

DEFORMATIONS OF $W_1(n) \otimes A$ AND MODULAR SEMISIMPLE LIE ALGEBRAS WITH A SOLVABLE MAXIMAL SUBALGEBRA

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ABSTRACT. In one of his last papers, Boris Weisfeiler proved that if a modular semisimple Lie algebra possesses a solvable maximal subalgebra which defines in it a long filtration, then the associated graded algebra is isomorphic to the one constructed from the Zassenhaus algebra tensored with the divided powers algebra. We completely determine such class of algebras, calculating in the process low-dimensional cohomology groups of the Zassenhaus algebra tensored with any associative commutative algebra.

INTRODUCTION

The ultimate goal of this paper is to describe semisimple finite-dimensional Lie algebras over an algebraically closed field of characteristic $p > 5$, having a solvable maximal subalgebra (that is, a maximal subalgebra which is solvable) which determines a “long filtration”, as defined below. We hope, however, that some intermediate results contained here are of independent interest.

Simple Lie algebras with a solvable maximal subalgebra were described by B. Weisfeiler [W]. We depend heavily on his results and to a certain extent this paper may be considered as a continuation of the Weisfeiler’s paper. Since its appearance the classification of modular simple Lie algebras has been completed (announced in [SW] and elaborated in a series of papers among which [Str] is the last one), and recently the approach to the classification problem has been reworked in a series of papers among which [PS] is the latest, including the low characteristic cases.

Though the classification provides a powerful tool for solutions of many problems in modular Lie algebras theory, the question considered here remains non-trivial even modulo this classification. Moreover, we hope that the result we obtain here in particular, and the cohomological technique we use to prove it in general, may simplify to certain degree the classification itself.

Let us recall the contents of Weisfeiler’s paper. He considers semisimple modular Lie algebra \mathfrak{L} with a solvable maximal subalgebra \mathfrak{L}_0 . \mathfrak{L}_0 defines a filtration in \mathfrak{L} via $\mathfrak{L}_{i+1} = \{x \in \mathfrak{L}_i \mid [x, \mathfrak{L}] = \mathfrak{L}_i\}$ (though in general the filtration can be prolonged also to the negative side, in the case under consideration we can let $\mathfrak{L}_{-1} = \mathfrak{L}$).

When the term \mathfrak{L}_1 of this filtration does not vanish, the filtration is called *long*, otherwise it is called *short*. Weisfeiler proved that when the filtration is long, the associated graded algebra is isomorphic to $S \otimes O_m + 1 \otimes \mathfrak{D}$, where S coincides either with $sl(2)$, the three-dimensional simple algebra, or with $W_1(n)$, the Zassenhaus algebra, O_m is the reduced polynomial ring in m variables, and \mathfrak{D} is a derivation algebra of O_m . The grading is “thick” in the sense that it is completely determined by the standard grading of $W_1(n)$ or $sl(2)$, containing therefore the whole tensor factor O_m in each component. In the short filtration case, Weisfeiler proved that the initial algebra \mathfrak{L} possesses a \mathbb{Z}_p -grading with very restrictive conditions. Then, considering the case of simple \mathfrak{L} , he derived that in the long filtration case, \mathfrak{L} is isomorphic either to $sl(2)$ or to $W_1(n)$ (in fact, this follows immediately from the results of Kuznetsov [K], which are also important for us here), and the short filtration case does not occur.

Here we study the long filtration case. We determine all filtered algebras whose associated graded algebra is $W_1(n) \otimes O_m + 1 \otimes \mathfrak{D}$ or $sl(2) \otimes O_m + 1 \otimes \mathfrak{D}$ with the above-mentioned “thick” grading. This is done in the framework of the deformation theory due to Gerstenhaber. In this theory the

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second cohomology group of a Lie algebra with coefficients in the adjoint module plays a significant role (for an excellent account of this subject, see [GS]). As it turns out that the “tail” $1 \otimes \mathfrak{D}$ is not important in these considerations, one needs to compute $H^2(W_1(n) \otimes O_m, W_1(n) \otimes O_m)$. In a (slightly) more general setting, we compute $H^2(W_1(n) \otimes A, W_1(n) \otimes A)$ for an arbitrary associative commutative algebra A with unit. ($H^2(sl(2) \otimes A, sl(2) \otimes A)$) was earlier computed by Cathelineau [C]). This calculation seems sufficiently interesting for its own sake, as a nontrivial example of the low-dimensional cohomology of current Lie algebras $L \otimes A$. This also may be considered as a complement to the Cathelineau’s computation of the second cohomology group of the current Lie algebra $\mathfrak{g} \otimes A$ extended over a classical simple Lie algebra \mathfrak{g} , as well as generalization of Dzhumadil’daev-Kostrikin computations of $H^2(W_1(n), W_1(n))$ [DK].

The knowledge of $H^2(W_1(n) \otimes O_m, W_1(n) \otimes O_m)$ allows to solve the problem of determining all filtered algebras associated with the graded structure mentioned above. The answer is not very surprising – all such algebras have a socle isomorphic to $W_1(k) \otimes O_l$ for some k and l .

The contents of this paper are as follows. §1 contains some preliminary material, the most significant of which is a representation of $W_1(n)$ as a deformation of $W_1(1) \otimes O_{n-1}$, due to Kuznetsov [K]. It turns out that it is much easier to perform cohomological calculations using this representation. Following Kuznetsov, we define a class of Lie algebras $\mathfrak{L}(A, D)$ which are certain deformations of $W_1(1) \otimes A$ defined by means of a derivation D of A . Then we compute $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ in two steps: first, in §2, we compute $H^2(W_1(1) \otimes A, W_1(1) \otimes A)$, and then, in §3, we determine $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$, using a spectral sequence abutting to $H^*(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ with the E_1 -term isomorphic to $H^*(W_1(1) \otimes A, W_1(1) \otimes A)$.

Parallel to the results for the second cohomology group, we state similar results for the first cohomology group, as well as for the second cohomology group with trivial coefficients, which are useful later, in §5.

In §4, using the Kuznetsov’s isomorphism, we transform the results about $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ into those about $H^2(W_1(n) \otimes A, W_1(n) \otimes A)$. This section contains also all necessary computations related to reduced polynomial rings, particularly, of their Harrison cohomology. After that, in §5 we formulate a theorem about filtered deformations of $W_1(n) \otimes A + 1 \otimes \mathfrak{D}$ and of $sl(2) \otimes A + 1 \otimes \mathfrak{D}$ and derive it almost immediately from preceding results. It turns out that each such deformation strictly related to the class $\mathfrak{L}(A, D)$ (for a different A), so in §6 we determine all semisimple algebras in this class up to isomorphism, completing therefore the consideration of the long filtration case (Theorem 6.4).

Since the present paper is overloaded with different kinds of computations, we omit some of them which are similar to those already presented, or just too tedious. We believe that this will not cause inconvenience to the reader.

1. PRELIMINARIES

In this section we recall all necessary notions, notation, definitions, results and theories, as well as define a class of algebras $\mathfrak{L}(A, D)$ important for further considerations.

The ground field K is assumed to be of characteristic $p > 3$, unless otherwise is stated explicitly. (When appealing to Weisfeiler’s results, we have to assume the ground field is algebraically closed of characteristic $p > 5$).

As we deal with modular Lie algebras, it is not surprising that the divided powers algebra $O_1(n)$ plays a significant role in our considerations. Recall that $O_1(n)$ is the commutative associative algebra with basis $\{x^i \mid 0 \leq i < p^n\}$ and multiplication $x^i x^j = \binom{i+j}{j} x^{i+j}$. It is isomorphic to the reduced polynomial ring $O_n = K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$, the isomorphism is given by

$$(1.1) \quad x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \mapsto \alpha_1! \alpha_2! \dots \alpha_n! x^{\alpha_1 + p\alpha_2 + \dots + p^{n-1}\alpha_n}.$$

The subalgebra $\{x^i \mid 1 \leq i < p^n\}$, denoted as $O_1(n)^+$ (or O_n^+) is the single maximal ideal of $O_1(n)$. The invertible elements of $O_1(n)$ are exactly those not lying in $O_1(n)^+$.

The Zassenhaus algebra $W_1(n)$ is the Lie algebra of derivations of $O_1(n)$ of the kind $u\partial$, where $u \in O_1(n)$ and $\partial(x^j) = x^{j-1}$. It possesses a basis $\{e_i = x^{i+1}\partial \mid -1 \leq i \leq p^n - 2\}$ with bracket

$$[e_i, e_j] = N_{ij}e_{i+j}, \quad \text{where } N_{ij} = \binom{i+j+1}{j} - \binom{i+j+1}{i}.$$

The grading $W_1(n) = \bigoplus_{i=-1}^{p^n-2} Ke_i$ is called *standard*. In the case $n = 1$ it coincides with the root space decomposition relative to the action of the semisimple element e_0 .

Notice the following properties of the coefficients N_{ij} :

- (1) $N_{ij} = 0$ if $-1 \leq i, j \leq p-2, i+j \geq p-2$
- (2) $N_{ij} = N_{i-1,j} + N_{i,j-1}$
- (3) $N_{i,j-p} = N_{ij}$ if $-1 \leq i \leq p-2 \leq j$

The first two are obvious, the third may be found, for example, in [DK]. Notice also that if $0 \leq i \leq j \leq p$, then $\binom{p-i}{p-j} = (-1)^{j-i} \binom{j-1}{i-1}$ and if $i = \sum_{n \geq 0} i_n p^n, j = \sum_{n \geq 0} j_n p^n$ are p -adic decompositions, then

$$\binom{i}{j} = \prod_{n \geq 0} \binom{i_n}{j_n}$$

(the latter is known as Lucas' theorem).

The derivation algebra $Der(W_1(n))$ is generated (linearly) by inner derivations of $W_1(n)$ and derivations $(ad e_{-1})^{p^t}, 1 \leq t \leq n-1$ (so the latter constitute a basis of $H^1(W_1(n), W_1(n))$ (cf. [B1] or [D2]).

The whole derivation algebra of $O_1(n) \simeq O_n$, known as a general Lie algebra of Cartan type W_n , is freely generated as $O_1(n)$ -module by $\{\partial^{p^i} \mid 0 \leq i \leq n-1\}$ (or, in terms of O_n , by $\{\partial/\partial x_i \mid 1 \leq i \leq n\}$) (cf. [BIO], [K], or [W]).

Let L be a Lie algebra and A an associative commutative algebra with unit. The Lie structure on the tensor product $L \otimes A$ is defined via $[x \otimes a, y \otimes b] = [x, y] \otimes ab$. If \mathfrak{D} is a subalgebra of $Der(A)$, then $L \otimes A + 1 \otimes \mathfrak{D}$ is defined as a semidirect product where $1 \otimes \mathfrak{D}$ acts on $L \otimes A$ by $[x \otimes a, 1 \otimes d] = x \otimes d(a)$. For $a \in A$, R_a stands for the multiplication on a in A .

We will need the following elementary results.

Proposition 1.1.

- (i) $Z(L \otimes A) = Z(L) \otimes A$
- (ii) $(1 \otimes Der(A)) \cap ad(L \otimes A) = 0$.

Proof. (i) Obviously $Z(L) \otimes A \subseteq Z(L \otimes A)$. Let $\sum z_i \otimes a_i \in Z(L \otimes A)$. We may assume that all a_i are linearly independent. Then

$$[\sum z_i \otimes a_i, x \otimes 1] = \sum [z_i, x] \otimes a_i = 0$$

for every $x \in L$, which together with our assumption implies $z_i \in Z(L)$ for all i .

(ii) Let $1 \otimes d = \sum ad x_i \otimes a_i \in (1 \otimes Der(A)) \cap ad(L \otimes A)$. Applying it to $y \otimes 1$, we get $\sum [y, x_i] \otimes a_i = 0$ whence $x_i \in Z(L)$ for all i , and $d = 0$. ■

Definition. Let $D \in Der(A)$. Define $\mathfrak{L}(A, D)$ to be a Lie algebra with the underlying vector space $W_1(1) \otimes A$ and Lie bracket $\{x, y\} = [x, y] + \Phi_D(x, y)$, where $[\cdot, \cdot]$ is the ordinary bracket on $W_1(1) \otimes A$, and

$$(1.2) \quad \Phi_D(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (aD(b) - bD(a)), & i = j = -1 \\ 0, & \text{otherwise} \end{cases}$$

for $a, b \in A$.

Remark (Referee). In [Re], Ree considered a class of Lie algebras which are subalgebras of $Der(B)$ for a commutative associative algebra B (freely) generated over B . The algebras $\mathfrak{L}(A, D)$ belong to this class. Indeed, let $D \in Der(A)$ and consider $d = \partial \otimes 1 + x^{p-1} \otimes D \in Der(O_1 \otimes A)$. One can easily see, by identifying O_1 and $W_1(1)$ as vector spaces via $x^i \mapsto e_{i-1}$, $0 \leq i < p$, that $\mathfrak{L}(A, D)$ is nothing else than the Lie algebra of derivations of $O_1 \otimes A$ of the form $\{bd \mid b \in O_1 \otimes A\}$ (i.e. freely generated, as a module over $O_1 \otimes A$, by a single derivation d).

