COMMUTATIVE LIE ALGEBRAS AND COMMUTATIVE COHOMOLOGY IN CHARACTERISTIC 2

VIKTOR LOPATKIN AND PASHA ZUSMANOVICH

ABSTRACT. We discuss a version of the Chevalley–Eilenberg cohomology in characteristic 2, where the alternating cochains are replaced by symmetric ones.

INTRODUCTION

Define a commutative Lie algebra as a commutative algebra satisfying the Jacobi identity. While in characteristic $\ne 2$ this definition gives rise to a very special class of locally nilpotent Jordan algebras (studied in the literature under the names "mock-Lie" and "Jacobi–Jordan", see [Z3] and references therein), in characteristic 2 the picture is entirely different: this class of algebras lies between ordinary Lie algebras (where commutativity is replaced by a stronger alternating property) and Leibniz algebras (where commutativity is dropped altogether), both inclusions are strict. The class of commutative Lie algebras admits a good cohomology theory: the cohomology is defined via the standard formula for the differential in the Chevalley–Eilenberg complex, with the alternating cochains being replaced by symmetric ones.

Why bother with such curiosity? We give four arguments, roughly in increasing degree of persuasiveness.

1) From the operadic viewpoint, a "natural" class of algebras should be defined by multilinear identities. Moreover, the class of commutative Lie algebras appears naturally in certain algebraic topological and categorical contexts.

2) The underlying complex based on symmetric cochains, unlike the usual one based on alternating cochains, does not necessary vanish in degrees larger than the dimension of the algebra. This situation is similar to those occurring in cohomology of Lie superalgebras or Leibniz algebras (in any characteristic), opens new possibilities, and poses new interesting questions.

3) Commutative cohomology provides a new invariant of ordinary Lie algebras.

4) Commutative cohomology of ordinary Lie algebras appears naturally in some problems related to classification of simple Lie algebras.

The present note is elucidation of points 2–4 (concerning point 1, see an interesting recent paper [Et] for an operadic context, [L] for an algebraic topological context, and [GV] for a categorical context). While elementary in nature, this elucidation captures, in our opinion, some important phenomena peculiar to characteristic 2 which will be important in the ongoing classification of simple Lie algebras in that characteristic.

Before we plunge into our considerations, a few remarks are in order.

• Commutative 2-cocycles of Lie algebras in arbitrary characteristic do appear naturally in some circumstances and were considered in [D], [DB], and [DZ], but, unlike in characteristic 2, they seemingly do not lead to any cohomology theory.

• For abelian (i.e., with trivial multiplication) Lie algebras, commutative cohomology may be defined in any characteristic. An instance of such second-degree cohomology appears in [Z1, §5] in the context

of calculating structure functions on manifolds of loops with values in compact Hermitian symmetric spaces. It seems to be worth to study this cohomology and associated structures further. (A more-than-decade-ago promise from [Z1] to develop a "symmetric analogue of Spencer cohomology related with symmetric analogue of Cartan prolongations and some Jordan algebras" remained, so far, unfulfilled).

- The phenomenon of appearance of not necessary alternating 2-cocycles in characteristic 2 was noted already in [J, §3.4].
- Another interesting (and more sophisticated) versions of cohomology theory of Lie (super)algebras attempting to fix deficiencies of the ordinary cohomology in characteristic 2 were suggested in [BGLL, §3]. These versions are based on a cochain complex defined on the divided powers instead of (super)alternating polynomials, with various values of the shearing parameters for each (co)homology theory. It seems to be interesting to combine the constructions of this note and of [BGLL].
- Everything here can be dualized to get commutative *homology*. This is left as an exercise to the reader. We are interested primarily in cohomology, due to its application in structure theory, as explained in §1.7 below.

1. DEFINITIONS

1.1. **Commutative Lie algebras.**
Throughout this note, the ground field K is assumed to be of characteristic 2, unless stated otherwise. A *commutative Lie algebra* is an algebra L over K with multiplication $[\cdot,\cdot]$ satisfying the commutative identity

$$[x,y] = [y,x]$$

and the Jacobi identity

$$[[x,y],z] + [[z,x],y] + [[y,z],x] = 0$$

for any $x,y,z \in L$. The usual Lie-algebraic notions of abelian algebra, simple algebra, center, ideal, quotient, derivations, deformations, module (including the notions of a trivial, adjoint, and dual module), are carried over commutative Lie algebras without any modification. When considered as an L-module, K is always understood as a trivial module.

1.2. **A note about terminology.**
As noted in the introduction, in characteristic different from 2, commutative Lie algebras appeared in the literature under different names, see [Z3] and references therein. Neither of these names ("mock-Lie", "Jacobi-Jordan", "Jordan algebras of nilindex 3", etc.) adequately reflects the characteristic 2 situation.

Algebras satisfying the anticommutative identity

$$[x,y] = -[y,x]$$

and the Jacobi identity, appeared in [L], [GV], and references therein under the name "quasi-Lie algebras". Quasi-Lie algebras in characteristic 2 are commutative Lie algebras in our terminology, and ordinary Lie algebras in all other characteristics.

1.3. **Relation to Lie and Leibniz algebras.**
As commutative Lie algebras form a subclass of Leibniz algebras, the relationships between the classes of commutative and ordinary Lie algebras follow the already established patterns. The Jacobi identity implies that in any commutative Lie algebra L, the squares $[x,x]$, where $x \in L$, linearly span the central ideal of L, denoted by L^{sq} (cf. [LP, §1.10], where in the case of Leibniz algebras this ideal is denoted by L^{ann}). More generally, L^{sq} acts trivially on any L-module M. The quotient L/L^{sq} is a Lie algebra, and one may study commutative Lie algebras by considering corresponding extensions of Lie algebras, like it is done, for example, in [DA].

In particular, in any simple commutative Lie algebra L this ideal vanishes, and hence L is a Lie algebra. Following [DA], one may consider the next possible minimal situation concerning ideals: a commutative Lie algebra will be called *almost simple* if each its proper ideal coincides with L^{sq}. For any almost simple commutative Lie algebra L, the quotient L/L^{sq} is simple. Note that, unlike in Leibniz setting, L^{sq} is central, so, in the case L is not Lie, L^{sq} is necessarily one-dimensional.

1.4. **Commutative cohomology.** Let L be a commutative Lie algebra and M an L-module, with the module action defined by $\bullet$. The *commutative cohomology* of L with coefficients in M, denoted by $\mathrm{H}^\bullet_{comm}(L, M)$, is defined as cohomology of the cochain complex

$$0 \to \mathrm{S}^0(L, M) \xrightarrow{\mathrm{d}} \mathrm{S}^1(L, M) \xrightarrow{\mathrm{d}} \mathrm{S}^2(L, M) \xrightarrow{\mathrm{d}} \ldots$$

where $\mathrm{S}^n(L, M)$ is the space of n-linear symmetric maps $f : \underbrace{L \times \cdots \times L}_{n \text{ times}} \to M$, i.e. n-linear maps satisfying

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(n)}) = f(x_1, \ldots, x_n)$$

for any permutation $\sigma \in S_n$. The differential is defined as

$$\mathrm{d}\,\varphi(x_1, \ldots, x_{n+1}) = \sum_{1 \le i < j \le n+1} \varphi([x_i, x_j], x_1, \ldots, \widehat{x_i}, \ldots, \widehat{x_j}, \ldots, x_{n+1}) + \sum_{i=1}^{n+1} x_i \bullet \varphi(x_1, \ldots, \widehat{x_i}, \ldots, x_{n+1}).$$

Note that this is the usual formula for differential in the Chevalley–Eilenberg complex of a Lie algebra in characteristic 2 (i.e., all the signs being dropped). The cocycles and coboundaries in this complex will be customarily denoted by $\mathrm{Z}^\bullet_{comm}(L, M)$ and $\mathrm{B}^\bullet_{comm}(L, M)$, respectively.

