

DEFORMATIONS OF CURRENT LIE ALGEBRAS. I. SMALL ALGEBRAS IN CHARACTERISTIC 2

ALEXANDER GRISHKOV AND PASHA ZUSMANOVICH

ABSTRACT. We compute low-degree cohomology of current Lie algebras extended over the 3-dimensional simple algebra, compute deformations of related semisimple Lie algebras, and apply these results to classification of simple Lie algebras of absolute toral rank 2 and having a Cartan subalgebra of toral rank one. Everything is in characteristic 2.

INTRODUCTION

This is the first in the planned series of papers, devoted to a unified approach to computation of deformations of current Lie algebras and algebras close to them, and applications of those deformations in the structure theory. This first paper is devoted to some particular characteristic 2 case.

Throughout the paper, the ground field K is assumed to be of characteristic 2 (sometimes with additional qualifications, such as being perfect or algebraically closed), unless stated otherwise. When referring to other results in modular Lie algebras theory, we customary refer to “big characteristics”, what should mean “characteristic $p > 2$ ”, or, depending on the context, “characteristic $p \neq 2$ ”, unless specified otherwise.

The classification of finite-dimensional simple Lie algebras over an algebraically closed field of characteristic $p > 3$ was completed relatively recently (see [St]). While there was a significant progress in understanding of an unwieldy zoo of existing examples of simple Lie algebras in characteristic 2 and 3 in a series of recent papers by Dimitry Leites and his collaborators (see [BGL] and [BLLS] and references therein), the classification problem in these characteristics remain widely open.

Our starting point is another remarkable paper – by Serge Skryabin [Sk1] – in which he proved, among other things, that in characteristic 2, there are no simple Lie algebras of absolute toral rank 1, and characterized simple Lie algebras having a Cartan subalgebra of toral rank 1 as certain filtered deformations of semisimple Lie algebras having the socle of the form $S \otimes \mathcal{O}$, where S belongs to a certain family of simple Lie algebras, and \mathcal{O} is a divided powers algebra.

In this paper we compute these deformations for the case where S is the smallest algebra in the family: the 3-dimensional simple Lie algebra. In doing so, we follow the standard nowadays approach by Murray Gerstenhaber (for a nice overview, see, for example, [GS, §8]) in which infinitesimal deformations are described by the second cohomology of the underlying Lie algebra with coefficients in the adjoint module, and obstructions for prolongations of infinitesimal deformations live in the third cohomology. Even this very particular, at the first glance, case is of considerable interest: it allows to complete the ongoing classification of simple Lie algebras of absolute toral rank 2, due to the first author and Alexander Premet. (In [Sk2] it is cautiously conjectured that any restricted simple Lie algebra of absolute toral rank 2 is isomorphic either to A_2 , or to G_2 .)

There are similarities between this paper and [Z1]: in both cases, driven by structure theory of modular Lie algebras (classification of simple Lie algebras with a solvable maximal subalgebra in big characteristics in [Z1], and classification of certain simple Lie algebras in characteristic 2 in the present paper), we compute some filtered deformations of semisimple Lie algebras, naturally appearing in the corresponding classification problems (the present paper is related to [Sk1] in more or less the same way as [Z1] is related to [W]). In fact, we do a bit more: we compute low-degree cohomology and deformations of the corresponding current Lie algebra $S \otimes A$, as well as its extensions by derivations,

Date: last revised March 18, 2021.
J. Algebra **473** (2017), 513–544.

where instead of divided powers algebra we take an arbitrary associative commutative algebra A . These intermediate computations lead to interesting formulae intertwining various cohomological invariants of S and A , complement investigations in [Z1], [Z2] and [Z3], and are of independent interest.

In the subsequent papers we plan to extend these computations to all the cases appearing in Skryabin's theorem, i.e., where S is either Zassenhaus or Hamiltonian algebra[†], and to obtain in this way a full description of simple Lie algebras having a Cartan subalgebra of toral rank 1, what should be an important intermediate step in classification efforts. However, the more or less direct computational approach of this paper will meet considerable difficulties if we will try to extend it to that generality. For example, as it will be clear from the discussion below, the second cohomology $H^2(S, S)$ is involved, and the latter cohomology in the case of general Zassenhaus algebra in characteristic 2 seems to be enormous (some relevant computations were made in 1980s by Askar Dzhumadil'daev, but no full account is available in the literature; and the case of Hamiltonian algebras appears to be even more cumbersome). Thus, different, more subtle, approaches will be needed.