The following is crucial for our considerations.

Proposition 1.2 (Kuznetsov). $W_1(n) \otimes A \simeq \mathfrak{L}(O_1(n-1) \otimes A, \partial \otimes 1)$.

Proof. This obviously follows from the isomorphism $W_1(n) \simeq \mathfrak{L}(O_1(n-1), \partial)$, noticed in [K]. A direct calculation shows that the mapping

$$e_{pk+i} \mapsto e_i \otimes x^k, \quad -1 \leq i \leq p-2, \quad 0 \leq k < p^{n-1}$$

provides the isomorphism desired. ■

The reason why we prefer to deal with such realization of $W_1(n)$ lies in the fact that $e_0 \otimes 1$ remains a semisimple element in $\mathfrak{L}(A, D)$ with root spaces $e_i \otimes A$. So we obtain the grading of length p , and not of length p^n as in the case of $W_1(n) \otimes A$. The significance of the "good" (=short) root space decomposition follows from the well-known theorem about the invariance of the Lie algebra cohomology under the torus action.

Introduce a filtration

$$(1.3) \quad \mathfrak{L}(A, D) = \mathfrak{L}_{-1} \supset \mathfrak{L}_0 \supset \mathfrak{L}_1 \supset \cdots \supset \mathfrak{L}_{p-2}$$

by $\mathfrak{L}_i = \bigoplus_{j \geq i} e_j \otimes A$.

In general, for a given decreasing filtration $\{\mathfrak{L}_i\}$ of a Lie algebra \mathfrak{L} , $gr \mathfrak{L} = \bigoplus_i \mathfrak{L}_i / \mathfrak{L}_{i+1}$ will denote its associated graded algebra.

The following is evident.

Proposition 1.3. *The graded Lie algebra $gr \mathfrak{L}(A, D)$ associated with filtration (1.3), is isomorphic to $W_1(1) \otimes A = \bigoplus_{i=-1}^{p-2} e_i \otimes A$.*

This is the place where deformation theory enters the game. It is known that each filtered algebra can be considered as a deformation of its associated graded algebra $L = \bigoplus L_i$ (for this fact as well as for all necessary background in the deformation theory we refer to [GS]). One calls such deformations *filtered deformations* (or $\{L_i\}$ -deformations in the terminology of [DK]). As the space of infinitesimal deformations coincides with the cohomology group $H^2(L, L)$, the space of infinitesimal filtered deformations coincides with its subgroup $H_+^2(L, L) = \{\bar{\phi} \in H^2(L, L) \mid \phi(L_i, L_j) \subset \bigoplus_{k \geq 1} L_{i+j+k}\}$. To describe all filtered deformations, one needs to investigate prolongations of infinitesimal ones, obstructions to which are described by Massey products $[\phi, \psi] \in H^3(L, L)$ defined as

$$[\phi, \psi](x, y, z) = \phi(\psi(x, y), z) + \psi(\phi(x, y), z) + \curlywedge.$$

(this product arises from the graded Lie (super)algebra structure on $H^*(L, L)$).

We formulate just a small part of this broad subject needed for our purposes.

Proposition 1.4 (cf. [GS] or direct verification). *Let L be a finitely graded Lie algebra such that Massey product of any two elements of $Z_+^2(L, L)$ is zero. Then any filtered Lie algebra \mathfrak{L} such that $gr \mathfrak{L} \simeq L$ (as graded algebras), is isomorphic to a Lie algebra with underlying vector space L and Lie bracket $\{\cdot, \cdot\} = [\cdot, \cdot] + \Phi$ for some $\Phi \in Z_+^2(L, L)$.*

Note in that connection that $[\Phi_D, \Phi_D] = 0$. We will see later that this holds also for other "positive" 2-cocycles on $W_1(1) \otimes A$ (and more generally, on $W_1(n) \otimes A$), so Proposition 1.4 will be applicable in our situation.

Now we formulate the Weisfeiler's main result [W]:

Theorem 1.5 (Weisfeiler). (*The ground field K is algebraically closed of characteristic $p > 5$*).

Let \mathfrak{L} be a semisimple Lie algebra with a solvable maximal subalgebra \mathfrak{L}_0 . Suppose that \mathfrak{L}_0 defines a long filtration in \mathfrak{L} . Then \mathfrak{L} is a filtered deformation of a graded Lie algebra $L = S \otimes O_m + 1 \otimes \mathfrak{D}$, where $S = sl(2)$ or $W_1(n)$ equipped with the standard grading $\bigoplus_i Ke_i$, $\mathfrak{D} \subset Der(O_m)$, and the graded components are:

$$L_i = \begin{cases} e_0 \otimes O_m + 1 \otimes \mathfrak{D}, & i = 0 \\ e_i \otimes O_m, & i \neq 0. \end{cases}$$

Further, the Harrison cohomology $Har^*(A, A)$ with coefficients in the adjoint module A plays a role in our considerations. Note that $Har^1(A, A) = Der(A)$ and Harrison 2-cocycles, denoted by $\mathcal{Z}^2(A, A)$, are just symmetrized Hochschild 2-cocycles (cf. [Ha] where this cohomology was introduced and [GS] for a more modern treatment). δ refers to the Harrison (=Hochschild) coboundary operator, i.e.

$$\begin{aligned} \delta G(a, b) &= aG(b) + bG(a) - G(ab) \\ \delta F(a, b, c) &= aF(b, c) - F(ab, c) + F(a, bc) - F(a, b)c \end{aligned}$$

for $G \in Hom(A, A)$ and $F \in Hom(A \otimes A, A)$. The action of $Der(A)$ on $Har^2(A, A)$ is defined via

$$D \star F(a, b) = F(D(a), b) + F(a, D(b)) - D(F(a, b)).$$

The same formula defines the action of $Der(L)$ on the cohomology $H^2(L, L)$ of the Lie algebra L .

Considering the L -action on $H^*(L, L)$, the well-known fact says that if T is an abelian subalgebra relative to which L decomposes into a sum of eigenspaces $L = \bigoplus L_\alpha$, then one can decompose the complex into the sum of subcomplexes

$$C_\alpha^m = \{\phi \in C^m(L, L) \mid \phi(L_{\alpha_1}, \dots, L_{\alpha_n}) \subseteq L_{\alpha_1 + \dots + \alpha_n + \alpha}\}$$

and, moreover, $H^*(C_\alpha) = 0$ for $\alpha \neq 0$ (cf. [F], Theorem 1.5.2).

Similarly, any \mathbb{Z} -grading $L = \bigoplus L_i$ induces a \mathbb{Z} -grading on the cohomology group $H^*(L, L)$, as the initial complex $C^*(L, L)$ splits into the sum of subcomplexes $C_i^*(L, L)$, where

$$(1.4) \quad C_i^m(L, L) = \{\phi \in C^m(L, L) \mid \phi(L_{i_1}, \dots, L_{i_n}) \subseteq L_{i_1 + \dots + i_n + i}\}.$$

The cocycles, coboundaries and cohomology of these subcomplexes form the modules denoted by $Z_i^n(L, L)$, $B_i^n(L, L)$ and $H_i^n(L, L)$ respectively. If there is an element $e \in L$ whose action on L_i is multiplication by i then $H^*(C_i) = 0$ for $i \neq 0 \pmod{p}$.

The symbol \curvearrowright after an expression refers to the sum of all cyclic permutations (in $S(3)$) of letters and indices occurring in that expression.

2. LOW-DIMENSIONAL COHOMOLOGY OF $W_1(1) \otimes A$

The aim of this section is to establish the following isomorphisms.

Proposition 2.1.

- (i) $H^1(W_1(1) \otimes A, W_1(1) \otimes A) \simeq Der(A)$
- (ii) $H^2(W_1(1) \otimes A, W_1(1) \otimes A)$
 $\simeq H^2(W_1(1), W_1(1)) \otimes A \oplus Der(A) \oplus Der(A) \oplus Har^2(A, A).$

Before beginning the proof, let us make several remarks.

Part (i) follows from [B2], Theorem 7.1 (formulated in terms of derivation algebras). Alternatively, one may prove it in a similar (and much easier) way as (ii). Perhaps it should be remarked only that the basic 1-cocycles on $W_1(1) \otimes A$ can be given as $1 \otimes D$ for $D \in Der(A)$.

So we will concentrate our attention on (ii).

The cohomology group $H^2(W_1(n), W_1(n))$ was computed in [DK]. Particularly, $\dim H^2(W_1(1), W_1(1)) = 1$ and the single basic cocycle can be chosen as:

$$(2.1) \quad \phi(e_i, e_j) = \begin{cases} N_{ij}/p \cdot e_{i+j-p}, & i+j \geq p-1 \\ 0, & \text{otherwise} \end{cases}$$

where N_{ij}/p denotes a (well defined) element of the field K which is obtained from N_{ij} by division by p and further reduction modulo p .

The appearance of the first and last terms in (ii) is evident: the corresponding parts of the cohomology group are spanned by the classes of cocycles

$$(2.2) \quad \Theta_{\phi, u} : x \otimes a \wedge y \otimes b \mapsto \phi(x, y) \otimes abu$$

$$(2.3) \quad \Upsilon_F : x \otimes a \wedge y \otimes b \mapsto [x, y] \otimes F(a, b)$$

respectively, where $\phi \in \text{Hom}(L \otimes L, L)$, $u \in A$, and $F \in \text{Hom}(A \otimes A, A)$. We will denote the cochains of type (2.2) with $u = 1$ as Θ_ϕ (so actually $\Theta_{\phi, u} = (1 \otimes R_u) \circ \Theta_\phi$).

We have the following simple proposition.

Proposition 2.2. *Let L be a Lie algebra which is not 2-step nilpotent. Then*

- (i) $(1 \otimes R_u) \circ \Theta_\phi \in Z^2(L \otimes A, L \otimes A)$ if and only if either $\phi \in Z^2(L, L)$ or $u = 0$
- (ii) $\Upsilon_F \in Z^2(L \otimes A, L \otimes A)$ if and only if $F \in Z^2(A, A)$.

Proof. We will prove the second part only, the first one is similar. The cocycle equation for Υ_F together with Jacobi identity gives

$$(2.4) \quad [[x, y], z] \otimes \delta F(a, c, b) + [[z, x], y] \otimes \delta F(a, b, c) = 0.$$

Since $[[L, L], L] \neq 0$ and $p \neq 3$, one may choose $x, y \in L$ such that $[[y, x], x] \neq 0$. Setting $z = x$, one gets $F \in Z^2(A, A)$. Conversely, the last condition implies (2.4). ■

It is possible to prove also that for any Lie algebra L these cocycles are cohomologically independent, whence $H^2(L \otimes A, L \otimes A)$ must contain $H^2(L, L) \otimes A$ and $\text{Har}^2(A, A)$ as direct summands.

Let us define now explicitly the remaining classes of basic cocycles: Φ_D is already defined by (1.2), and

$$(2.5) \quad \Psi_D(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j} \otimes (({}^{i+j+1}_j bD(a) - ({}^{i+j+1}_i aD(b))), & -2 < i+j < p-1 \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2.3. *For any $D \in \text{Der}(A)$, $\Psi_D, \Phi_D \in Z^2(W_1(1) \otimes A, W_1(1) \otimes A)$.*

Proof. We perform necessary calculations for Ψ_D , leaving the easier case of Φ_D to the reader (in fact, that Φ_D is a 2-cocycle on $W_1(1) \otimes A$ follows from the Jacobi identity in $\mathfrak{L}(A, D)$).

Isolating the coefficient of $e_{i+j+k} \otimes abD(c)$ in the cocycle equation for Ψ_D , we get

$$\begin{aligned} -N_{ij} \binom{i+j+k+1}{i+j} + N_{jk} \binom{i+j+k+1}{i} + N_{ki} \binom{i+j+k+1}{j} \\ + N_{i,j+k} \binom{j+k+1}{j} - N_{j,k+i} \binom{k+i+1}{i} = 0. \end{aligned}$$

The last relation can be verified immediately. ■

The element $e_0 \otimes 1$ acts semisimply on $W_1(1) \otimes A$, as well as on $\mathfrak{L}(A, D)$. The roots of $ad(e_0 \otimes 1)$ -action lie in the prime subfield and the root spaces are:

$$(2.6) \quad L_{[i]} = e_i \otimes A, \quad [i] \in \mathbb{Z}_p, \quad -1 \leq i \leq p-2.$$

Thus any cocycle in $Z^2(W_1(1) \otimes A, W_1(1) \otimes A)$ is cohomologous to a cocycle in

$$(2.7) \quad Z_{[0]}^2(W_1(1) \otimes A, W_1(1) \otimes A) = Z_{-p}^2 \oplus Z_0^2 \oplus Z_p^2$$

as noted above.