1.5. **"De quadratum nihilo exaequari"**[†]**.** The equality $\mathrm{d}^2 = 0$ may be established by applying verbatim the same standard arguments used in the case of the usual Chevalley–Eilenberg cohomology. Namely, for each $x \in L$ let $i(x)$ be endomorphism of the vector space $\mathrm{S}^\bullet(L, M) = \bigoplus_{n \ge 0} \mathrm{S}^n(L, K)$ which maps $\mathrm{S}^n(L, M)$ to $\mathrm{S}^{n-1}(L, M)$ by the formula

$$(i(x)f)(x_1, \ldots, x_n) = f(x, x_1, \ldots, x_n),$$

and let θ be the natural representation of L in $\mathrm{S}^n(L, M)$. Then, for any $x, y \in L$, the usual Cartan formulas hold:

(1.1)
$$\begin{aligned} \theta(x)i(y) + i(y)\theta(x) &= i([x, y]) \\ i(x)\,\mathrm{d} + \mathrm{d}\,i(x) &= \theta(x) \\ \theta(x)\,\mathrm{d} &= \mathrm{d}\,\theta(x) \end{aligned}$$

from what the desired equality $\mathrm{d}^2 = 0$ follows.

Here is a nice heuristic explanation why this works, due to Alexei Lebedev. In the proof of the equality $\mathrm{d}^2 = 0$ in the Lie-algebraic (i.e., alternating) case, we would need an alternating property, and not merely a commutativity, of the Lie algebra bracket, only in the case where the formula for d^2 would involve expressions of the form $[u, u]$, where u is some expression involving $x_1, \ldots, x_{n+1}$. Similarly, the alternating, and not merely symmetric, property of cochains φ's would be required only in the case where d^2 would involve expressions of the form $\varphi(u, u, \ldots)$. Neither of these is the case, and hence the commutativity of the Lie bracket, and the symmetry of cochains is enough.

1.6. **No derived functor?** The similarity with the Chevalley–Eilenberg cohomology, however, does have its limits: it is interesting to see where the standard proof that the Chevalley–Eilenberg cohomology is the derived functor of the functor of taking the module invariants $M \mapsto M^L$ (cf., e.g. [We, §7.7]), fails in the case of commutative cohomology.

First we should find a suitable replacement of the universal enveloping algebra in the commutative case. As L^{sq} acts trivially on any module, the usual universal enveloping algebra $U(L/L^{sq})$ should serve the purpose: the categories of representations of L and of $U(L/L^{sq})$ are the same. Define the chain complex

(1.2)
$$\ldots \xrightarrow{\delta} U(L/L^{sq}) \otimes \bigvee^3 (L) \xrightarrow{\delta} U(L/L^{sq}) \otimes \bigvee^2 (L) \xrightarrow{\delta} U(L/L^{sq}) \otimes L \xrightarrow{\delta} U(L/L^{sq}) \xrightarrow{\varepsilon} K \to 0$$

[†]From Henri Cartan laudatio on then occasion of receiving Doctor Honoris Causa from the Oxford University.

where $\bigvee^n(L)$ is the n-fold symmetric product of L, and ε is the augmentation map with the kernel $U^+(L/L^{sq})$. The differential is defined exactly by the same formula as in the Lie-algebraic (alternating) case:

$$\delta\big(u \otimes (x_1 \vee \cdots \vee x_n)\big)$$

$$= \sum_{1 \le i < j \le n} u \otimes ([x_i, x_j] \vee x_1 \vee \cdots \vee \widehat{x_i} \vee \cdots \vee \widehat{x_j} \vee \cdots \vee x_n) + \sum_{i=1}^{n} u x_i \otimes (x_1 \vee \cdots \vee \widehat{x_i} \vee \cdots \vee x_n),$$

where $u \in U(L/L^{sq})$, and $x_1, \ldots, x_n \in L$.

By the same arguments as in the Lie-algebraic case – involving a version of Cartan formulas (1.1) for the complex (1.2) – we have $\delta^2 = 0$. However, the complex (1.2) is not exact, so, unlike in the Lie-algebraic case, it is not a free resolution of the trivial module K. It is not exact already in the case of abelian L (what, in the Lie-algebraic case, constitute the Koszul complex and essentially serves as an E_0 page of the spectral sequence abutting to the homology in the general case): for example, the chain $1 \otimes (x \vee x)$, for nonzero $x \in L$, belongs to $\operatorname{Ker}\delta$, but not to $\operatorname{Im}\delta$, since the latter in the second degree lies in $U^+(L/L^{sq}) \otimes \mathrm{S}^2(L)$.

Replacing in the complex (1.2) the symmetric product by the "alternating" one, i.e., by the quotient of the tensor algebra $T^\bullet(L)$ by the ideal generated by elements of the form $x \vee x$, $x \in L$, will not work either: in characteristic 2, this "alternating" product is isomorphic to the exterior one, $\bigwedge^\bullet(L)$, and for the finite-dimensional L, the so obtained complex is finite, while the symmetric cohomology a priori may not vanish in an arbitrarily large degree (and it does not vanish indeed in all examples computed below).

1.7. Motivation.
We have encountered commutative cohomology when started a project of description of simple finite-dimensional Lie algebras having a Cartan subalgebra of toral rank 1, of which [GZ] is the beginning. In the process, one needs to compute various low-degree cohomology of current Lie algebras, i.e. Lie algebras of the form $L \otimes A$ where L is a Lie algebra and A is a commutative associative algebra, for certain particular instances of L and A. When one tries to extend the known formulas for such cohomology in characteristics $\ne 2, 3$ from [Z1] to the case of characteristic 2, one naturally encounters low-degree commutative cohomology of L. In [GZ], where we dealt with the case where L is the 3-dimensional simple algebra, commutative cohomology appears in disguise in Proposition 2.1. The results of this note will be used in subsequent classification efforts of simple Lie algebras in characteristic 2.

2. Elementary observations

2.1. Cohomology of low degree.
The usual interpretations of low-degree cohomology are trivially carried over from Lie (and Leibniz) algebras to the commutative Lie case: $\mathrm{H}^0_{comm}(L, M) = M^L$, the module of invariants, $\mathrm{H}^1_{comm}(L, K) \simeq (L/[L, L])^*$, $\mathrm{H}^1_{comm}(L, L)$ coincides with outer derivations of L, $\mathrm{H}^2_{comm}(L, M)$ describes equivalence classes of abelian extensions

$$0 \to M \to \cdot \to L \to 0,$$

$\mathrm{H}^2_{comm}(L, L)$ describes infinitesimal deformations of a commutative Lie algebra L, whereas obstructions to prolongability of infinitesimal deformations to global ones live in $\mathrm{H}^3_{comm}(L, L)$.