The contents of the paper are as follows. In the preliminary §1 we fix notation and recall the necessary notions (current algebras, various cohomology theories, etc.). In §§2 and 3 we obtain results about the second cohomology with the coefficients in the trivial and adjoint module, respectively, of the current Lie algebra extended over the 3-dimensional simple Lie algebra. In §4 we glue these results together to compute the positive part of the second cohomology, and filtered deformations of the corresponding semisimple Lie algebra. In §5 we apply the preceding results to derive the main result of the paper: there are no "new" simple Lie algebras of absolute toral rank 2 and with Cartan subalgebra of toral rank 1. This result will be used in the forthcoming classification of simple Lie algebras of absolute toral rank 2. The last §6 contains a brief discussion of a family of 15-dimensional simple Lie algebras appearing during the proof.

1. PRELIMINARIES

1.1. Zassenhaus algebras. Recall a construction of the ubiquitous Zassenhaus algebras (see, for example, [St, Vol. I, §7.6]).

Let A be an associative commutative algebra with unit, and D a derivation of A . Then the set of derivations $aD = \{aD \mid a \in A\}$ is a Lie algebra of derivations of A . Assuming that this Lie algebra is also a free (one-dimensional) A -module, and identifying aD with a , aD can be considered as a Lie algebra structure on A with the bracket $[a, b] = aD(b) - bD(a)$ for any $a, b \in A$. In characteristic 2, the latter formula is equivalent to

$$(1.1) \quad [a, b] = D(ab).$$

(Note that algebras with multiplication (1.1) were considered also in big characteristics, but, of course, then they are no longer Lie algebras. They belong to the class of so-called Novikov–Jordan algebras – commutative algebras satisfying a certain identity of degree 4, see [D]. In characteristic 2 the classes of Lie algebras and Novikov–Jordan algebras have nontrivial intersection – for example, all algebras with multiplication (1.1) – but these classes do not coincide).

Recall that the divided powers algebra $\mathcal{O}_1(n)$ over a field of characteristic p is defined as a p^n -dimensional algebra having a basis $\{x^{(i)}\}$, $0 \leq i < p^n$, with multiplication $x^{(i)}x^{(j)} = \binom{i+j}{j}x^{(i+j)}$. The special derivation ∂ of $\mathcal{O}_1(n)$ is defined as

$$\partial(x^{(i)}) = \begin{cases} x^{(i-1)} & \text{if } i > 0 \\ 0 & \text{if } i = 0. \end{cases}$$

[†]In characteristic 2, unlike in big characteristics, there are two types of simple Hamiltonian algebras, alternate and non-alternate, depending on the type of the differential 2-form the algebra preserves; see [BLLS, §1.1.2] and references therein. The Hamiltonian algebras we are dealing with are of alternate type (denoted by $\mathfrak{h}_{\Pi}(n, N)$ in [BLLS]).

Specializing the bracket (1.1) to the case $A = \mathcal{O}_1(n)$, and $D = \partial$, we arrive at the Lie algebra $W_1(n)$ of dimension 2^n with the basis $\{e_i = x^{(i+1)}\partial \mid -1 \leq i \leq 2^n - 2\}$ and multiplication

$$[e_i, e_j] = \begin{cases} \binom{i+j+2}{i+1} e_{i+j} & \text{if } -1 \leq i+j \leq 2^n - 2 \\ 0 & \text{otherwise.} \end{cases}$$

(The reader may be puzzled for a second by the unusual coefficient $\binom{i+j+2}{i+1}$ instead of the usual one $\binom{i+j+1}{j} - \binom{i+j+1}{i}$, but it is obvious that in characteristic 2 the latter is equal to the former; we adopt the usual convention that $\binom{i}{j} = 0$ if $i < j$.)