Lemma 2.4. *Let $\{u_i\}$ be linearly independent elements of A , $\{D_i\}$ be linearly independent derivations of A , $\{F_i\}$ be cohomologically independent cocycles in $Z^2(A, A)$.*

Then cocycles $(1 \otimes R_{u_i}) \circ \Theta_\phi$, Ψ_{D_i} , Υ_{F_i} , Φ_{D_i} (defined in (2.2), (2.5), (2.3) and (1.2) respectively), are cohomologically independent.

Proof. As the cocycles of the first type belong to the $(-p)$ th component of $Z^2(W_1(1) \otimes A, W_1(1) \otimes A)$, the cocycles of the second and third type – to the zero component, the cocycles of the fourth type – to the p th component, and the degree of any coboundary is in the range between $1 - p$ and $p - 1$, one needs only to show the independence of cocycles of the form Ψ_{D_i} and Υ_{F_i} .

Suppose that there is a linear combination of the above-mentioned cocycles equal to a coboundary $d\omega$. Clearly this condition can be written as

$$(2.8) \quad \Psi_D + \Upsilon_F = d\omega$$

where D, F are some linear combinations of D_i 's and F_i 's, respectively.

Due to the $e_0 \otimes 1$ -action on $\mathfrak{L}(A, D)$, we may assume that ω preserves the root space decomposition (2.6), i.e.

$$\omega(e_i \otimes a) = e_i \otimes X_i(a)$$

for some $X_i \in \text{Hom}(A, A)$.

Evaluating the left and right sides of (2.8) for the pair $e_0 \otimes a, e_0 \otimes 1$, one gets $D = 0$. Then (2.8) reduces to

$$(2.9) \quad F(a, b) = aX_j(b) + bX_i(a) - X_{i+j}(ab)$$

for all i, j such that $N_{ij} \neq 0$.

Substituting in (2.9) $j = 0$ and using the symmetry of F , we get $F = \delta X_0$. Since F is a linear combination of cohomologically independent Harrison cocycles, $F = 0$. We see that all elements entering (2.8) vanish, whence all coefficients in the initial linear combinations of cocycles are equal to zero. ■

Now, to prove Proposition 2.1(ii), one merely needs to show that each cocycle $\phi \in Z_{[0]}^2(W_1(1) \otimes A, W_1(1) \otimes A)$ is cohomologous to the sum of the previous cocycles.

Let

$$(2.10) \quad \phi = \phi_{-p} + \phi_0 + \phi_p, \quad \phi_k(e_i \otimes A, e_j \otimes A) \subseteq e_{i+j+k} \otimes A, \quad k = -p, 0, p$$

be a decomposition corresponding to (2.7). It is immediate that $d\phi = 0 \iff d\phi_{-p} = d\phi_0 = d\phi_p = 0$.

The next three lemmas elucidate the form of cocycles $\phi_{-p}, \phi_0, \phi_p$ respectively. Two of them are formulated in a slightly more general setting which will be used later, in §3.

Lemma 2.5. $\phi_{-p} = (1 \otimes R_u) \circ \Theta_\phi$ for some $u \in A$.

Proof. Write

$$\phi_{-p}(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j-p} \otimes X_{ij}(a, b), & i + j \geq p - 1 \\ 0, & \text{otherwise} \end{cases}$$

for certain $X_{ij} \in \text{Hom}(A \otimes A, A)$. Writing the cocycle equation for triples $e_i \otimes a, e_j \otimes b, e_{-1} \otimes 1$ and $e_i \otimes a, e_j \otimes 1, e_0 \otimes c$, one obtains respectively:

$$(2.11) \quad X_{ij}(a, b) = X_{i-1, j}(a, b) + X_{i, j-1}(a, b), \quad i + j > p - 1$$

$$(2.12) \quad X_{ij}(a, c) = \frac{i+j}{j} cX_{ij}(a, 1) - \frac{i}{j} X_{ij}(ac, 1), \quad i + j \geq p - 1$$

The last equality in the case $i + j = p$ entails

$$X_{ij}(a, c) = X_{ij}(ac, 1), \quad i + j = p.$$

Now the cocycle equation for the triple $e_i \otimes a, e_j \otimes b, e_1 \otimes 1, i + j = p - 1, i, j \neq 1$ implies

$$X_{ij}(a, b) = -N_{1,i}(X_{i+1,j}(a, b) + X_{i,j+1}(a, b)), \quad i + j = p - 1.$$

Substitution of the last but one equality into the last one yields

$$(2.13) \quad X_{ij}(a, b) = Y_{ij}(ab), \quad i + j = p - 1, \quad i, j \neq 1$$

where $Y_{ij} = -N_{1,i}(X_{i+1,j}(a, 1) + X_{i,j+1}(a, 1))$. Substituting this in its turn, in (2.12) (with $i + j = p - 1$), one gets

$$Y_{ij}(ac) = cY_{ij}(a)$$

which implies $Y_{ij}(a) = au_{ij}$ for some $u_{ij} \in A$. Hence

$$X_{ij}(a, b) = abu_{ij}, \quad i + j = p - 1, \quad i, j \neq 1.$$

Writing the cocycle equation for the triple $e_1 \otimes a, e_1 \otimes 1, e_{p-2} \otimes 1$, one obtains

$$X_{1,p-2}(a, 1) = aX_{1,p-2}(1, 1).$$

Substituting this in (2.12) under the particular case $i = 1, j = p - 2$, one deduces (2.13) also in this case, with $u_{1,p-2} = X_{1,p-2}(1, 1)$. Then writing the cocycle equation for triple $e_i \otimes 1, e_j \otimes 1, e_1 \otimes 1, i + j = p - 2, i, j \neq 0$, and taking into account (2.13), one obtains

$$N_{1,i}u_{i+1,j} + N_{1,j}u_{i,j+1} - N_{ij}u_{1,p-2} = 0, \quad i + j = p - 2, \quad i, j \neq 0.$$

The last relation for $i = 2, 3, \dots, p - 4$ ($i = 1$ and $p - 3$ give trivial relations) together with the equality $u_{\frac{p-1}{2}, \frac{p-1}{2}} = 0$ (which follows from (2.13)) gives $p - 5$ equations for $p - 5$ unknowns $u_{2,p-3}, \dots, u_{p-3,2}$. One easily checks that

$$u_{ij} = N_{ij}/p \cdot u, \quad i + j = p - 1$$

for a certain $u \in A$ (actually, $u = -\frac{2}{3}u_{1,p-2}$), provides a unique solution.

With the aid of (2.11) this equality can be extended to all $i, j, i + j \geq p - 1$. ■

Lemma 2.6. *Let $d\phi_0 = \xi$, where $\phi_0 \in C_0^2(W_1(1) \otimes A, W_1(1) \otimes A)$ and $\xi \in C_0^3(W_1(1) \otimes A, W_1(1) \otimes A)$ such that $\xi(e_i \otimes a, e_j \otimes b, e_k \otimes c)$ is (possibly) nonzero only when one of the indices i, j, k is equal to -1 and the sum of two others is equal to $p - 1$.*

Then $\xi = 0$ and ϕ_0 is a cocycle which is cohomologous to $\Upsilon_F + \Psi_D$ for some $F \in \mathcal{Z}^2(A, A)$ and $D \in \text{Der}(A)$.

Proof. Write

$$\phi_0(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j} \otimes X_{ij}(a, b), & -2 < i + j < p - 1 \\ 0, & \text{otherwise.} \end{cases}$$

Define $\omega \in C^1(W_1(1) \otimes A, W_1(1) \otimes A)$ as follows:

$$\omega(e_{-1} \otimes a) = 0, \quad \omega(e_i \otimes a) = e_i \otimes \sum_{j=0}^i X_{-1,j}(1, a), \quad i \geq 0.$$

Then $d\omega(e_{-1} \otimes 1, e_i \otimes a) = e_{i-1} \otimes X_{-1,i}(1, a) = \phi_0(e_{-1} \otimes 1, e_i \otimes a)$ and replacing ϕ_0 by $\phi_0 - d\omega$ (without changing the notation), one can assume that

$$(2.14) \quad X_{-1,j}(1, a) = 0.$$

Writing the equation $d\phi_0 = \xi$ for the triple $e_{-1} \otimes a, e_{-1} \otimes b, e_i \otimes c$, one obtains

$$(2.15) \quad X_{-1,i-1}(b, ac) - X_{-1,i-1}(a, bc) = aX_{-1,i}(b, c) - bX_{-1,i}(a, c).$$

Setting here $b = 1$ and using (2.14), one gets $X_{-1,i-1}(b, c) = X_{-1,i}(b, c)$, which implies

$$(2.16) \quad X_{-1,i}(a, b) = X_{-1,0}(a, b).$$

Together with (2.15) this gives

$$(2.17) \quad X_{-1,0}(b, ac) - X_{-1,0}(a, bc) - aX_{-1,0}(b, c) + bX_{-1,0}(a, c) = 0.$$

Writing the equation $d\phi_0 = \xi$ for the triple $e_i \otimes a, e_j \otimes b, e_{-1} \otimes c, i + j \leq p - 2$, one obtains

$$(2.18) \quad N_{ij}X_{-1,i+j}(c, ab) - X_{i-1,j}(ac, b) - X_{i,j-1}(a, bc) + cX_{ij}(a, b) \\ - N_{i,j-1}aX_{-1,j}(c, b) - N_{i-1,j}bX_{-1,i}(c, a) = 0, \quad i + j \leq p - 2, \quad i, j \geq 0.$$

Setting in the last equality $c = 1$, one gets

$$X_{ij}(a, b) = X_{i-1,j}(a, b) + X_{i,j-1}(a, b).$$

The last relation together with (2.16) permits to prove, by induction on $i + j$, the following equality:

$$(2.19) \quad X_{ij}(a, b) = \binom{i+j+1}{j} X_{-1,0}(a, b) - \binom{i+j+1}{i} X_{-1,0}(b, a)$$

Setting in (2.18) $i = j = 0$ and using the fact that $X_{00}(a, b) = X_{-1,0}(a, b) - X_{-1,0}(b, a)$ (which follows from (2.19)), one obtains

$$(2.20) \quad X_{-1,0}(bc, a) - X_{-1,0}(ac, b) - bX_{-1,0}(c, a) - cX_{-1,0}(b, a) \\ + cX_{-1,0}(a, b) + aX_{-1,0}(c, b) = 0.$$

Set

$$F(a, b) = \frac{1}{2}(X_{-1,0}(a, b) + X_{-1,0}(b, a) - X_{-1,0}(ab, 1)) = X_{-1,0}(b, a) - aX_{-1,0}(b, 1) \\ D(a) = X_{-1,0}(a, 1).$$

Using (2.17) and (2.20) it is easy to see that $F \in \mathcal{Z}^2(A, A)$ and $D \in \text{Der}(A)$, and hence (2.19) implies

$$X_{ij}(a, b) = N_{ij}F(a, b) + \binom{i+j+1}{j} bD(a) - \binom{i+j+1}{i} aD(b).$$

Thus ϕ_0 is a cocycle, whence $\xi = 0$. ■

Lemma 2.7. *Let $d\phi_p = \xi$, where $\phi_p \in C_p^2(W_1(1) \otimes A, W_1(1) \otimes A)$ and $\xi \in C_p^3(W_1(1) \otimes A, W_1(1) \otimes A)$ such that the only possibly nonzero values of ξ are given by*

$$\xi(e_{-1} \otimes a, e_{-1} \otimes b, e_0 \otimes c) = e_{p-2} \otimes (aG(b, c) - bG(a, c))$$

for some $G \in \text{Hom}(A \otimes A, A)$.

Then G is a Harrison 2-coboundary and $\phi_p = \Phi_D$ for some $D \in \text{End}(A)$. If $G = 0$, then $D \in \text{Der}(A)$.

Proof. Write

$$\phi_p(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes X(a, b), & i = j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

Obviously X is skew-symmetric. Writing the equation $d\phi_p = \xi$ for the triples $e_{-1} \otimes a, e_{-1} \otimes b, e_{-1} \otimes 1$ and $e_{-1} \otimes a, e_{-1} \otimes 1, e_0 \otimes b$, one gets respectively:

$$(2.21) \quad X(a, b) = aX(b, 1) - bX(a, 1)$$

and

$$(2.22) \quad -X(ab, 1) + X(b, a) + 2bX(a, 1) = aG(1, b) - G(a, b).$$

Setting $D(a) = X(1, a)$, we obtain $\phi_p = \Phi_D$. Substitution of (2.21) into (2.22) gives

$$G(a, b) - aG(1, b) = \delta D(a, b).$$

Symmetrizing the last equality, one gets

$$G(a, b) = \delta D(a, b) + abG(1, 1) = \delta(D + R_{G(1,1)})(a, b).$$

If $G = 0$ then $\delta D = 0$, i.e. $D \in \text{Der}(A)$. ■

This completes the proof of Proposition 2.1(ii).

Similar but more elementary computations can be utilized to prove

Proposition 2.8.

- (i) $H^1(\mathfrak{sl}(2) \otimes A, \mathfrak{sl}(2) \otimes A) \simeq \text{Der}(A)$
- (ii) $H^2(\mathfrak{sl}(2) \otimes A, \mathfrak{sl}(2) \otimes A) \simeq \text{Har}^2(A, A)$.