In particular, the problem of the description of almost simple commutative Lie algebras reduces to the determination of 1-dimensional central extensions $0 \to Q^{sq} \to Q \to L \to 0$, and hence to the computation of $\mathrm{H}^2_{comm}(L, K)$ of all simple Lie algebras L.

For any Lie algebra L defined over a field of characteristic $\ne 2$, there is a useful exact sequence

$$(2.1) \qquad\qquad 0 \to \mathrm{H}^2(L, K) \to \mathrm{H}^1(L, L^*) \to \mathrm{B}(L) \to \mathrm{H}^3(L, K)$$

which goes back to classical works of Koszul and Hochschild–Serre (see, for example, [DZ, §1] and references therein). Here $\mathrm{H}^\bullet(L, M)$ is the usual Chevalley–Eilenberg cohomology with coefficients in an

L-module M, and $B(L)$ is the space of symmetric invariant bilinear forms on L, i.e. symmetric bilinear maps $\varphi : L \times L \to K$ such that

$$(2.2) \qquad \qquad \varphi([x,y],z) = \varphi([z,x],y)$$

for any $x,y,z \in L$. In characteristic 2, however, (2.1) is no longer true, but we have instead

Proposition 1. *For any commutative Lie algebra L, there is a short exact sequence*

$$0 \to H^2_{comm}(L,K) \to H^1_{comm}(L,L^*) \to B_{alt}(L) \to H^3_{comm}(L,K).$$

Here $B_{alt}(L)$ denotes the space of all *alternating* bilinear maps satisfying (2.2).

Proof. The proof repeats the standard arguments used in establishing the exact sequence (2.1) or its commutative analog in characteristic $\neq 2$ (see, for example, [DZ, Proof of Proposition 1.1]).   □

2.2. **Relation to Chevalley–Eilenberg and Leibniz cohomology.** The natural inclusion of alternating maps to symmetric ones induces, for any Lie algebra L, L-module M, and $n \in \mathbb{N}$, a commutative diagram

$$
\begin{array}{ccc}
C^n(L,M) & \xrightarrow{\ d\ } & C^{n+1}(L,M) \\
\downarrow & & \downarrow \\
S^n(L,M) & \xrightarrow{\ d\ } & S^{n+1}(L,M)
\end{array}
$$

where $C^n(L,M)$ is the usual space of alternating cochains, and d is the usual Chevalley-Eilenberg differential. This, in its turn, induces the map

$$(2.3) \qquad \qquad H^n(L,M) \to H^n_{comm}(L,M).$$

Similarly, the natural inclusion of symmetric maps to all multilinear maps induces, for any commutative Lie algebra L and an L-module M, a commutative diagram

$$
\begin{array}{ccc}
S^n(L,M) & \xrightarrow{\ d\ } & S^{n+1}(L,M) \\
\downarrow & & \downarrow \\
\mathrm{Hom}_K(L^{\otimes n}, M) & \xrightarrow{\ d\ } & \mathrm{Hom}_K(L^{\otimes n+1}, M)
\end{array}
$$

Here d in the bottom row denotes the differential in the Leibniz complex. This, in its turn, induces the map

$$(2.4) \qquad \qquad H^n_{comm}(L,M) \to HL^n(L,M),$$

where $HL^\bullet(L,M)$ denotes the Leibniz cohomology.

Obviously, for $n = 0,1$ the maps (2.3) and (2.4) are isomorphisms (there is nothing to "symmetrize" or "alternate" for cochains in 0 or 1 arguments). For any Lie algebra L, any 2-coboundary with arbitrary coefficients

$$(2.5) \qquad \qquad d\,\varphi(x,y) = \varphi([x,y]) + x \bullet \varphi(y) + y \bullet \varphi(x),$$

and any 3-coboundary with trivial coefficients

$$d\,\varphi(x,y,z) = \varphi([x,y],z) + \varphi([z,x],y) + \varphi([y,z],x)$$

is alternating, and hence the map (2.3) is an embedding for $n = 2$, and for $n = 3$ and $M = K$. Similarly, for any commutative Lie algebra L, the Leibniz 2-coboundary is given by the same formula (2.5), and hence the map (2.4) is an embedding for $n = 2$. In general, however, neither of the maps (2.3) and (2.4) is an embedding or a surjection.

2.3. **Extension of the base field.** The standard arguments based on the universal coefficient theorem, the same as in the case of ordinary Chevalley–Eilenberg cohomology, imply that the commutative cohomology does not change under field extension: if L is a commutative Lie algebra over a field K, and $K \subset K'$ is a field extension, then

$$\mathrm{H}^n_{comm}(L \otimes_K K', M \otimes_K K') \simeq \mathrm{H}^n_{comm}(L, M) \otimes_K K'.$$

3. The cup product

For a commutative Lie algebra L over a field K, define the bilinear map

$$\smile : \mathrm{S}^\bullet(L, K) \times \mathrm{S}^\bullet(L, K) \to \mathrm{S}^\bullet(L, K)$$

by the formula

$$(3.1) \qquad (\varphi \smile \psi)(x_1, \dots, x_{p+q}) = \sum_{IJ} \varphi(x_{i_1}, \dots, x_{i_p}) \cdot \psi(x_{j_1}, \dots, x_{j_q}),$$

where the sum is taken over all shuffles, i.e. partitions of the sequence $\{1, \dots, p + q\}$ into two disjoint increasing subsequences $I = \{i_1, \dots, i_p\}$ and $J = \{j_1, \dots, j_q\}$.

It is obvious that the so defined $\smile$ turns $\mathrm{S}^\bullet(L, K)$ into a (graded) associative ring.

Proposition 2. *The differential* d *is a derivation of the ring* $\mathrm{S}^\bullet(L, K)$ *with respect to the product* $\smile$.

Proof. We need to prove that for any $\varphi \in \mathrm{S}^p(L, K)$ and $\psi \in \mathrm{S}^q(L, K)$, the following relation holds:

$$(3.2) \qquad \mathrm{d}(\varphi \smile \psi) = \mathrm{d}\varphi \smile \psi + \varphi \smile \mathrm{d}\psi.$$

This is verified by direct computation: we have

$$
\begin{aligned}
(\mathrm{d}\varphi \smile \psi)(x_1, \dots, x_{p+q}) &= \sum_{IJ} (\mathrm{d}\varphi)(x_{i_1}, \dots, x_{i_p}) \cdot \psi(x_{j_1}, \dots, x_{j_q}) \\
&= \sum_{IJ} \sum_{1 \le r < s \le p} \varphi([x_{i_r}, x_{i_s}], x_{i_1}, \dots, \widehat{x_{i_r}}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p}) \cdot \psi(x_{j_1}, \dots, x_{j_q}),
\end{aligned}
$$

$$
\begin{aligned}
(\varphi \smile \mathrm{d}\psi)(x_1, \dots, x_{p+q}) &= \sum_{IJ} \varphi(x_{i_1}, \dots, x_{i_p}) \cdot (\mathrm{d}\psi)(x_{j_1}, \dots, x_{j_q}) \\
&= \sum_{IJ} \sum_{1 \le l < t \le q} \varphi(x_{i_1}, \dots, x_{i_p}) \cdot \psi([x_{j_l}, x_{j_t}], x_{j_1}, \dots, \widehat{x_{j_l}}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q}),
\end{aligned}
$$