The algebra $W_1(n)$ is a subalgebra of the whole derivation algebra $\text{Der}(\mathcal{O}_1(n))$, the latter is freely generated as an $\mathcal{O}_1(n)$ -module by $\partial, \partial^2, \dots, \partial^{2^n-1}$.

In characteristic 2, unlike in big characteristics, the algebra $W_1(n)$ is not simple, but its commutant $W'_1(n)$ of dimension $2^n - 1$, linearly spanned by elements $\{e_i \mid -1 \leq i \leq 2^n - 3\}$, is. By abuse of terminology, we will refer to both algebras $W_1(n)$ and $W'_1(n)$ as *Zassenhaus algebras*. (In [Sk1] the latter algebra is denoted by $K'_1(n)$.)

In the first nontrivial case $n = 2$, the algebra $W'_1(2)$ is 3-dimensional. This is the case we will mainly deal with, so we adopt a special notation for this algebra, S , and its basic elements: $e = e_{-1}$, $h = e_0$, and $f = e_1$. The algebra S has multiplication table

$$(1.2) \quad [e, h] = e, \quad [f, h] = f, \quad [e, f] = h,$$

and is an analog of $\mathfrak{sl}(2)$ in big characteristics. Further, denoting $g = e_2$, we get a basis $\{e, h, f, g\}$ of $W_1(2)$ where, in addition to (1.2),

$$[e, g] = f, \quad [h, g] = 0, \quad [f, g] = 0.$$

1.2. Current and semisimple algebras. Given a Lie algebra L and associative commutative algebra A , the Lie algebra $L \otimes A$ with the bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for any $x, y \in L$, $a, b \in A$, is referred to as *current Lie algebra*. If A contains the unit 1, then there is an obvious embedding of Lie algebras $L \hookrightarrow L \otimes A$ induced by the map $x \mapsto x \otimes 1$, $x \in L$.

We can extend current Lie algebras by derivations. For example, if \mathfrak{D} is a Lie algebra of outer derivations of S , \mathfrak{E} is a Lie algebra of derivations of A , and U is an \mathfrak{E} -invariant subspace of A , we can consider an extension of the current algebra $S \otimes A$ of the form

$$(1.3) \quad S \otimes A + \mathfrak{D} \otimes U + \mathfrak{E},$$

(here, and in the subsequent similar constructions, “+” refers to the semidirect sum, while the sign \oplus is reserved for the direct sum of vector spaces), where $\mathfrak{D} \otimes U$ and \mathfrak{E} act on $S \otimes A$ via the first and the second tensor factor, respectively:

$$\begin{aligned} [x \otimes a, D \otimes u] &= D(x) \otimes au \\ [x \otimes a, E] &= x \otimes E(a) \end{aligned}$$

for any $x \in S$, $a \in A$, $u \in U$, $D \in \mathfrak{D}$, $E \in \mathfrak{E}$, and the Lie bracket between $\mathfrak{D} \otimes U$ and \mathfrak{E} is defined by taking their commutator as maps acting on $S \otimes A$:

$$[D \otimes u, E] = D \otimes E(u).$$

(Of course, more complex extensions are possible, where derivations are “mixed”, i.e., involve non-splitable sums of terms from $\text{Der}(S) \otimes B$ and $\text{Der}(A)$, but the “homogeneous” case above will be enough for our purposes here.)

The significance of such constructions in the structure theory of modular Lie algebras stems from the fact that, according to the classical Block theorem, every finite-dimensional semisimple Lie algebra over a field of positive characteristic is sandwiched between the direct sum of current Lie algebras of the form $S \otimes \mathcal{O}$, where S is a simple Lie algebra, and \mathcal{O} is a divided powers algebra, and its Lie

algebra of derivations, i.e., the direct sum of algebras of the form $\text{Der}(S) \otimes \mathcal{O} + \text{Der}(\mathcal{O})$ (see [St, Vol. I, Corollary 3.3.5]). In particular, it happens often in the structure theory that description of some classes of Lie algebras reduces to elucidation of the structure of some classes of deformations of semisimple Lie algebras of the kind (1.3), or similar algebras. The present paper is an instance of such elucidation.