Proof. We refer to the paper of Cathelineau [C]. Though formally it contains a slightly different result – namely, the computation of $H_2(\mathfrak{g} \otimes A, \mathfrak{g} \otimes A)$ for classical simple Lie algebra \mathfrak{g} over a field of characteristic zero, the methods employed there can be easily adapted to our case. Alternatively, one may go along the lines of our proof for the case $W_1(1) \otimes A$. All basic cocycles turn out to be of the type (2.3). ■

3. LOW-DIMENSIONAL COHOMOLOGY OF $\mathfrak{L}(A, D)$

Theorem 3.1.

- (i) $H^2(\mathfrak{L}(A, D), K) \simeq (A^*)^D$
- (ii) $H^1(\mathfrak{L}(A, D), \mathfrak{L}(A, D)) \simeq \text{Der}(A)^D$
- (iii) $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D)) \simeq A^D \oplus \text{Der}(A)_D \oplus \text{Der}(A)^D \oplus \text{Har}^2(A, A)^D$.

All super- and subscripts here denote the kernel and cokernel respectively of the corresponding action of D (which is, for (i), given by $Df(a) \mapsto f(D(a))$ for $f \in A^*$, and for (ii) and (iii) is the standard action on Harrison (=Hochschild) cocycles described in §1).

Part (i) borrowed from [Z], where it is proved along the lines of the present paper (though the computations are easier).

We will give also an explicit basis of $H^1(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ and $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$.

There are at least three ways to compute the cohomology of deformed algebra knowing the cohomology of an initial one. The first way is the Coffee-Gerstenhaber lifting theory (cf. [GS]) which tells how to determine obstructions to lifting of cocycles on a Lie algebra L to its deformation \mathfrak{L} .

The second way is applicable when \mathfrak{L} is a filtered deformation of L , i.e. \mathfrak{L} is a filtered Lie algebra with descending filtration $\{\mathfrak{L}_i\}$ and $L = \text{gr}\mathfrak{L}$. One can define a descending filtration in the Chevalley-Eilenberg complex $C^*(\mathfrak{L}, \mathfrak{L})$:

$$C_i^n(\mathfrak{L}, \mathfrak{L}) = \{\phi \in C^n(\mathfrak{L}, \mathfrak{L}) \mid \phi(\mathfrak{L}_{i_1}, \dots, \mathfrak{L}_{i_n}) \subseteq \mathfrak{L}_{i_1 + \dots + i_n}\}.$$

Then the associated graded complex will be $C^*(L, L)$ with the grading defined by (1.4), and the general theory about filtered complexes says that there is a spectral sequence abutting to $H^*(\mathfrak{L}, \mathfrak{L})$ whose E_1 -term is $H^*(L, L)$.

The third way is applicable in a special situation when \mathfrak{L} is a "1-step" deformation of L , i.e. multiplication in \mathfrak{L} is given by

$$\{x, y\} = [x, y] + \phi(x, y)t$$

where $[\cdot, \cdot]$ is a multiplication in L and $\phi \in Z^2(L, L)$. Then we have three complexes defined on the underlying module $C^*(L, L)$: the first one responsible for the cohomology of L with differential d , the second one – with differential $b = [\cdot, \phi]$ (Massey bracket), and the third one is responsible for the cohomology of \mathfrak{L} with differential $b + d$. Moreover, the Jacobi identity for $\{\cdot, \cdot\}$ implies $bd + db = 0$. In this situation it is possible to define a double complex on $C^*(L, L)$ whose horizontal arrows are d and vertical ones are b . The total complex \mathcal{T} of this double complex is closely related to the Chevalley–Eilenberg complex $\mathcal{C} = C^*(\mathfrak{L}, \mathfrak{L})$ responsible for the cohomology of \mathfrak{L} . Namely, there is a surjection

$$\mathcal{T}^n = \bigoplus_{i=1}^n C^n(L, L) \rightarrow C^n(\mathfrak{L}, \mathfrak{L})$$

defined by the summation of all coordinates, whose kernel \mathcal{K} is closely related to the shifted complex $\mathcal{T}[-1]$. So one can determine the cohomology $H^n(\mathfrak{L}, \mathfrak{L})$ from the long exact sequence associated with the short exact sequence of complexes $0 \rightarrow \mathcal{K} \rightarrow \mathcal{T} \rightarrow \mathcal{C} \rightarrow 0$.

However, in our even more specific situation we will use the fourth method employing the special \mathbb{Z} -grading. Its advantage is that we will be able not only to determine $H^1(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ and $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ as modules, but also to find explicit expressions for cocycles.

As noted in §1, when considering the cohomology both of $W_1(1) \otimes A$ and $\mathfrak{L}(A, D)$, we may restrict our attention to a subcomplex preserving the \mathbb{Z}_p -grading of $W_1(1) \otimes A$:

$$C_{[0]}^n(W_1(1) \otimes A, W_1(1) \otimes A) = \bigoplus_{i \in \mathbb{Z}} C_{ip}^n(W_1(1) \otimes A, W_1(1) \otimes A).$$

Let d and d_D be the differentials in the Chevalley-Eilenberg complexes $C^*(W_1(1) \otimes A, W_1(1) \otimes A)$ and $C^*(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$, respectively. We obviously have

$$(3.1) \quad d_D = d + [\cdot, \Phi_D]$$

where $[\cdot, \cdot]$ denotes the graded Lie (super)algebra structure (Massey brackets) on $H^*(W_1(1) \otimes A, W_1(1) \otimes A)$.

Since $\Phi_D \in C_p^2(W_1(1) \otimes A, W_1(1) \otimes A)$, the bracket $b = [\cdot, \Phi_D]$ acts as a differential of bidegree $(1, p)$ on the bigraded module $C_*^*(W_1(1) \otimes A, W_1(1) \otimes A)$ (the first grading is the usual cohomology grading, the second one comes from the \mathbb{Z} -grading on $W_1(1) \otimes A$). Denoting for convenience the module $C_{ip}^n(W_1(1) \otimes A, W_1(1) \otimes A)$ as \widehat{C}_i^n , we have a double complex

$$\begin{array}{ccccccc} \dots & & \dots & & \dots & & \dots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \widehat{C}_1^0 & \xrightarrow{d} & \widehat{C}_1^1 & \xrightarrow{d} & \widehat{C}_1^2 & \xrightarrow{d} & \widehat{C}_1^3 \longrightarrow \dots \\ & & \uparrow b & & \uparrow b & & \uparrow b \\ & & \widehat{C}_0^0 & \xrightarrow{d} & \widehat{C}_0^1 & \xrightarrow{d} & \widehat{C}_0^2 \longrightarrow \dots \\ & & & & \uparrow b & & \uparrow b \\ & & & & \widehat{C}_{-1}^0 & \xrightarrow{d} & \widehat{C}_{-1}^1 \longrightarrow \dots \\ & & & & \uparrow & & \\ & & & & \dots & & \end{array}$$

In view of (3.1), the total complex of this double complex is exactly the Chevalley-Eilenberg complex computing the cohomology $H^*(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$. Therefore the first spectral sequence $\{E_r^{st}\}$ associated with it has the E_1 -term

$$E_1^{st} \simeq H_{ps}^{s+t}(W_1(1) \otimes A, W_1(1) \otimes A).$$

A necessary condition for $\widehat{C}_i^n \neq 0$ is that there exists a solution to $-1 \leq i_1 + \dots + i_n + ip \leq p - 2$ for $-1 \leq i_k \leq p - 2$. This implies the inequalities $-n + \frac{2n-1}{p} \leq i \leq 1 + \frac{n-2}{p}$ (so for $n = 1$, $i = 0$, for $n = 2$, $-1 \leq i \leq 1$, for $n = 3$, $-2 \leq i \leq 1$). Thus in each degree there is finite number of nonvanishing components and the spectral sequence converges to $H^{s+t}(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$.

The only possibly nonzero terms responsible for the cohomology of low degree are:

$$E_r^{01}, E_r^{-1,3}, E_r^{02}, E_r^{11}, E_r^{-2,5}, E_r^{-1,4}, E_r^{03}, E_r^{12}.$$

Hence the only possibly nonzero differentials affecting the values of $H^1(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ and $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ are:

$$\begin{aligned} d_1^{01} &: E_1^{01} \rightarrow E_1^{11} \\ d_1^{-1,3} &: E_1^{-1,3} \rightarrow E_1^{03} \\ d_1^{02} &: E_1^{02} \rightarrow E_1^{12} \\ d_2^{-1,3} &: E_2^{-1,3} \rightarrow E_2^{12}. \end{aligned}$$

Consequently,

$$(3.2) \quad \begin{aligned} E_\infty^{01} &= E_1^{01} = \text{Ker } d_1^{01} \\ E_\infty^{-1,3} &= E_3^{-1,3} = \text{Ker } d_2^{-1,3}; \quad E_2^{-1,3} = \text{Ker } d_1^{-1,3} \\ E_\infty^{02} &= E_2^{02} = \text{Ker } d_1^{02} \\ E_\infty^{11} &= E_2^{11} = E_1^{11} / \text{Im } d_1^{01} \end{aligned}$$

and $H^1(\mathfrak{L}(A, D), \mathfrak{L}(A, D)) \simeq E_\infty^{01}$, $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D)) \simeq E_\infty^{-1,3} \oplus E_\infty^{02} \oplus E_\infty^{11}$. Proposition 2.1 (strictly speaking, the explicit basic cocycles provided in its proof) yields

$$\begin{aligned} E_1^{01} &\simeq H_0^1(W_1(1) \otimes A, W_1(1) \otimes A) \simeq \text{Der}(A) \\ E_1^{-1,3} &\simeq H_{-p}^2(W_1(1) \otimes A, W_1(1) \otimes A) \simeq H^2(W_1(1), W_1(1)) \otimes A \simeq A \\ E_1^{02} &\simeq H_0^2(W_1(1) \otimes A, W_1(1) \otimes A) \simeq \text{Der}(A) \oplus \text{Har}^2(A, A) \\ E_1^{11} &\simeq H_p^2(W_1(1) \otimes A, W_1(1) \otimes A) \simeq \text{Der}(A). \end{aligned}$$

In the next lemmas we will determine all necessary kernels and images in (3.2).

Lemma 3.2. $E_\infty^{-1,3} \simeq A^D$.

Proof. In order to determine $\text{Ker } d_1^{-1,3}$, one needs to consider the equation

$$(3.3) \quad [(1 \otimes R_u) \circ \Theta_\phi, \Phi_D] = d\Lambda_u$$

for some $\Lambda_u \in C_0^2(W_1(1) \otimes A, W_1(1) \otimes A)$. The terms of $E_2^{-1,3}$ will be of the form $(1 \otimes R_u) \circ \Theta_\phi - \Lambda_u$ for appropriate solutions of (3.3).

Direct computations show

$$(3.4) \quad [(1 \otimes R_u) \circ \Theta_\phi, \Phi_D] = (1 \otimes R_u) \circ [\Theta_\phi, \Phi_D] + (1 \otimes R_{D(u)}) \circ \Gamma$$

where $\Gamma \in C^3(W_1(1) \otimes A, W_1(1) \otimes A)$ defined as (assuming $i \leq j \leq k$)

$$\Gamma(e_i \otimes a, e_j \otimes b, e_k \otimes c) = \begin{cases} e_{p-2} \otimes N_{jk}/p \cdot abc, & i = -1, j+k = p-1 \\ 0, & \text{otherwise} \end{cases}$$

and

$$(3.5) \quad [\Theta_\phi, \Phi_D](e_i \otimes a, e_j \otimes b, e_k \otimes c) = \begin{cases} e_{p-2} \otimes N_{jk}/p (bcD(a) - aD(bc)), & i = -1, j+k = p-1 \\ 0, & \text{otherwise.} \end{cases}$$

(Notice that $N_{jk}/p = (-1)^k \frac{2k+1}{k(k+1)}$ if $j+k = p-1$).

Define

$$\Theta'(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j} \otimes (\lambda_{ij} aD(b) - \lambda_{ji} bD(a)), & -2 < i+j < p-1 \\ 0, & \text{otherwise} \end{cases}$$

for some $\lambda_{ij} \in K$.

Let us verify first that there are λ_{ij} such that $\Theta_\phi + \Theta' \in E_2^{-1,3}$. Writing the equation (3.3) under conditions $u = 1$ and $\Lambda_u = -\Theta'$ for the triple $e_{-1} \otimes a, e_j \otimes b, e_k \otimes c, j+k = p-1$ and for all remaining cases, we obtain respectively:

$$(3.6) \quad \lambda_{j-1,k} + \lambda_{j,k-1} = (-1)^k \frac{2k+1}{k(k+1)}$$

$$(3.7) \quad \lambda_{j,k-1} - \lambda_{k,j-1} = 2(-1)^k \lambda_{j,-1} + 2(-1)^{k+1} \lambda_{k,-1} + (-1)^{k+1} \frac{2k+1}{k(k+1)}$$

where $j+k = p-1$, and

$$(3.8) \quad N_{ij} \lambda_{i+j,k} - N_{jk} \lambda_{i,j+k} + N_{ik} \lambda_{j,i+k} + N_{j+k,i} \lambda_{jk} - N_{i+k,j} \lambda_{ik} = 0, \quad i, j, k \geq 0, \quad i+j+k < p-1.$$

(the left-hand side in the latter is a basic expression for the coefficient of $abD(c)$ in $d\Theta'$).