and

$$
\begin{aligned}
\left(\mathrm{d}(\varphi \smile \psi) \right)(x_1, \dots, x_{p+q}) &= \sum_{1 \le \alpha < \beta \le p+q} (\varphi \smile \psi)([x_\alpha, x_\beta], x_1, \dots, \widehat{x_\alpha}, \dots, \widehat{x_\beta}, \dots, x_{p+q}) \\
&= \sum_{IJ} \sum_{1 \le r < s \le p} \varphi([x_{i_r}, x_{i_s}], x_{i_1}, \dots, \widehat{x_{i_r}}, \dots, \widehat{x_{i_s}}, \dots, x_{i_p}) \cdot \psi(x_{j_1}, \dots, x_{j_q}) \\
&\quad + \sum_{IJ} \sum_{1 \le l < t \le q} \varphi(x_{i_1}, \dots, x_{i_p}) \cdot \psi([x_{j_l}, x_{j_t}], x_{j_1}, \dots, \widehat{x_{j_l}}, \dots, \widehat{x_{j_t}}, \dots, x_{j_q}),
\end{aligned}
$$

and the equality (3.2) follows. $\qquad \square$

It is obvious that the derivation d preserves the grading of $\mathrm{S}^\bullet(L, K)$.

As for any ring with a derivation D, the kernel $\mathrm{Ker}\,D$ is a subring, and the image of D is an ideal in $\mathrm{Ker}\,D$, we get:

Corollary. *For any commutative Lie algebra L:*
  (i) *The space $\mathrm{Z}^\bullet_{comm}(L, K)$ of commutative cocycles is a subring of the ring $\mathrm{S}^\bullet(L, K)$.*
  (ii) *The space $\mathrm{B}^\bullet_{comm}(L, K)$ of commutative coboundaries is an ideal of the ring $\mathrm{Z}^\bullet_{comm}(L, K)$.*
  (iii) *The commutative cohomology $\mathrm{H}^\bullet_{comm}(L, K)$ is a graded associative ring with respect to the product* $\smile$.

4. Examples

In this section we compute the commutative cohomology in several interesting cases.

4.1. Abelian algebra.

If L is an abelian (commutative) Lie algebra, the differential in the complex $S^\bullet(L, K)$ vanishes, and $H^n_{comm}(L, K) = S^n(L, K)$ for any n.

4.2. 1-dimensional algebra.

Obviously, the 1-dimensional commutative Lie algebra is abelian (and hence is a Lie algebra). For any module M over the 1-dimensional algebra Kx, $[x, x] = 0$, we have $S^n(Kx, M) = \mathrm{Hom}_K((Kx)^{\otimes n}, M) \simeq M$. The differential $d : S^n(Kx, M) \to S^{n+1}(Kx, M)$ reduces to $d\,\varphi(x, \dots, x) = nx \bullet \varphi(x, \dots, x)$, and hence both Lie commutative and Leibniz complexes are reduced to the complex

$$0 \to M \xrightarrow{0} M \xrightarrow{x} M \xrightarrow{0} M \xrightarrow{x} \dots$$

whose cohomology is

$$H^n_{comm}(Kx, M) = HL^n(Kx, M) \simeq \begin{cases} \mathrm{Ker}(x|_M) & \text{if } n \text{ is even} \\ \mathrm{Coker}(x|_M) & \text{if } n \text{ is odd.} \end{cases}$$

4.3. 2-dimensional algebra.

Let L be the 2-dimensional nonabelian Lie algebra with the basis $\{a, b\}$, $[a, b] = a$. Choose a basis in $S^n(L, K)$ consisting of the cochains χ_{pq}, $p + q = n$, defined by

$$\chi_{pq}(\underbrace{a, \dots, a}_{r}, \underbrace{b, \dots, b}_{s}) = \begin{cases} 1 & \text{if } p = r \text{ and } q = s \\ 0 & \text{otherwise.} \end{cases}$$

We have:

$$d\chi_{pq}(\underbrace{a, \dots, a}_{r}, \underbrace{b, \dots, b}_{s}) = rs\,\chi_{pq}(\underbrace{a, \dots, a}_{r-1}, a, \underbrace{b, \dots, b}_{s-1}),$$

and hence

$$d\chi_{pq} = p(q + 1)\chi_{p,q+1}.$$

It follows that $B^n_{comm}(L, K)$ has a basis consisting of χ_{pq}, where $p + q = n$, and both p, q are odd; and $Z^n_{comm}(L, K)$ has a basis consisting of χ_{pq}, where $p + q = n$, and either p is even, or q is odd.

Therefore, the cocycles χ_{pq}, where $p + q = n$, and p is even, can be chosen as basic cocycles whose representatives span $H^n_{comm}(L, K)$.

To determine the cup product in terms of this basis, note that by (3.1),

$$\chi_{pq} \smile \chi_{rs}(\underbrace{a, \dots, a}_{p+r}, \underbrace{b, \dots, b}_{q+s}) = \binom{p + r}{p}\binom{q + s}{q},$$

and hence

$$\chi_{pq} \smile \chi_{rs} = \binom{p + r}{p}\binom{q + s}{q}\chi_{p+r,q+s}.$$

In particular,

$$\chi_{p0} \smile \chi_{0s} = \chi_{ps},$$

which shows that the basic cocycles of the form χ_{p0} and χ_{0s} generate the whole $H^\bullet_{comm}(L, K)$ as a ring.

4.4. Heisenberg algebra. The $(2\ell + 1)$-dimensional Lie algebra with the basis $a, b_1, \ldots, b_\ell, c_1, \ldots, c_\ell$, and multiplication

$$[b_i, a] = [c_i, a] = 0, \quad [b_i, c_j] = \begin{cases} a & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

is called the *Heisenberg algebra*, and is denoted by $\mathcal{H}_\ell$.

To compute commutative cohomology of $\mathcal{H}_\ell$ with coefficients in the trivial module, we will use algebraic discrete Morse theory, briefly recalled in Appendix (which should be consulted for all undefined notions and notation in this section). A very similar in spirit computation of the usual Chevalley–Eilenberg *homology* of the Heisenberg algebra in characteristic 2, was performed earlier in [S1].

