primitive superalgebras with superinvolution*, J. Algebra **206** (1998), no.2, 588–614.

[W] K.G. Wolfson, *An ideal-theoretic characterization of the ring of all linear transformations*, Amer. J. Math. **75** (1953), no.2, 358–386.

University of Ostrava, Czech Republic
Email address: pasha.zusmanovich@osu.cz

[†] This is a trivial, albeit a crucial point in our reasoning. Compare with the condition of so-called totality in [J1, Chapter IX, §11, Theorem 7], which, in our notation, amounts to saying, in a sense, a dual thing: for any nonzero $v \in V$ there is $f \in \Pi$ such that $f(v) \neq 0$. The latter condition is equivalent to the density of rings under consideration, and allows one to use Jacobson's density theorem.