Lemma 3.3.

$$\lambda_{ij} = \sum_{k=1}^i \binom{i+j+1-k}{j+1} \frac{k+2}{k(k+1)}, \quad -1 \leq i, j \leq p-2$$

provides solution for (3.6)–(3.8).

Proof. Note the following properties of the just defined coefficients λ_{ij} :

- (i) $\lambda_{-1,j} = \lambda_{0j} = 0; \quad \lambda_{1,j} = \frac{3}{2}$
- (ii) $\lambda_{i,-1} = \sum_{k=1}^i \frac{k+2}{k(k+1)}$
- (iii) $\lambda_{ij} = \lambda_{i-1,j} + \lambda_{i,j-1}$.

Now, (3.6) may be reformulated as

$$\lambda_{ij} = (-1)^j \frac{2j+1}{j(j+1)}, \quad i+j = p-1, \quad i, j \geq 1$$

which can be proved with the help of simple transformations of binomial coefficients in the spirit of the first few pages of [Ri].

(3.7) is proved by induction on j , using (3.6) in the induction step.

Finally, (3.8) is proved by induction on $i+j+k$. The induction step is:

$$\begin{aligned} & N_{ij} \lambda_{i+j,k} - N_{jk} \lambda_{i,j+k} + N_{ik} \lambda_{j,i+k} + N_{j+k,i} \lambda_{jk} - N_{i+k,j} \lambda_{ik} \\ &= N_{ij} \lambda_{i+j-1,k} + N_{ij} \lambda_{i+j,k-1} - N_{jk} \lambda_{i-1,j+k} - N_{jk} \lambda_{i,j+k-1} + N_{ik} \lambda_{j-1,i+k} \\ &+ N_{ik} \lambda_{j,i+k-1} + N_{j+k,i-1} \lambda_{jk} + N_{j+k-1,i} \lambda_{jk} - N_{i+k,j-1} \lambda_{ik} - N_{i+k-1,j} \lambda_{ik} \\ &= (N_{i-1,j} \lambda_{i+j-1,k} - N_{jk} \lambda_{i-1,j+k} + N_{i-1,k} \lambda_{j,i+k-1} + N_{j+k,i-1} \lambda_{jk} - N_{i+k-1,j} \lambda_{i-1,k}) \\ &+ (N_{i,j-1} \lambda_{i+j-1,k} - N_{j-1,k} \lambda_{i,j+k-1} + N_{ik} \lambda_{j-1,i+k} + N_{j+k-1,i} \lambda_{j-1,k} - N_{i+k,j-1} \lambda_{ik}) \\ &+ (N_{ij} \lambda_{i+j,k-1} - N_{j,k-1} \lambda_{i,j+k-1} + N_{i,k-1} \lambda_{j,i+k-1} + N_{j+k-1,i} \lambda_{j,k-1} - N_{i+k-1,j} \lambda_{i,k-1}) \\ &= 0 \end{aligned}$$

where the first equality follows from recurrent relations for λ_{ij} , the second one from those for N_{ij} , and the third one from the induction assumption for triples $(i-1, j, k)$, $(i, j-1, k)$ and $(i, j, k-1)$.

■

Continuation of the proof of Lemma 3.2. Now consider a general solution of (3.3). Taking into account (3.4), the partial solution $[\Theta_\phi, \Phi_D] = -d\Theta'$, and the commutativity of operators d and R_u , (3.3) can be rewritten as

$$d(\Lambda_u + (1 \otimes R_u) \circ \Theta') = (1 \otimes R_{D(u)}) \circ \Gamma.$$

By Lemma 2.6, $D(u) = 0$ and $\Lambda_u = -(1 \otimes R_u) \circ \Theta'$ up to elements from $Z_0^2(W_1(1) \otimes A, W_1(1) \otimes A)$. Hence $E_2^{-1,3}$ consists of elements of the form

$$\bar{\Theta}_u = (1 \otimes R_u) \circ (\Theta_\phi + \Theta'), \quad u \in A^D$$

and $E_2^{-1,3} \simeq A^D$.

To compute $\text{Ker } d_2^{-1,3}$, take a look at $\text{Im } d_2^{-1,3}$, i.e. on elements of the form $[(1 \otimes R_u) \circ (\Theta_\phi + \Theta'), \Phi_D], u \in A^D$. The latter expression is equal to $(1 \otimes R_u) \circ [\Theta', \Phi_D]$ up to elements from $B^3(W_1(1) \otimes A, W_1(1) \otimes A)$. Direct computations show

$$\begin{aligned} [\Theta', \Phi_D](e_i \otimes a, e_j \otimes b, e_k \otimes c) &= \begin{cases} e_{p-2} \otimes \lambda_{p-2,0}(aD(b) - bD(a))D(c), & i = j = -1, k = 0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

But

$$\lambda_{p-2,0} = -\sum_{k=1}^{p-2} \left(1 + \frac{2}{k}\right) = -(p-2) - 2 \sum_{k=1}^{p-1} k + \frac{2}{p-1} = 0.$$

Consequently, $d_2^{-1,3}$ is zero and $E_\infty^{-1,3} = E_3^{-1,3} = E_2^{-1,3} \simeq A^D$. ■

Lemma 3.4. $E_\infty^{02} \simeq \text{Der}(A)^D \oplus \text{Har}^2(A, A)^D$.

Proof. To determine $\text{Ker } d_p^{02}$, one needs to solve two equations

$$(3.9) \quad [\Psi_E, \Phi_D] = d\Lambda_E$$

$$(3.10) \quad [\Upsilon_F, \Phi_D] = d\Lambda_F$$

for some $\Lambda_E, \Lambda_F \in C_p^2(W_1(1) \otimes A, W_1(1) \otimes A)$. Let

$$\Psi'_E(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (D(a)E(b) - E(a)D(b)), & i = j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

By means of direct computations one gets

$$[\Psi_E, \Phi_D] = d\Psi'_E + \Gamma_E$$

where the only possibly nonzero values of Γ_E are given by

$$\Gamma_E(e_{-1} \otimes a, e_{-1} \otimes b, e_{-1} \otimes c) = e_{p-2} \otimes (a[E, D](b) - b[E, D](a))c.$$

But (3.9) implies that Γ_E is a coboundary $d(\Lambda_E - \Psi'_E)$. One easily checks that each coboundary vanishes on the triple $e_{-1} \otimes a, e_{-1} \otimes 1, e_0 \otimes 1$, which implies $[E, D] = 0$ and $\Gamma_E = 0$. Consequently, $\Lambda_E = \Psi'_E$ up to elements from $Z_p^2(W_1(1) \otimes A, W_1(1) \otimes A)$ and the set of elements $\{\Psi_E - \Psi'_E \mid E \in \text{Der}(A)^D\}$ embeds into E_2^{02} .

To solve equation (3.10), define

$$\Upsilon'_F(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (F(D(a), b) - F(a, D(b))), & i = j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

By means of direct computations one gets

$$(3.11) \quad [\Upsilon_F, \Phi_D] = d\Upsilon'_F + \Gamma_F$$

where the only possibly nonzero values of Γ_F are given by

$$\Gamma_F(e_{-1} \otimes a, e_{-1} \otimes b, e_0 \otimes c) = e_{p-2} \otimes (aD \star F(b, c) - bD \star F(a, c)).$$

By (3.11), $\Gamma_F = d(\Lambda_F - \Upsilon'_F)$. According to Lemma 2.7, $\Lambda_F - \Upsilon'_F = \Phi_H$ for some $H \in \text{End}(A)$ and moreover, $D \star F = \delta H$ (we may suppose that $D \star F(1, 1) = 0$).

Conversely, if $D \star F = \delta H$, then $\Gamma_F = -d\Phi_H$, which in view of (3.11) leads to the solution $\Lambda_F = \Upsilon'_F + \Phi_H$ of (3.10).

So, $E_\infty^{02} = E_2^{02}$ is the direct sum of two subspaces consisting of elements of the form $\Psi_E - \Psi'_E$ and $\Upsilon_F - \Upsilon'_F - \Phi_H$ for appropriate E, F and H , and isomorphic to $Der(A)^D$ and $Har^2(A, A)^D$ respectively. ■

Lemma 3.5.

- (i) $E_\infty^{01} \simeq Der(A)^D$
- (ii) $E_\infty^{11} \simeq Der(A)_D$.

Proof. d_1^{01} acts on the space $E_1^{01} \simeq Der(A)$ as $1 \otimes E \mapsto [1 \otimes E, \Phi_D] = \Phi_{[D, E]}$, $E \in Der(A)$. Hence $Ker d_1^{01} \simeq Der(A)^D$, proving (i).

$Im d_1^{01} \simeq [D, Der(A)]$, $E_\infty^{11} = E_2^{11} \simeq Der(A)_D$ and E_∞^{11} consists of elements Φ_E for $E \in Der(A)$ which are independent modulo $[D, Der(A)]$, proving (ii). ■

Putting all these calculations together, we get statements (ii) and (iii) of Theorem 3.1. (Lemma 3.5(i) is used to get a formula for cohomology of degree 1, while all the rest is used to get a formula for cohomology of degree 2).

For convenience we summarize here the cocycles whose cohomology classes constitute a basis of $H^1(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$ and $H^2(\mathfrak{L}(A, D), \mathfrak{L}(A, D))$.

Basic cocycles of degree 1 are just mappings of the form $1 \otimes E$, $E \in Der(A)^D$.

All cocycles of degree 2 constructed here have their counterparts in $Z^2(W_1(1) \otimes A, W_1(1) \otimes A)$ (in fact, they are liftings, in the Gerstenhaber's terminology [GS], of 2-cocycles on $W_1(1) \otimes A$). Each class of cocycles denoted by overlined capital Greek letter is lifted from the corresponding class of §2 denoted by the same letter.

So, let $\overline{\Theta}_u$, $\overline{\Upsilon}_{F, H}$, $\overline{\Psi}_E$ and $\overline{\Phi}_E$ be 2-cochains on $\mathfrak{L}(A, D)$ defined by the following formulas, where the top line comes from the appropriate cocycle of §2 (the "regular" components), and the second line represent a new component coming from the deformation:

$$\overline{\Theta}_u(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{i+j-p} \otimes N_{ij}/p abu, & i+j \geq p-1 \\ e_{i+j} \otimes (\lambda_{ij} a D(b) - \lambda_{ji} b D(a))u, & -2 < i+j < p-1 \\ 0, & \text{otherwise} \end{cases}$$

where $u \in A^D$ and the coefficients λ_{ij} defined as in Lemma 3.3 (the regular component is (2.2)),

$$\begin{aligned} & \overline{\Upsilon}_{F, H}(e_i \otimes a, e_j \otimes b) \\ &= \begin{cases} e_{i+j} \otimes N_{ij} F(a, b), & -2 < i+j < p-1 \\ e_{p-2} \otimes (bH(a) - aH(b) - F(D(a), b) + F(a, D(b))), & i=j=-1 \\ 0, & i+j \geq p-1 \end{cases} \end{aligned}$$

where $F \in \mathcal{Z}^2(A, A)^D$ and $H \in End(A)$ such that $D \star F = \delta H$ (the regular component is (2.3)),

$$\overline{\Psi}_E(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (E(a)D(b) - E(b)D(a)), & i=j=-1 \\ e_{i+j} \otimes ((\binom{i+j+1}{j})bE(a) - (\binom{i+j+1}{i})aE(b)), & -2 < i+j < p-1 \\ 0, & i+j \geq p-1 \end{cases}$$

where $E \in Der(A)^D$ (the regular component is (2.5)), and, finally, $\overline{\Phi}_E = \Phi_E$ (the regular component is (1.2); there is no deformation component).

Lemma 2.4 (stating the independence of initial cocycles on $W_1(1) \otimes A$) together with the spectral sequence construction assure the independence of the corresponding cocycles on $\mathfrak{L}(A, D)$. More precisely, the following is true:

Proposition 3.6. *Let $\{u_i\}$ be linearly independent elements of A , $\{F_i\}$ be cohomologically independent cocycles in $\mathcal{Z}^2(A, A)$, $\{H_i\}$ be elements in $Hom(A, A)$ linearly independent modulo $Der(A)$*

and such that $D \star F_i = \delta H_i$, $\{E_i\}$ be linearly independent elements in $Z_D(\text{Der}(A))$, and $\{E'_i\}$ be derivations of A linearly independent modulo $[D, \text{Der}(A)]$.

Then the cocycles $\overline{\Theta}_{u_i}$, $\overline{\Upsilon}_{F_i, H_i}$, $\overline{\Psi}_{E_i}$, $\overline{\Phi}_{E'_i}$ are cohomologically independent.