Any cochain $\varphi \in S^n(\mathcal{H}_\ell, K)$ is determined uniquely by its values on the basic elements:

(4.1)
$$\varphi(\underbrace{a, \ldots, a}_{\alpha}, \underbrace{b_1, \ldots, b_1}_{\beta_1}, \ldots, \underbrace{b_\ell, \ldots, b_\ell}_{\beta_\ell}, \underbrace{c_1, \ldots, c_1}_{\gamma_1}, \ldots, \underbrace{c_\ell, \ldots, c_\ell}_{\gamma_\ell}),$$

where

(4.2)
$$\alpha + \beta_1 + \cdots + \beta_\ell + \gamma_1 + \cdots + \gamma_\ell = n.$$

Assuming $\beta = (\beta_1, \ldots, \beta_\ell)$ and $\gamma = (\gamma_1, \ldots, \gamma_\ell)$, the following shorthand notation will be used: $\varphi(\alpha; \beta; \gamma)$ will denote the corresponding value (4.1), and $\alpha + \beta + \gamma$ will denote the left-hand side of (4.2). At the same time, $\beta \pm \beta'$ denotes the vector obtained by the usual coordinate-wise addition or subtraction of vectors in $\mathbb{Z}_{\geq 0}^\ell$, i.e. $(\beta_1 \pm \beta_1', \ldots, \beta_\ell \pm \beta_\ell')$, similarly for γ's. The vector of length ℓ having 1 at the ith place, and 0 at all other places, will be denoted by $\mathbb{1}_i$. Further, define

$$I_0(\beta) = \{ i \in \{1, \ldots, \ell\} \mid \beta_i \text{ is even} \}$$
$$I_1(\beta) = \{ i \in \{1, \ldots, \ell\} \mid \beta_i \text{ is odd} \}.$$

For any triple $(\alpha; \beta; \gamma)$ such that $\alpha + \beta + \gamma = n + 1$, we have:

(4.3)
$$d\,\varphi(\alpha; \beta; \gamma) = \sum_{\substack{1 \leq i \leq \ell \\ \beta_i > 0, \gamma_i > 0}} \beta_i \gamma_i \, \varphi(\alpha + 1; \beta - \mathbb{1}_i; \gamma - \mathbb{1}_i).$$

Now choose a basis X_n in $S^n(\mathcal{H}_\ell, K)$ consisting of the cochains $\chi_{(\alpha;\beta;\gamma)}$, $\alpha + \beta + \gamma = n$, defined by

$$\chi_{(\alpha;\beta;\gamma)}(\tilde{\alpha}; \tilde{\beta}; \tilde{\gamma}) = \begin{cases} 1 & \text{if } \tilde{\alpha} = \alpha, \tilde{\beta} = \beta, \tilde{\gamma} = \gamma \\ 0 & \text{otherwise.} \end{cases}$$

The formula (4.3) implies then $d\chi_{(0;\beta;\gamma)} = 0$, and

$$d\chi_{(\alpha;\beta;\gamma)}(\alpha - 1; \beta + \mathbb{1}_i; \gamma + \mathbb{1}_i) = (\beta_i + 1)(\gamma_i + 1)$$

for any $\alpha > 0$ and $1 \leq i \leq \ell$. This, in its turn, implies

(4.4)
$$d\chi_{(\alpha;\beta;\gamma)} = \begin{cases} \sum_{i=1}^\ell (\beta_i + 1)(\gamma_i + 1) \chi_{(\alpha-1;\beta+\mathbb{1}_i;\gamma+\mathbb{1}_i)} & \text{if } \alpha > 0 \\ 0 & \text{if } \alpha = 0. \end{cases}$$
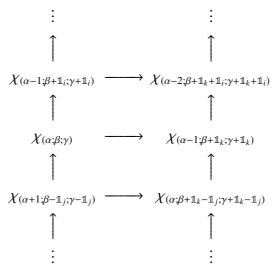
Now we are in the position to apply algebraic discrete Morse theory to the cochain complex $(S^\bullet(\mathcal{H}_\ell, K), d)$. In the graph $\Gamma(S^\bullet(\mathcal{H}_\ell, K))$ constructed from this complex with the chosen basis $\bigcup_{n \geq 0} X_n$, define the set M consisting of all edges of the form

$$\chi_{(\alpha;\beta;\gamma)} \to \chi_{(\alpha-1;\beta+\mathbb{1}_k;\gamma+\mathbb{1}_k)},$$

where $k = \max(I_0(\beta) \cap I_0(\gamma))$ (so both β_k, γ_k are even), and

$$k = \max(I_0(\beta) \cap I_0(\gamma)) > \max(I_1(\beta) \cap I_1(\gamma)).$$

The set M can be depicted as horizontal arrows in the following graph (where it is assumed that $i, j \in I_1(\beta) \cap I_1(\gamma)$):

$$
\begin{array}{ccc}
\vdots & & \vdots \\
\uparrow & & \uparrow \\
\chi_{(\alpha-1;\beta+\mathbb{1}_i;\gamma+\mathbb{1}_i)} & \longrightarrow & \chi_{(\alpha-2;\beta+\mathbb{1}_k+\mathbb{1}_i;\gamma+\mathbb{1}_k+\mathbb{1}_i)} \\
\uparrow & & \uparrow \\
\chi_{(\alpha;\beta;\gamma)} & \longrightarrow & \chi_{(\alpha-1;\beta+\mathbb{1}_k;\gamma+\mathbb{1}_k)} \\
\uparrow & & \uparrow \\
\chi_{(\alpha+1;\beta-\mathbb{1}_j;\gamma-\mathbb{1}_j)} & \longrightarrow & \chi_{(\alpha;\beta+\mathbb{1}_k-\mathbb{1}_j;\gamma+\mathbb{1}_k-\mathbb{1}_j)} \\
\uparrow & & \uparrow \\
\vdots & & \vdots
\end{array}
$$

It is clear that after flipping all the horizontal arrows, the new graph $\Gamma^M(S^\bullet(\mathcal{H}_\ell, K))$ does not contain directed cycles. Also, no vertex is incident to more than one edge in M. Therefore, M is an acyclic matching.

The set of vertices in $V = \bigcup_{n \geq 0} X_n$ which do not serve as a tail for any arrow in M, is equal to

(4.5) $$\{ \chi_{(0;\beta;\gamma)} \} \cup \{ \chi_{(\alpha;\beta;\gamma)} \mid \alpha > 0, \ \max(I_0(\beta) \cap I_0(\gamma)) < \max(I_1(\beta) \cap I_1(\gamma)) \},$$

while the set of vertices which do not serve as a head for any arrow in M, is equal to

(4.6) $$\{ \chi_{(\alpha;\beta;\gamma)} \mid \max(I_0(\beta) \cap I_0(\gamma)) > \max(I_1(\beta) \cap I_1(\gamma)) \}.$$

Thus $\bigcup_{n \geq 0} X_n^M$, being the intersection of the sets (4.5) and (4.6), is equal to the set $\mathscr{C}_0 \cup \mathscr{C}_1$, where

$$\mathscr{C}_0 = \{ \chi_{(0;\beta;\gamma)} \mid \max(I_0(\beta) \cap I_0(\gamma)) > \max(I_1(\beta) \cap I_1(\gamma)) \},$$

and

$$\mathscr{C}_1 = \{ \chi_{(\alpha;\beta;\gamma)} \mid I_0(\beta) \cap I_0(\gamma) = I_1(\beta) \cap I_1(\gamma) = \varnothing \}.$$

By (4.4), all cochains from both $\mathscr{C}_0$ and $\mathscr{C}_1$ are cocycles, and then by the theorem from Appendix, $\mathscr{C}_0 \cup \mathscr{C}_1$ forms a basis of the cohomology $H^\bullet_{comm}(\mathcal{H}_\ell, K)$. (To be more precise, a basis of the nth degree cohomology $H^n_{comm}(\mathcal{H}_\ell, K)$ is formed by cocycles from $\mathscr{C}_0$ with $\beta + \gamma = n$, and by cocycles from $\mathscr{C}_1$ with $\alpha + \beta + \gamma = n$).