4. LOW-DIMENSIONAL COHOMOLOGY OF $W_1(n) \otimes A$

Now our objective is to transform the results obtained so far for $\mathfrak{L}(A, D)$ into those for $W_1(n) \otimes A$.

For this, take $A = O_1(n-1) \otimes B$ and $D = \partial \otimes 1$. By Proposition 1.2, $\mathfrak{L}(A, D)$ in this case is isomorphic to $W_1(n) \otimes B$, and Theorem 3.1 entails

$$(4.1) \quad \begin{aligned} H^2(W_1(n) \otimes B, W_1(n) \otimes B) \\ \simeq (O_1(n-1) \otimes B)^{\partial \otimes 1} \oplus \text{Der}(O_1(n-1) \otimes B)_{\partial \otimes 1} \oplus \text{Der}(O_1(n-1) \otimes B)^{\partial \otimes 1} \\ \oplus \text{Har}^2(O_1(n-1) \otimes B, O_1(n-1) \otimes B)^{\partial \otimes 1}. \end{aligned}$$

The next lemmas collect all necessary information for evaluation of four summands appearing on the right side of this isomorphism. (Just for notational convenience, we put $m = n - 1$).

Lemma 4.1.

- (i) $(O_1(m) \otimes B)^{\partial \otimes 1} = 1 \otimes B$
- (ii) $\text{Der}(O_1(m) \otimes B)_{\partial \otimes 1} \simeq \langle x^{p^m-1} \partial^{p^k} \mid 0 \leq k \leq m-1 \rangle \otimes B \oplus x^{p^m-1} \otimes \text{Der}(B)$
- (iii) $\text{Der}(O_1(m) \otimes B)^{\partial \otimes 1} \simeq \langle \partial^{p^k} \mid 0 \leq k \leq m-1 \rangle \otimes B \oplus 1 \otimes \text{Der}(B)$.

Proof. (i) Obvious, as $\text{Ker}_{O_1(m)} \partial = K1$.

(ii) Since

$$\text{Der}(O_1(m) \otimes B) \simeq \text{Der}(O_1(m)) \otimes B + O_1(m) \otimes \text{Der}(B)$$

and $\text{Der}(O_1(m))$ is a free $O_1(m)$ -module with basis $\{\partial^{p^k} \mid 0 \leq k \leq m-1\}$,

$$\begin{aligned} [\partial \otimes 1, \text{Der}(O_1(m) \otimes B)] &\simeq [\partial, \text{Der}(O_1(m))] \otimes B \oplus \partial(O_1(m)) \otimes \text{Der}(B) \\ &= \langle x^i \partial^{p^k} \mid 0 \leq i < p^m - 1, 0 \leq k \leq m-1 \rangle \otimes B \oplus \langle x^i \mid 0 \leq i < p^m - 1 \rangle \otimes \text{Der}(B). \end{aligned}$$

As $\langle x^{p^m-1} \partial^{p^k} \mid 0 \leq k < m-1 \rangle$ is a complement in $\text{Der}(O_1(m))$ to the tensor factor in the first summand, and $\langle x^{p^m-1} \rangle$ is a complement in $O_1(m)$ to those in the second summand, we get the isomorphism desired.

(iii) Analogous to (ii). ■

Further, according to [Ha], Theorem 5,

$$(4.2) \quad \begin{aligned} \text{Har}^2(O_1(m) \otimes B, O_1(m) \otimes B)^{\partial \otimes 1} \\ \simeq \text{Har}^2(O_1(m), O_1(m))^{\partial} \otimes B \oplus O_1(m)^{\partial} \otimes \text{Har}^2(B, B) \end{aligned}$$

(as $O_1(m)^{\partial} \simeq K$, the second summand is actually just $\text{Har}^2(B, B)$).

So we need to compute the second Harrison cohomology of the divided powers algebra $O_1(m)$. First we determine its Hochschild cohomology. It is more convenient to work with reduced polynomial ring O_m .

Note that O_m is a factor-algebra of a polynomial algebra as well as the group algebra of an elementary abelian group, and for both class of algebras all sort of cohomological computations have been done. Instead of digging the result we need out of the literature (which will require some additional computations anyway, see e.g. [L], §7.4 and [Ho] and references therein), we give a direct simple proof suited for our needs.

We use multi-index notations: $F_m = \{\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}^m \mid 0 \leq \alpha_i < p\}$, $x^\alpha = x_1^{\alpha_1} \dots x_m^{\alpha_m}$, ε_i denotes element in F_m of the form $(0, \dots, 1, \dots, 0)$ (1 in the i th position).

Proposition 4.2. $H^i(O_m, O_m)$ is a free O_m -module of dimension $\binom{i+m-1}{i}$.

Proof. Consider first the case $m = 1$, i.e., an algebra $O_1 = K[x]/(x^p)$. We use a very simple (and nice) free O_1^e -resolution of O_1 presented in [RS] (as O_1 is commutative, $O_1^e \simeq O_1 \otimes O_1$):

$$\dots \xrightarrow{d_1} O_1 \otimes O_1 \xrightarrow{d_0} O_1 \otimes O_1 \xrightarrow{m} O_1 \longrightarrow 0$$

where m is the multiplication in O_1 and

$$d_i(a \otimes b) = \begin{cases} a \otimes xb - ax \otimes b, & i \text{ even} \\ \sum_{k=0}^{p-1} ax^k \otimes x^{p-1-k}b, & i \text{ odd.} \end{cases}$$

Applying the functor $\text{Hom}_{O_1^e}(-, O_1)$, we get a (deleted) complex whose all but first differentials are zero:

$$0 \longrightarrow O_1 \xrightarrow{id} O_1 \xrightarrow{0} O_1 \xrightarrow{0} \dots$$

Therefore, for each i , $H^i(O_1, O_1) \simeq O_1$.

Now the general case is proved via induction on m by applying the Künneth formula to the decomposition $O_m \simeq O_{m-1} \otimes O_1$. ■

Lemma 4.3.

- (i) $\text{Har}^2(O_m, O_m)$ is a free O_m -module of dimension m . The basic cocycles (over O_m) can be chosen as

$$F_i(x^\alpha, x^\beta) = \begin{cases} x^{\alpha+\beta-p\varepsilon_i}, & \alpha_i + \beta_i \geq p \\ 0, & \alpha_i + \beta_i < p \end{cases}$$

for $1 \leq i \leq m$.

- (ii) $\dim \text{Har}^2(O_1(m), O_1(m))^\partial = m$. The basic ∂ -invariant cocycles can be chosen as

$$(4.3) \quad F_i(x^\alpha, x^\beta) = \begin{cases} \binom{\alpha+\beta}{\beta}/p \cdot x^{\alpha+\beta-p^i}, & \alpha_i + \beta_i \geq p \\ 0, & \alpha_i + \beta_i < p \end{cases}$$

where $\alpha = \sum_{i \geq 1} \alpha_i p^{i-1}$, $\beta = \sum_{i \geq 1} \beta_i p^{i-1}$ are p -adic decompositions.

Proof. (i) By Proposition 4.2, $H^2(O_m, O_m)$ is a free O_m -module of dimension $\frac{m(m+1)}{2}$. We assert that the two classes of cocycles, $F_i, 1 \leq i \leq m$ and $\partial/\partial x_i \cup \partial/\partial x_j, 1 \leq i < j \leq m$, form a basis of this module. Indeed, the cocycle condition is verified immediately. As we have $m + \frac{m(m-1)}{2} = \frac{m(m+1)}{2}$ cocycles, it remains to check their independence. Since F_i are symmetric and $\partial/\partial x_i \cup \partial/\partial x_j$ are skew, one suffices to do this only for F_i (remember that 2-coboundaries are symmetric). Suppose

$$(4.4) \quad \sum_{i=1}^m u_i F_i = \delta G$$

for some $G \in \text{Hom}(O_m, O_m)$ and $u_i \in O_m$.

Then $\delta G(x^\alpha, x^\beta) = 0$ if $\alpha_i + \beta_i < p$ for each i . This implies that G acts as derivation on products $x^\alpha x^\beta$ if $\alpha_i + \beta_i < p$ for all i , hence

$$G(x^\alpha) = \sum_{i=1}^m \alpha_i x^{\alpha - \varepsilon_i} G(x^{\varepsilon_i})$$

what in its turn entails that G is a derivation, and thus $\delta G(x^\alpha, x^\beta) = 0$ for all α, β . Then evaluating the left side of (4.4) for all pairs (x^α, x^β) such that $\alpha_j + \beta_j = \delta_j p$ for each j and a fixed i , we get $u_i = 0$. This shows that cocycles F_i are independent.

Now picking from the basic cocycles of $H^2(O_m, O_m)$ those which are symmetric, we obtain a basis $\{F_i \mid 1 \leq i \leq m\}$ of $\text{Har}^2(O_m, O_m)$ (as a module over O_m). The freeness of $\text{Har}^2(O_m, O_m)$ follows either from the previous reasonings or from the fact that the Harrison cohomology is a direct summand of the Hochschild one (cf. [GS]).

(ii) Using the isomorphism (1.1), the cocycles of part (i) may be rewritten as (4.3). Direct easy check shows that the cocycles F_i are ∂ -invariant (in fact, $\partial \star F_i = 0$). The identity $\partial \star (uF) = u \star \partial F - (\partial u)F$ shows that the equality $\partial \star (u_1 F_1 + \cdots + u_k F_k) = 0$ implies

$$(\partial u_1)F_1 + \cdots + (\partial u_k)F_k = 0$$

which due to the freeness of $Har^2(O_1(m), O_1(m))$ over $O_1(m)$ entails that all $u_i \in K1$, and the assertion desired follows. ■

Now, collecting (4.1), (4.2) and Lemmas 4.1 and 4.3(ii), we get an isomorphism

$$(4.5) \quad H^2(W_1(n) \otimes B, W_1(n) \otimes B) \simeq \mathcal{H} \otimes B \oplus Der(B) \oplus Der(B) \oplus Har^2(B, B)$$

where \mathcal{H} is a vector space with basis $\{1, x^{p^{n-1}-1} \partial^{p^k}, \partial^{p^k}, F_{k+1} \mid 0 \leq k \leq n-2\}$.

To obtain an explicit basis of this cohomology group, let us regroup the basis of $H^2(\mathcal{L}(A, D), \mathcal{L}(A, D))$, exhibited in §3, according to the direct summands in (4.5) as follows.

The classes of $3n-2$ cocycles

$$\bar{\Theta}_{1 \otimes u}, \bar{\Upsilon}_{F_{i+1} \otimes R_u, 0}, \bar{\Psi}_{\partial^{p^i} \otimes R_u}, \bar{\Phi}_{x^{p^{n-1}-1} \partial^{p^i} \otimes R_u}, \quad 0 \leq i \leq n-2, \quad u \in B$$

form a module denoted in (4.5) as $\mathcal{H} \otimes B$. It is easy to see that all these cocycles are of the form $(1 \otimes R_u) \circ \Theta_\phi$ for appropriate $\phi \in Z^2(W_1(n), W_1(n))$. As by Proposition 3.6 all these cocycles are independent, the corresponding $3n-2$ cocycles on $W_1(n)$ are also independent. But according to [DK], $\dim H^2(W_1(n), W_1(n)) = 3n-2$, whence $\mathcal{H} \simeq H^2(W_1(n), W_1(n))$. It should be noted that the 2-cocycles on $W_1(n)$ derived here do not wholly coincide with basic cocycles presented in [DK].

The classes of cocycles $\bar{\Psi}_{1 \otimes D}$ and $\bar{\Phi}_{x^{p^{n-1}-1} \otimes D}$, $D \in Der(B)$, denoted from now for the convenience as ψ_D and ϕ_D respectively, form two modules isomorphic to $Der(B)$. They are just obvious generalizations of cocycles Ψ_D and Φ_D to arbitrary n :

$$(4.6) \quad \psi_D(e_i \otimes a, e_j \otimes b) = e_{i+j} \otimes \left(\binom{i+j+1}{j} bD(a) - \binom{i+j+1}{i} aD(b) \right),$$

$$-1 \leq i, j \leq p^n - 2$$

$$(4.7) \quad \phi_D(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p^n-2} \otimes (aD(b) - bD(a)), & i = j = -1 \\ 0, & \text{otherwise.} \end{cases}$$

And finally, the classes of cocycles $\bar{\Upsilon}_{1 \otimes F, 0}$ where $F \in Z^2(B, B)$, generate a module isomorphic to $Har^2(B, B)$. These cocycles are of the form Υ_F (cf. Proposition 2.2).

Thus we get a generalization of Proposition 2.1:

Theorem 4.4. *For an arbitrary associative commutative unital algebra B ,*

$$H^2(W_1(n) \otimes B, W_1(n) \otimes B) \simeq H^2(W_1(n), W_1(n)) \otimes B \oplus Der(B) \oplus Der(B) \oplus Har^2(B, B).$$

The basic cocycles can be chosen among $(1 \otimes R_u) \circ \Theta_\phi$ for $\phi \in Z^2(W_1(n), W_1(n))$, ψ_D , ϕ_D for $D \in Der(B)$, and Υ_F for $F \in Z^2(B, B)$, given by formulas (2.2), (4.6), (4.7) and (2.3) respectively.