Let us look now at the ring structure of $H^\bullet_{comm}(\mathcal{H}_\ell, K)$. For any two triples $(\alpha;\beta;\gamma)$ and $(\alpha';\beta';\gamma')$ we have:

$$\chi_{(\alpha;\beta;\gamma)} \smile \chi_{(\alpha';\beta';\gamma')} = \binom{\alpha + \alpha'}{\alpha} \binom{\beta + \beta'}{\beta} \binom{\gamma + \gamma'}{\gamma} \chi_{(\alpha+\alpha';\beta+\beta';\gamma+\gamma')},$$

where $\binom{\beta'}{\beta}$ is a shorthand for the product $\binom{\beta'_1}{\beta_1} \cdots \binom{\beta'_\ell}{\beta_\ell}$, similarly for γ's. From this formula it is clear that $\mathscr{C}_0 \smile \mathscr{C}_0 \subseteq \mathscr{C}_0$, $\mathscr{C}_0 \smile \mathscr{C}_1 \subseteq \mathscr{C}_1$, and $\mathscr{C}_1 \smile \mathscr{C}_1 \subseteq \mathscr{C}_1$, and therefore, as a ring, $H^\bullet_{comm}(\mathcal{H}_\ell, K)$ is decomposed into the semidirect sum of two subrings:

$$H^\bullet_{comm}(\mathcal{H}_\ell, K) \simeq K\mathscr{C}_0 \oplus K\mathscr{C}_1,$$

where $K\mathscr{C}_0$ acts on $K\mathscr{C}_1$.

4.5. Zassenhaus algebras.

The algebra $W_1(n)$ is defined as an algebra of special derivations $\mathcal{O}_1(n)\partial$ of the divided powers algebra $\mathcal{O}_1(n)$ (see, e.g., [DA], [J], or [GZ] for details). It has the basis $\{e_i = x^{(i+1)}\partial \mid -1 \leq i \leq 2^n - 2\}$ with multiplication

$$[e_i, e_j] = \begin{cases} \binom{i+j+2}{i+1} e_{i+j} & \text{if } -1 \leq i + j \leq 2^n - 2 \\ 0 & \text{otherwise.} \end{cases}$$

In characteristic 2, unlike in bigger characteristics, the algebra $W_1(n)$ is not simple, but its commutant $W'_1(n)$ of dimension $2^n - 1$, linearly spanned by the elements $\{e_i \mid -1 \le i \le 2^n - 3\}$, is. The algebras $W'_1(n)$ are referred as *Zassenhaus algebras*. The basic elements provide the standard grading

$$W'_1(n) = \bigoplus_{i=-1}^{2^n-3} Ke_i.$$

In the first nontrivial case $n = 2$, the algebra $W'_1(2)$ is 3-dimensional, with multiplication table

$$[e_{-1}, e_0] = e_{-1}, \quad [e_1, e_0] = e_1, \quad [e_{-1}, e_1] = e_0,$$

and is an analog of $\mathsf{sl}(2)$ in big characteristics.

Another realization of the algebra $W_1(n)$ is defined over the field $\mathsf{GF}(2^n)$ as the algebra with the basis $\{f_\alpha \mid \alpha \in \mathsf{GF}(2^n)\}$ and multiplication

$$[f_\alpha, f_\beta] = (\alpha + \beta)f_{\alpha+\beta}$$

for $\alpha, \beta \in \mathsf{GF}(2^n)$. Again, in characteristic 2 this algebra is not simple, but its commutant $\{f_\alpha \mid \alpha \in \mathsf{GF}(2^n)^*\}$, isomorphic to $W'_1(n)$, is.

For any k elements $\alpha_1, \ldots, \alpha_k \in \mathsf{GF}(2^n)^*$ such that the sum of any number of these elements is nonzero, the $2^k - 1$ elements $f_{\alpha_{i_1} + \cdots + \alpha_{i_\ell}}$, $1 \le i_1 \le \cdots \le i_\ell \le k$, span a subalgebra $L(\alpha_1, \ldots, \alpha_k)$ of $W'_1(n)$ isomorphic to $W'_1(k)$.

Proposition 3. $\mathrm{H}^2_{comm}(W'_1(n), K)$ *has dimension n. The basic cocycles can be chosen as*

(4.7)
$$e_i \vee e_j \mapsto \begin{cases} 1 & \text{if } i = j = 2^k - 2, \text{ or } \{i, j\} = \{-1, 2^{k+1} - 3\} \\ 0 & \text{otherwise.} \end{cases}$$

for $k = 0, \ldots, n - 1$.

Proof. It is straightforward to check that the maps (4.7) are indeed commutative 2-cocycles (that boils down to the fact that if $i, j \ge 0$ and $i + j = 2^k - 2$, then $[e_i, e_j] = \binom{2^k}{i+1} e_{2^k-2} = 0$). Since these cocycles are non-alternating, and 2-coboundaries are alternating, their cohomological independence is equivalent to the linear independence, and the latter follows from the fact that all they have different weights with respect to the standard grading of $W'_1(n)$. Thus $\dim \mathrm{H}^2_{comm}(W'_1(n), K) \ge n$. To prove that we have here an equality, we will switch to the basis $\{f_\alpha\}$.

We shall prove that the basic cocycles in $\mathrm{H}^2_{comm}(W'_1(n), K)$ can be chosen as

(4.8)
$$f_\alpha \vee f_\beta \mapsto \begin{cases} \lambda_\alpha & \text{if } \alpha = \beta \\ 0 & \text{if } \alpha \ne \beta \end{cases}$$

where $\alpha, \beta \in \mathsf{GF}(2^n)^*$, $\lambda_\alpha \in K$, subject to linear relations

(4.9)
$$\alpha \lambda_\alpha + \beta \lambda_\beta + (\alpha + \beta)\lambda_{\alpha+\beta} = 0$$

for any $\alpha, \beta \in \mathsf{GF}(2^n)^*$, $\alpha \ne \beta$.

We proceed similarly to [DB] where, in order to prove the vanishing of commutative 2-cocycles on simple classical Lie algebras in characteristic > 2, first the rank 2 case is established, and the general case follows easily.

So, first consider the cases $n = 2$ and $n = 3$. In that cases the statement follows from direct computations, similar to those performed in [D, Theorem 6.5] and [DB]. These computations can be also performed on computer, using a simple GAP program for computations of the space of commutative 2-cocycles on a given Lie algebra (see [DZ, footnote at §3]).

In the general case $n \ge 3$, take arbitrary $\alpha, \beta, \gamma \in \mathsf{GF}(2^n)^*$, $\alpha + \beta + \gamma \ne 0$, and restrict an arbitrary cocycle $\varphi \in \mathrm{Z}^2_{comm}(W'_1(n), K)$ to the 7-dimensional subalgebra $L(\alpha, \beta, \gamma)$ linearly spanned by f_α, f_β, f_γ, $f_{\alpha+\beta}$, $f_{\alpha+\gamma}$, $f_{\beta+\gamma}$, $f_{\alpha+\beta+\gamma}$. Obviously, this restriction is a commutative 2-cocycle on $L(\alpha, \beta, \gamma) \simeq W'_1(3)$, and by the just established case $n = 3$, we have that first,

$$\varphi(f_\alpha, f_\beta) = (\alpha + \beta)\omega_{\alpha,\beta,\gamma}(f_{\alpha+\beta})$$

for some linear map $\omega_{\alpha,\beta,\gamma} : L(\alpha,\beta,\gamma) \to K$, and second, that the relation (4.9) holds for $\lambda_\alpha = \varphi(f_\alpha, f_\alpha)$. Embedding the pair α,β into another triple α,β,γ', we see that $\omega_{\alpha,\beta,\gamma}$ does not depend on γ. In the same vein, it does not depend neither on α, nor on β, so $\varphi(f_\alpha, f_\beta) = \mathrm{d}\,\omega([f_\alpha, f_\beta])$ for any $\alpha,\beta \in \mathsf{GF}(2^n)^*$, $\alpha \ne \beta$, and some linear map $\omega : W_1'(n) \to K$. Consequently, φ can be represented as the sum of $\mathrm{d}\,\omega$ and a map of the form (4.8). The latter maps are obviously commutative 2-cocycles, and we are done.