We conclude this section with formulation of all necessary results needed for our further purposes, which are obtained in a similar (and much simpler) way as Theorem 4.4 and/or can be found elsewhere (cf. [B2], [C], [Z]):

$$(4.8) \quad H^1(W_1(n) \otimes B, W_1(n) \otimes B) \simeq H^1(W_1(n), W_1(n)) \otimes B \oplus Der(B)$$

$$(4.9) \quad H^2(W_1(n) \otimes B, K) \simeq H^2(W_1(n), K) \otimes B^*$$

$$(4.10) \quad H^2(sl(2) \otimes A, K) \simeq HC^1(A).$$

5. FILTERED DEFORMATIONS OF $W_1(n) \otimes A + 1 \otimes \mathfrak{D}$

As explained in §1, we are interested in filtered Lie algebras whose associated graded algebra is $S \otimes O_m + 1 \otimes \mathfrak{D}$ for $S = W_1(n)$ or $sl(2)$, where \mathfrak{D} is a subalgebra of $Der(O_m)$. First, basing on Theorem 4.4, we shall compute the second cohomology group of such algebras.

Lemma 5.1. *Let L be a Lie algebra which can be written as the semidirect product $L = I \oplus Q$, where I is a centerless perfect ideal of L , Q is a subalgebra, and $Q \cap ad(I) = 0$ (in the last equality, Q and $ad(I)$ considered as subspaces of $End(I)$). Then the terms relevant to the cohomology group $H^2(L, L)$ in the Hochschild-Serre spectral sequence of L with respect to I with the general E_2 -term $E_2^{pq} = H^p(Q, H^q(I, L))$, are the following:*

$$\begin{aligned} E_\infty^{20} &= 0 \\ E_\infty^{11} &= E_2^{11} = H^1(Q, H^1(I, I)/Q) \\ E_2^{02} &= H^2(I, I)^Q \oplus (Ker F)^Q \\ E_\infty^{02} &= E_3^{02} = Ker d_2^{02} \end{aligned}$$

where $F : H^2(I) \otimes Q \rightarrow H^3(I, I)$ is induced by the mapping

$$\begin{aligned} C^2(I, Q) &\rightarrow C^3(I, I) \\ \phi &\mapsto (x \wedge y \wedge z \mapsto [x, \phi(y, z)] + \curvearrowright). \end{aligned}$$

Proof. One has $E_2^{p0} = H^p(Q, L^I)$. The condition $Q \cap ad(I) = 0$ entails $L^I = Z(I) = 0$, so $E_2^{p0} = 0$. Thus $E_\infty^{20} = 0$ and $E_\infty^{02} = E_3^{02}$ follow from standard considerations. As d_2 maps E_2^{11} to $E_2^{30} = 0$, $E_\infty^{11} = E_2^{11}$. We also have

$$\begin{aligned} E_2^{02} &= H^0(Q, H^2(I, L)) = H^2(I, L)^Q \\ E_2^{11} &= H^1(Q, H^1(I, L)) \\ E_2^{21} &= H^2(Q, H^1(I, L)). \end{aligned}$$

Consider a piece of the cohomology long exact sequence associated with the short exact sequence $0 \rightarrow I \rightarrow L \rightarrow Q \rightarrow 0$ of I -modules (Q considered as a trivial I -module):

$$(5.1) \quad \begin{aligned} H^0(I, L) \rightarrow H^0(I, Q) \rightarrow H^1(I, I) \rightarrow H^1(I, L) \rightarrow H^1(I, Q) \rightarrow H^2(I, I) \\ \rightarrow H^2(I, L) \rightarrow H^2(I, Q) \xrightarrow{F} H^3(I, I) \end{aligned}$$

(F is connecting homomorphism).

We obviously have: $H^0(I, L) = L^I = 0$, $H^0(I, Q) = Q^I = Q$, $H^1(I, Q) = H^1(I) \otimes Q = 0$, $H^2(I, Q) = H^2(I) \otimes Q$. Hence

$$\begin{aligned} H^1(I, L) &\simeq H^1(I, I)/Q \\ H^2(I, L) &\simeq H^2(I, I) \oplus Ker F \end{aligned}$$

(note that since $Q \cap ad(I) = 0$, Q consists of outer derivations of I , and therefore embeds in $H^1(I, I)$).

As $L = I \oplus Q$ as Q -modules and differential commutes with each $adx, x \in Q$, the Q -action commutes with inclusion and projection arrows in (5.1) (but not necessarily with connecting homomorphism), and we get

$$H^2(I, L)^Q \simeq H^2(I, I)^Q \oplus (Ker F)^Q.$$

Putting all this together, we obtain the asserted equalities. ■

Passing to our specific case, define a grading on $L = S \otimes B + 1 \otimes \mathfrak{D}$ as in Theorem 1.5, i.e.

$$(5.2) \quad L_i = \begin{cases} e_0 \otimes B + 1 \otimes \mathfrak{D}, & i = 0 \\ e_i \otimes B, & i \neq 0 \end{cases}$$

and consider induced grading on $H^2(L, L)$. $H_+^2(L, L)$ denotes a positive part of that induced grading.

Proposition 5.2. *Let $L = S \otimes B + 1 \otimes \mathfrak{D}$, $\mathfrak{D} \subseteq \text{Der}(B)$. Then:*

- (i) *if $S = W_1(n)$, then $H_+^2(L, L) \simeq H_+^2(S, S) \otimes B^\mathfrak{D} \oplus \text{Der}(B)^\mathfrak{D}$*
- (ii) *if $S = \text{sl}(2)$, then $H_+^2(L, L) = 0$.*

Proof. (i) Proposition 1.1 ensures that Lemma 5.1 is applicable here if we put $I = S \otimes B$ and $Q = 1 \otimes \mathfrak{D}$. Using Theorem 4.4, (4.8) and (4.9) and considering the action of $1 \otimes \mathfrak{D}$ on appropriate cohomology groups on the level of explicit cocycles, one gets

$$\begin{aligned} E_2^{11} &\simeq H^1(S, S) \otimes H^1(\mathfrak{D}, B) \oplus H^1(\mathfrak{D}, \text{Der}(B)/\mathfrak{D}) \\ E_2^{02} &\simeq H^2(S, S) \otimes B^\mathfrak{D} \oplus \text{Der}(B)^\mathfrak{D} \oplus \text{Har}^2(B, B)^\mathfrak{D} \oplus (\text{Ker } F)^\mathfrak{D}. \end{aligned}$$

The grading (5.2) induces a \mathbb{Z} -grading on each term of the spectral sequence (cf. [F]).

The knowledge of $H^1(W_1(n), W_1(n))$, $H^2(W_1(n), K)$ (cf. [B1] or [D2]) and $H^2(W_1(n), W_1(n))$ (cf. [DK]), allows to write down all nonzero graded components of $E_2 = E_2^{11} \oplus E_2^{02}$ with respect to grading (5.2):

$$\begin{aligned} (E_2)_{-p^n} &\simeq H_{-p^n}^2(W_1(n), W_1(n)) \otimes B^\mathfrak{D} \oplus (\text{Ker } F)^\mathfrak{D} \\ (E_2)_{-p^t} &\simeq H_{-p^t}^2(W_1(n), W_1(n)) \otimes B^\mathfrak{D} \oplus H_{-p^t}^1(W_1(n), W_1(n)) \otimes H^1(\mathfrak{D}, B) \\ (E_2)_0 &\simeq \text{Der}(B)^\mathfrak{D} \oplus \text{Har}^2(B, B)^\mathfrak{D} \oplus H^1(\mathfrak{D}, \text{Der}(B)/\mathfrak{D}) \\ (E_2)_{p^n-p^t} &\simeq H_{p^n-p^t}^2(W_1(n), W_1(n)) \otimes B^\mathfrak{D} \\ (E_2)_{p^n} &\simeq \text{Der}(B)^\mathfrak{D} \end{aligned}$$

where $1 \leq t \leq n-1$.

The last two classes constitute $(E_2)_+$ and generated by cocycles $(1 \otimes R_u) \circ \Theta_{\psi_t}$ where $u \in B^\mathfrak{D}$,

$$\psi_t(e_i, e_j) = \begin{cases} e_{p^n-2}, & i = -1, j = p^t - 1 \\ 0, & \text{otherwise} \end{cases}$$

and ϕ_D where $D \in \text{Der}(B)^\mathfrak{D}$, respectively. (Strictly speaking, these cocycles are extended from the corresponding cocycles from $Z^2(W_1(n) \otimes B, W_1(n) \otimes B)$ by letting them vanish on $W_1(n) \otimes B \wedge 1 \otimes \mathfrak{D}$ and $1 \otimes \mathfrak{D} \wedge 1 \otimes \mathfrak{D}$).

According to Lemma 5.1, the corresponding E_3 -term is

$$(E_3^{02})_+ = \text{Ker}((E_2^{02})_+ \xrightarrow{(d_2^{02})_+} (E_2^{21})_+).$$

Theorem 4.4 shows that all \mathfrak{D} -invariant cohomology classes in $(E_2^{02})_+ \simeq H_+^2(S, S) \otimes B^\mathfrak{D} \oplus (\text{Der}(B))^\mathfrak{D}$ can be represented by \mathfrak{D} -invariant cocycles, what implies $(d_2^{02})_+ = 0$ and the positive part of the spectral sequence collapses in the relevant range.

(ii) Quite analogous (and simpler). ■

Remark. In principle, one may compute the whole cohomology group $H^2(S \otimes B + 1 \otimes \mathfrak{D}, S \otimes B + 1 \otimes \mathfrak{D})$ by the following scheme: first, it is possible to evaluate $(\text{Ker } F)^\mathfrak{D}$ in the spirit of §2 or §3, and, particularly, to show that in this case the Q -action commutes with F , what in its turn implies

$$E_2^{02} = H^2(S \otimes B, S \otimes B)^\mathfrak{D} \oplus (\text{Ker } F \cap (H^2(S \otimes B) \otimes \mathfrak{D}))^\mathfrak{D}.$$

Then the same reasoning as at the end of the proof of Proposition 5.2 shows that $d_2^{02} = 0$ in general.

As all cocycles constituting the basis of $H_+^2(L, L)$, being of the types $(1 \otimes R_u) \circ \Theta_{\psi_t}$ and ϕ_D , are possibly nonzero only on the $(-1)st$, $1st$ and $(p^t - 1)st$ ($1 \leq t \leq n-1$) graded components of L , with values in the $(p^n - 2)th$ graded component, each Massey product of two such cocycles is obviously zero, and by Proposition 1.4 we have

Theorem 5.3. *Let L be as in Proposition 5.2 with grading defined by (5.2). Let \mathfrak{L} be a filtered algebra whose associated graded algebra is isomorphic (as graded algebra) to L . Then*

(i) if $S = W_1(n)$, then \mathfrak{L} is determined by brackets

$$\{\cdot, \cdot\} = [\cdot, \cdot] + \sum_{t=1}^{n-1} (1 \otimes R_{u_t}) \circ \Theta_{\psi_t} + \phi_D$$

for some $u_t \in B^{\mathfrak{D}}$ and $D \in \text{Der}(B)^{\mathfrak{D}}$

(ii) if $S = \mathfrak{sl}(2)$, then $\mathfrak{L} \simeq L$.

Note that the basic cocycles in $H_+^2(L, L)$ are of the form Φ_d for appropriate $d \in \text{Der}(O_1(n-1) \otimes B)$: $\phi_D = \Phi_{x^{p^{n-1}-1} \otimes D}$ and $(1 \otimes R_u) \circ \Theta_{\psi_t} = \Phi_{x^{p^{n-1}-1} \partial^{p^t-1} \otimes R_u}$, $1 \leq t \leq n-1$. Therefore the algebras appearing in part (i) of Theorem 5.3 are all of the kind $\mathfrak{L}(A, D) + 1 \otimes \mathfrak{D}$ for $A = O_1(n-1) \otimes B$ and $\mathfrak{D} \subseteq \text{Der}(A)$. (Note that from the Jacobi identity follows $[D, \mathfrak{D}] = 0$). Combining this fact (in the particular case $B = O_m$) with Theorem 1.5, we conclude:

Corollary 5.4. *(The ground field is algebraically closed of characteristic $p > 5$).*

Let \mathfrak{L} be a semisimple Lie algebra with a solvable maximal subalgebra defining in \mathfrak{L} a long filtration. Then either $\mathfrak{L} \simeq \mathfrak{sl}(2) \otimes O_m + 1 \otimes \mathfrak{D}$, or $\mathfrak{L} \simeq \mathfrak{L}(O_m, D) + 1 \otimes \mathfrak{D}$ for some $m \in \mathbb{N}$, $D \in \text{Der}(O_m)$ and a solvable subalgebra \mathfrak{D} in $\text{Der}(O_m)$ such that $[D, \mathfrak{D}] = 0$ and O_m has no $\langle \mathfrak{D}, D \rangle$ -invariant ideals.

Remarks.

(i) The close inspection of Weisfeiler's results shows that if \mathfrak{L}_0 is a solvable maximal subalgebra in Theorem 1.5, then after passing to the associated graded algebra, \mathfrak{L}_0 goes to $\langle e_0, e_1 \rangle \otimes O_m + 1 \otimes \mathfrak{D}$ (in the case $S = \mathfrak{sl}(2)$) or to $W_1(n)_0 \otimes O_m + 1 \otimes \mathfrak{D}$ (in the case $S = W_1(n)$). Our computations of filtered deformations show that actually \mathfrak{L}_0 coincides with these algebras (as they do not change under deformations).

(ii) Since $[D, \mathfrak{D}] = 0$, the algebra $\langle \mathfrak{D}, D \rangle \subseteq \text{Der}(O_m)$ either coincides with \mathfrak{D} (if $D \in \mathfrak{D}$), or is 1-dimensional abelian extension of \mathfrak{D} (if $D \notin \mathfrak{D}$).

So, to classify semisimple Lie algebras with a solvable maximal subalgebra occurring in Theorem 1.5, it remains to describe algebras appearing in Corollary 5.4 up to isomorphism and to identify them with the known semisimple Lie algebras. We accomplish this task in the next section.

6. CLASSIFICATION OF SEMISIMPLE ALGEBRAS $\mathfrak{L}(A, D) + 1 \otimes \mathfrak{D}$

The object of this section is the class of Lie algebras $\mathfrak{L}(A, D) + 1 \otimes \mathfrak{D}$, where $1 \otimes \mathfrak{D}$ acts on $\mathfrak{L}(A, D)$ as on $W_1(1) \otimes A$ and $[D, \mathfrak{D}] = 0$. The case $A = O_m$ is of particular importance.

All algebras throughout this section assumed to be finite-dimensional.

Note that consideration of dimensions immediately implies that no algebra of the form $\mathfrak{L}(O_n, D)$ is isomorphic to some $\mathfrak{sl}(2) \otimes O_m$.

Lemma 6.1. *Each ideal of $\mathfrak{L}(A, D) + 1 \otimes \mathfrak{D}$ is of the form $\mathfrak{L}(I, D) + 1 \otimes \mathfrak{E}$ where I is a $\langle \mathfrak{D}, D \rangle$ -invariant ideal of A , \mathfrak{E} is an ideal of \mathfrak{D} , and $\mathfrak{E}(A) \subseteq I$.*

Proof. Let \mathfrak{J} be an ideal of $\mathfrak{L}(A, D) + 1 \otimes \mathfrak{D}$, then $\mathfrak{J} \cap \mathfrak{L}(A, D)$ is an ideal of $\mathfrak{L}(A, D)$. Passing to the associated graded algebra (as in Proposition 1.3), we get that $gr(\mathfrak{J} \cap \mathfrak{L}(A, D))$ is an ideal of $W_1(1) \otimes A$. Either a direct calculation in $W_1(1)$, or the general result of [Ste], yields that

$$(6.1) \quad gr(\mathfrak{J} \cap \mathfrak{L}(A, D)) = W_1(1) \otimes I$$

for some ideal I of A . Particularly, $e_{p-2} \otimes I \subset \mathfrak{J} \cap \mathfrak{L}(A, D)$. Multiplying elements from $e_{p-2} \otimes I$ a necessary number of times by $e_{-1} \otimes 1$, one gets $e_i \otimes I \subset \mathfrak{J} \cap \mathfrak{L}(A, D)$ for each $-1 \leq i \leq p-2$, that is, $W_1(1) \otimes I \subseteq \mathfrak{J} \cap \mathfrak{L}(A, D)$. Due to (6.1) this inclusion is actually an equality (of vector spaces): $\mathfrak{J} \cap \mathfrak{L}(A, D) = W_1(1) \otimes I$. Particularly, $W_1(1) \otimes I$ is closed under brackets $\{\cdot, \cdot\}$ (cf. Definition in §1), what is equivalent to $D(I) \subseteq I$.

Now, taking an arbitrary element $\sum_{i=-1}^{p-2} e_i \otimes a_i + 1 \otimes d \in \mathfrak{J}$, and multiplying it by $e_0 \otimes 1$ and $e_{-1} \otimes 1$, we get $\sum_i i e_i \otimes a_i \in \mathfrak{J} \cap \mathfrak{L}(A, D)$ and $\sum_i e_{i-1} \otimes a_i \in \mathfrak{J} \cap \mathfrak{L}(A, D)$ respectively, showing therefore that all $a_i \in I$. This proves that $\mathfrak{J} = \mathfrak{L}(I, D) + 1 \otimes \mathfrak{E}$ for some subalgebra $\mathfrak{E} \subseteq \mathfrak{D}$. The rest of the conditions in the assertion follow immediately. ■

Lemma 6.2. *Let a Lie algebra $\mathfrak{L} = \mathfrak{L}(A, D) + 1 \otimes \mathfrak{D}$ is semisimple. Then the following hold:*

- (i) $A \simeq \bigoplus_i O_{n_i}$ for some $n_i \in \mathbb{N}$, each O_{n_i} has no $\langle \mathfrak{D}, D \rangle$ -invariant ideals
- (ii) $\mathfrak{L}(A, D) \simeq \bigoplus_i S_i \otimes O_{m_i}$ for some $m_i \in \mathbb{N}$ and simple Lie algebras S_i
- (iii) $S_i \simeq \mathfrak{L}(O_{k_i}, d_i)$ for some $k_i \in \mathbb{N}$ and $d_i \in \text{Der}(O_{k_i})$, O_{k_i} has no d_i -invariant ideals.

Proof. The proof merely consists of multiple applications of classical Block's results [B2].

By Lemma 6.1, A has no $\langle \mathfrak{D}, D \rangle$ -invariant nilpotent ideals, i.e., A is $\langle \mathfrak{D}, D \rangle$ -semisimple in the terminology of Block [B2]. According to [B2], Main Theorem and Corollary 8.3, A is isomorphic to the direct sum $\bigoplus_i O_{n_i}$ of reduced polynomial rings having no $\langle \mathfrak{D}, D \rangle$ -invariant ideals. Hence $D = \sum_i D_i$, where each D_i acts as derivation on O_{n_i} and zero on O_{n_j} , $j \neq i$. Obviously

$$\mathfrak{L}\left(\bigoplus_i O_{n_i}, \sum_i D_i\right) \simeq \bigoplus_i \mathfrak{L}(O_{n_i}, D_i)$$

and by Lemma 6.1 each minimal ideal of \mathfrak{L} coincides with one of $\mathfrak{L}(O_{n_i}, D_i)$. Thus by [B2], Theorem 1.3, $\mathfrak{L}(O_{n_i}, D_i) \simeq S_i \otimes O_{m_i}$ for some simple Lie algebra S_i and $m_i \in \mathbb{N}$.

Applying Lemma 6.1 again, we see that each ideal of $\mathfrak{L}(O_{n_i}, D_i)$ is of the form $\mathfrak{L}(I, D_i)$ for some D_i -invariant ideal I of O_{n_i} , and by [Ste] each ideal of $S_i \otimes O_{m_i}$ is of the form $S_i \otimes J$ for some ideal J of O_{m_i} . But $O_{m_i}^+$ is the greatest ideal of O_{m_i} whence there is a greatest D_i -invariant ideal I_i of O_{n_i} , $\mathfrak{L}(I_i, D_i) \simeq S_i \otimes O_{m_i}^+$, and

$$\mathfrak{L}(O_{n_i}, D_i) / \mathfrak{L}(I_i, D_i) \simeq (S_i \otimes O_{m_i}) / (S_i \otimes O_{m_i}^+) \simeq S_i.$$

It is easy to see that the left side here is isomorphic to $\mathfrak{L}(O_{n_i}/I_i, d_i)$, $d_i \in \text{Der}(O_{n_i}/I_i)$ being induced from D_i . Since S_i is simple, O_{n_i}/I_i has no d_i -invariant ideals and again by Block's theorem, $O_{n_i}/I_i \simeq O_{k_i}$ for some $k_i \in \mathbb{N}$. ■

Now we determine simple Lie algebras in the class $\mathfrak{L}(A, D)$.

Lemma 6.3. *(The ground field is perfect of characteristic $p > 3$).*

$\mathfrak{L} = \mathfrak{L}(A, D)$ is simple if and only if $\mathfrak{L} \simeq W_1(n)$ for some $n \in \mathbb{N}$.

Proof. The "if" part contained in Proposition 1.2. So suppose that $\mathfrak{L}(A, D)$ is simple. According to Lemmas 6.1 and 6.2, $A \simeq O_n$ for some $n \in \mathbb{N}$. Hence \mathfrak{L} has a subalgebra $\mathfrak{L}_0 = e_{-1} \otimes O_n^+ + \langle e_0, \dots, e_{p-2} \rangle \otimes O_n$ of codimension 1. Then by [D1], \mathfrak{L} is isomorphic to either $sl(2)$ or $W_1(n)$, the first case is impossible by dimension consideration. ■

Remarks.

(i) If the ground field is algebraically closed, one may deduce the assertion of the Lemma from many other results in the literature, e.g. [Re] (by utilizing the fact that algebras under consideration are Ree's algebras, see remark after definition of $\mathfrak{L}(A, D)$ in §1), or [K] or [W] (by noting that that \mathfrak{L}_0 is solvable).

(ii) Combining Theorem 5.3(i) (with remark after it) and Lemma 6.3, we recover the fact that each filtered deformation (with respect to the standard grading) of $W_1(n)$ is isomorphic to $W_1(n)$. This fact is important in consideration of some classes of Lie algebras with given properties of subalgebras or elements and was proved by Benkart, Isaacs and Osborn in [BIO], §3 and Dzhumadil'daev in [D1].

Now summarizing all our results, we obtain the final classification of the long filtration case.

Theorem 6.4. *(The ground field is algebraically closed of characteristic $p > 5$).*

\mathfrak{L} is a semisimple Lie algebra with a solvable maximal subalgebra defining in it a long filtration, if and only if either $\mathfrak{L} \simeq sl(2) \otimes O_m + 1 \otimes \mathfrak{D}$, or $W_1(n) \otimes O_m \subset \mathfrak{L} \subset \text{Der}(W_1(n)) \otimes O_m + 1 \otimes W_m$, where \mathfrak{D} in the first case, and $\text{pr}_{W_m} \mathfrak{L}$ in the second, are solvable subalgebras of W_m such that O_m has no \mathfrak{D} - or $\text{pr}_{W_m} \mathfrak{L}$ -invariant ideals.

Proof. “only if” part. Summarizing results of Lemmas 6.2 and 6.3, we obtain that semisimple Lie algebras of the form $\mathfrak{L}(A, D) + 1 \otimes \mathfrak{D}$ are exactly those whose socle is a direct sum of algebras $W_1(n) \otimes O_m$ for some $n, m \in \mathbb{N}$. By Corollary 5.4, these algebras (with solvable \mathfrak{D}), along with $sl(2) \otimes O_m + 1 \otimes \mathfrak{D}$, exhaust all possible semisimple Lie algebras with a solvable maximal subalgebra defining in it a long filtration. Obviously a socle of such algebra should consist of only one minimal ideal, and the assertion desired follows.

“if” part. In the $sl(2)$ case it is evident that $\mathfrak{L}_0 = \langle e_0, e_1 \rangle \otimes O_m + 1 \otimes \mathfrak{D}$ is a solvable maximal subalgebra.

In the $W_1(n)$ case, we have $W_1(n) \otimes O_m \subset \mathfrak{L} \subset Der(W_1(n) \otimes O_m) \simeq Der(W_1(n)) \otimes O_m + 1 \otimes Der(O_m)$. By Proposition 1.2, identify $W_1(n) \otimes O_m$ with $\mathfrak{L}(O_1(n-1) \otimes O_m, \partial \otimes 1)$. By Theorem 3.1(ii), $\mathfrak{L} = \mathfrak{L}(O_1(n-1) \otimes O_m, \partial \otimes 1) + 1 \otimes \mathfrak{D}$ for some solvable subalgebra $\mathfrak{D} \subset Der(O_1(n-1) \otimes O_m)$ (a further elucidation of the structure of \mathfrak{D} is possible due to conditions imposed on $pr_{Der(O_m)}\mathfrak{L}$ and Theorem 4.1(iii), but we don’t need it here).

Consider a maximal subalgebra \mathfrak{L}_0 containing a subalgebra $\langle e_0, e_1, \dots, e_{p-2} \rangle \otimes O_1(n-1) \otimes O_m + 1 \otimes \mathfrak{D}$. Obviously

$$\mathfrak{L}_0 = e_{-1} \otimes I + \langle e_0, e_1, \dots, e_{p-2} \rangle \otimes O_1(n-1) \otimes O_m + 1 \otimes \mathfrak{D}$$

for some $I \triangleleft O_1(n-1) \otimes O_m$. Since $O_1(n-1) \otimes O_m$ is a reduced polynomial ring itself, each its ideal is nilpotent, whence \mathfrak{L}_0 is solvable. ■

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