It remains to determine the dimension of $\mathrm{H}^2_{comm}(W_1'(n), K)$. The relation (4.9) can be expanded as

$$\lambda_{\alpha_1 + \cdots + \alpha_k} = \frac{\alpha_1}{\alpha_1 + \cdots + \alpha_k}\lambda_{\alpha_1} + \cdots + \frac{\alpha_k}{\alpha_1 + \cdots + \alpha_k}\lambda_{\alpha_k}$$

for any $\alpha_1, \ldots, \alpha_k \in \mathsf{GF}(2^n)^*$, $\alpha_1 + \cdots + \alpha_k \ne 0$, which means that $\dim \mathrm{H}^2_{comm}(W_1'(n), K)$ is equal to the number of the generators of the additive group of $\mathsf{GF}(2^n)$. The latter number is equal to dimension of $\mathsf{GF}(2^n)$ as a vector space over $\mathsf{GF}(2)$, and hence is equal to n. □

Note that since $\dim \mathrm{B}^2(W_1'(n), K) = \dim W_1'(n) = 2^n - 1$, we have $\dim \mathrm{Z}^2_{comm}(W_1'(n), K) = 2^n + n - 1$.

Note also, that the calculations above imply that every alternating 2-cocycle on $W_1'(n)$ is a coboundary, and hence $\mathrm{H}^2(W_1'(n), K) = 0$.

4.6. Eick's algebras.
Commutative cohomology may serve as another invariant helping to distinguish algebras. In [Ei], a computer-generated list of simple Lie algebras over $\mathsf{GF}(2)$ of dimension ≤ 20 was presented, and sophisticated (nonlinear) methods were used to establish non-isomorphism of algebras in the list.

For example, computer calculations with GAP show that the degree 2 commutative cohomology with trivial coefficients of the two new 15-dimensional simple Lie algebras in Eick's list, number 7 and 8, is of dimension 1 and 2 respectively. All the other "conventional" "linear" invariants of these two algebras we can think of (dimension of low-degree Chevalley–Eilenberg cohomology with trivial and adjoint coefficients, dimension of the p-envelope and of the sandwich subalgebra, the absence of nondegenerate symmetric invariant forms) do coincide.

5. Further questions

Finally, we take a liberty to indicate some avenues for further research. Some of the questions listed here seem to be of a purely technical character, while others seem to be difficult and probably will require new nontrivial approaches.

5.1. Is it possible to represent the commutative cohomology as a derived functor? (This question seems to be tricky, as it is hard to imagine what the other candidate for the role of the universal enveloping algebra in the commutative case could be, see §1.6).

5.2. To compute commutative cohomology for various "interesting" algebras. In particular, for the three-dimensional simple Lie algebra, and for free Lie algebras.

5.3. To get a formula relating $\mathrm{H}^2_{comm}(\mathsf{sl}_n(A), K)$ and (a version of) cyclic cohomology of A in the spirit of [KL]. A glance at [GZ, Proposition 2.1] may suggest that the version of cyclic cohomology, peculiar to characteristic 2, which should appear here, is those where the (skew)symmetric cochains are replaced by alternating ones.

5.4. Establish an analog of the Hopf formula for the second degree commutative *homology* with trivial coefficients.

5.5. Define the cup product in §3 the same standard way as it is done for the Chevalley–Eilenberg cohomology and other classic cohomology theories, i.e. as a composition of the isomorphism provided by the Künneth formula, and the map between cohomology of $L \oplus L$ and L (see, for example, [We, Exercise 7.3.8]). For this, of course, we will need (a version of) the Künneth formula for commutative cohomology.

5.6. The classical Stallings-Swan theorem says that groups of cohomological dimension 1 are free. In characteristics 0 and 2 it is an open question whether Lie algebras of cohomological dimension 1 are free. What about commutative Lie algebras? (Note that since we do not have a definition of commutative cohomology as a derived functor, the very notion of cohomological dimension in this case is a bit problematic).

5.7. It is well known (and easy to see) that the Euler-Poincaré characteristic of cohomology of a finite-dimensional Lie algebra, i.e. the alternating sum of dimensions of cohomology in all degrees, vanishes. The very notion of the Euler–Poincaré characteristic of the commutative cohomology does not make sense, as the sum

$$\dim \mathrm{H}^0_{comm}(L, M) - \dim \mathrm{H}^1_{comm}(L, M) + \dim \mathrm{H}^2_{comm}(L, M) - \ldots$$

is, generally, infinite and thus diverges. Can this sum be assigned a reasonable value using the theory of divergent series, similarly how it was (partially) done for cohomology of Lie superalgebras in [Z2]?

5.8. As shown in §4.5, the dimension of the space of commutative 2-cocycles with trivial coefficients on the Zassenhaus algebra $W'_1(n)$, is equal to $2^n + n - 1$. Find a link with a combinatorial interpretation of this number as the shortest length of a sequence of 0 and 1 containing all subsequences of length n (see [OEIS, A052944]).

5.9. Whether the variety of commutative Lie algebras is Schreier, i.e., whether a subalgebra of a free commutative Lie algebra is free?

Let us note at the end that recently Friedrich Wagemann, [Wa], has constructed several spectral sequences pertaining the commutative cohomology: the Hochschild–Serre-like spectral sequence (the construction more or less repeats the construction of the Hochschild–Serre spectral sequence for the Chevalley–Eilenberg cohomology), and spectral sequences elucidating relationship with Chevalley–Eilenberg and Leibniz cohomology (see §2.2).

Appendix. Algebraic discrete Morse theory

Algebraic discrete Morse theory is an algebraic version of discrete Morse theory developed independently by Sköldberg, [S2], and by Jöllenbeck and Welker, [JW]. It allows to construct, starting from a chain complex, a new homotopically equivalent smaller complex using directed graphs. Here, for the convenience of the reader, we present a short version of this machinery adapted for cochain, rather than chain, complexes (this can be done formally by considering cochain complexes as chain complexes with negative indices and reverting arrows, but we prefer to write down everything explicitly). We follow closely [JW, Chapter 2], with minor simplifications and variations in notation.

Let

$$\mathrm{C} : \mathrm{C}_0 \xrightarrow{\mathrm{d}_0} \mathrm{C}_1 \xrightarrow{\mathrm{d}_1} \mathrm{C}_2 \xrightarrow{\mathrm{d}_2} \ldots$$

be a cochain complex of vector spaces over a field K (which is assumed here to be of arbitrary characteristic; in fact, the whole theory is generalized, with slight modifications, to the case of arbitrary complexes of free modules over an arbitrary associative ring).

Let X_n be a basis of the vector space C_n. Write the differentials $\mathrm{d}_n : \mathrm{C}_n \to \mathrm{C}_{n+1}$ with respect to these bases:

$$\mathrm{d}_n(c) = \sum_{c' \in X_{n+1}} [c : c'] \cdot c',$$

where $c \in X_n$, and $[c : c']$ are coefficients from K.

From this data, we construct a directed weighted graph $\Gamma(C) = (V, E)$. The set of vertices V of $\Gamma(C)$ is the basis $V = \bigcup_{n \geq 0} X_n$, and the set E of weighted edges consists of triples

$$\{ (c, c', [c : c']) \mid c \in X_n, c' \in X_{n+1}, [c : c'] \neq 0 \}.$$

A finite subset $M \subseteq E$ of the set of edges is called an *acyclic matching*, if it satisfies the following two conditions:

(Matching) Each vertex $v \in V$ lies in at most one edge $e \in M$.

(Acyclicity) The subgraph $\Gamma^M(C) = (V, E^M)$ of the graph $\Gamma(C)$ has no directed cycles, where

$$E^M = (E \setminus M) \cup \{ (c', c, -\frac{1}{[c : c']}) \mid (c, c', [c : c']) \in M \}.$$

For an acyclic matching M on the graph $\Gamma(C)$, we introduce the following notation:

(1) Define
$$X_n^M = \{ c \in X_n \mid c \text{ does not lie in any edge } e \in M \}.$$

(2) Write $c' \leq c$ if $c \in X_n$, $c' \in X_{n+1}$, and $[c : c'] \neq 0$.

(3) Path(c, c') is the set of paths from c to c' in $\Gamma^M(C)$.

(4) The weight $w(p)$ of a path $p = c_1 \to \ldots \to c_r \in \text{Path}(c_1, c_r)$ is defined as

$$w(c_1 \to \ldots \to c_r) = \prod_{k=1}^{r-1} w(c_k \to c_{k+1})$$

$$w(c \to c') = \begin{cases} -\dfrac{1}{[c : c']} & \text{if } c \leq c' \\ [c : c'] & \text{if } c' \leq c. \end{cases}$$

The following is a cohomological version of [JW, Theorem 2.2]:

Theorem. *Given an acyclic matching M, the cochain complex (C, d) is homotopy equivalent to the complex (C^M, d^M), where C_n^M is the vector space linearly spanned by X_n^M, and the differential $d_n^M : C_n^M \to C_{n+1}^M$ is defined as*

$$d_n^M(c) = \sum_{c' \in X_{n+1}^M} \sum_{p \in \text{Path}(c, c')} w(p)\, c',$$

where $c \in C_n^M$.

References

[BGLL] S. Bouarroudj, P. Grozman, A. Lebedev, and D. Leites, *Divided power (co)homology. Presentations of simple finite dimensional modular Lie superalgebras with Cartan matrix*, Homology Homotopy Appl. **12** (2010), no.1, 237–278; arXiv:0911.0243.

[D] A.S. Dzhumadil'daev, *Algebras with skew-symmetric identity of degree* 3, Sovremennaya Matematika i ee Prilozheniya **60** (2008) (E.B. Vinberg Festschrift), 13–31 (in Russian); J. Math. Sci. **161** (2009), no.1, 11–30 (English translation).

[DA] ―――― and S.A. Abdykassymova, *Leibniz algebras in characteristic p*, Comptes Rendus Acad. Sci. Paris **332** (2001), no.12, 1047–1052.

[DB] ―――― and A.B. Bakirova, *Simple two-sided anti-Lie-admissible algebras*, Sovremennaya Matematika i ee Prilozheniya **60** (2008) (E.B. Vinberg Festshrift), 32–37 (in Russian); J. Math. Sci. **161** (2009), no.1, 31–36 (English translation).

[DZ] ―――― and P. Zusmanovich, *Commutative 2-cocycles on Lie algebras*, J. Algebra **324** (2010), no.4, 732–748; arXiv:0907.4780.

[Ei] B. Eick, *Some new simple Lie algebras in characteristic 2*, J. Symb. Comput. **45** (2010), no.9, 943–951.

[Et] P. Etingof, *Koszul duality and the PBW theorem in symmetric tensor categories in positive characteristic*, Adv. Math. **327** (2018), 128–160; arXiv:1603.08133.

[GV] X. García-Martínez and T. Van der Linden, *A characterisation of Lie algebras via algebraic exponentiation*, Adv. Math. **341** (2019), 92–117; arXiv:1711.00689.

[GZ]   A. Grishkov and P. Zusmanovich, *Deformations of current Lie algebras. I. Small algebras in characteristic* 2, J. Algebra **473** (2017), 513–544; arXiv:1410.3645.

[JW]   M. Jöllenbeck and V. Welker, *Minimal Resolutions via Algebraic Discrete Morse Theory*, Memoirs Amer. Math. Soc. **197** (2009), no.923.

[J]   G. Jurman, *A family of simple Lie algebras in characteristic two*, J. Algebra **271** (2004), no.2, 454–481.

[KL]   C. Kassel and J.-L. Loday, *Extensions centrales d'algèbres de Lie*, Ann. Inst. Fourier **32** (1982), no.4, 119–142.

[L]   J. Levine, *Labeled binary planar trees and quasi-Lie algebras*, Algebr. Geom. Topol. **6** (2006), no.2, 935–948; arXiv:math/0504278.

[LP]   J.-L. Loday and T. Pirashvili, *Universal enveloping algebras of Leibniz algebras and (co)homology*, Math. Ann. **296** (1993), no.1, 139–158.

[S1]   E. Sköldberg, *The homology of Heisenberg Lie algebra over fields of characteristic two*, Math. Proc. Roy. Irish Acad. **105A** (2005), no.2, 47–49; arXiv:math/0312124.

[S2]   ———— , *Morse theory from an algebraic viewpoint*, Trans. Amer. Math. Soc. **358** (2006), no.1, 115–129.

[Wa]   F. Wagemann, *Spectral sequences for commutative Lie algebras*, Comm. Math., to appear; arXiv:1908.06764.

[We]   C. Weibel, *An Introduction to Homological Algebra*, Cambridge Univ. Press, 1994.

[Z1]   P. Zusmanovich, *Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds*, Lin. Algebra Appl. **407** (2005), 71–104; arXiv:math/0302334.

[Z2]   ———— , *How Euler would compute the Euler-Poincaré characteristic of a Lie superalgebra*, Expos. Math. **29** (2011), no.3, 345–360; arXiv:0812.2255.

[Z3]   ———— , *Special and exceptional mock-Lie algebras*, Lin. Algebra Appl. **518** (2017), 79–96; arXiv:1608.05861.

Software and online repositories

[G]   The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.7.8, 2015; https://www.gap-system.org/

[OEIS] The On-Line Encyclopedia of Integer Sequences; http://oeis.org/

(Viktor Lopatkin) Laboratory of Modern Algebra and Applications, St. Petersburg State University and St. Petersburg Department of Steklov Mathematical Institute, St. Petersburg, Russia

Email address: wickktor@gmail.com

(Pasha Zusmanovich) University of Ostrava, Ostrava, Czech Republic

Email address: pasha.zusmanovich@gmail.com