1.3. Gradings. The basic elements of the Zassenhaus algebra provide the gradings

$$W'_1(n) = \bigoplus_{i=-1}^{2^n-3} Ke_i \quad \text{and} \quad W_1(n) = \bigoplus_{i=-1}^{2^n-2} Ke_i,$$

which will be referred to as *standard gradings*.

Any grading $S = \bigoplus_i S_i$ on an algebra S induces a grading on the current algebra $S \otimes A = \bigoplus_i (S_i \otimes A)$. In the cases $S = \mathfrak{S}$ and $S = W_1(2)$, this induced grading on the respective current algebra will be also referred to as *standard*.

The extended algebras of the form (1.3) acquire the grading induced from the standard grading on $S \otimes A$, assigning the respective weights to elements of $\mathfrak{D} \otimes U$ according to weights of the external derivations in \mathfrak{D} , and putting \mathfrak{E} to the zero component.

Recall that for \mathbb{Z} -gradings, *depth* is the minimal index of a negative nonzero component, and *length* is the maximal index of a positive nonzero component. For example, the standard gradings on \mathfrak{S} and $W_1(2)$ have length 1 and 2, respectively, and both of them have depth 1.

1.4. Lie algebra cohomology. Given a Lie algebra L and an L -module M , the corresponding n th cohomology will be denoted by $H^n(L, M)$ (we will be concerned exclusively with the cases $n = 1, 2, 3$, and the trivial module K or the adjoint module L). In evaluating cohomology, we will use repeatedly the following fact. If $L = \bigoplus_{\alpha \in G} L_\alpha$ is a grading of a Lie algebra L by an additively written abelian group G , and $M = \bigoplus_{\alpha \in G} M_\alpha$ is a G -graded L -module (in the case of the trivial module, all the grading is concentrated in degree 0), then the Chevalley–Eilenberg complex computing $H^n(L, M)$ decomposes into the direct sum of complexes:

$$(1.4) \quad H^n(L, M) \simeq \bigoplus_{\lambda \in G} H_\lambda^n(L, M),$$

where $H_\lambda^n(L, M)$ is linearly spanned by the classes of cocycles φ satisfying the condition

$$\varphi(L_{\alpha_1}, \dots, L_{\alpha_n}) \subseteq M_{\alpha_1 + \dots + \alpha_n - \lambda}$$

for any $\alpha_1, \dots, \alpha_n, \lambda \in G$. We will call n and λ the *cohomology degree* and *weight*, respectively.

In the case $G = \mathbb{Z}$, the *positive cohomology* is defined as

$$H_+^n(L, M) = \bigoplus_{\lambda > 0} H_\lambda^n(L, M).$$

When speaking about positive cohomology of algebras $S \otimes A$ and their extended algebras of the form (1.3), we always mean the standard grading.

When specifying cocycles (and, more generally, cochains) on algebras by giving the cocycle values on all appropriate combinations of basic elements, we will omit the zero values. For example, if we say (as, for example, in case (i) of Proposition 2.1 below) that a cocycle $\Phi : (S \otimes A) \wedge (S \otimes A) \rightarrow K$ is given by

$$(f \otimes a) \wedge (f \otimes b) \mapsto \xi(ab),$$

that implicitly assumes that Φ vanishes on $(e \otimes A) \wedge (e \otimes A)$, $(e \otimes A) \wedge (f \otimes A)$, $(e \otimes A) \wedge (h \otimes A)$, $(h \otimes A) \wedge (h \otimes A)$, and $(h \otimes A) \wedge (f \otimes A)$.

1.5. Deformations. The significance of the positive cohomology stems from the fact that it is responsible for description of filtered deformations of the graded algebra L , i.e., filtered Lie algebras such that their associated graded algebra is isomorphic to L . Infinitesimal deformations lie in $H_+^2(L, L)$, and obstructions to their prolongability are described by the Massey brackets, defined as cohomology classes in $H_+^3(L, L)$ of a cocycle $[[\varphi, \psi]] + [[\psi, \varphi]]$, where

$$[[\varphi, \psi]](x, y, z) = \varphi(\psi(x, y), z) + \varphi(\psi(z, x), y) + \varphi(\psi(y, z), x)$$

for any $x, y, z \in L$. (For details, see [GS] or [BGL, §2.1]. The latter reference addresses the peculiarities of characteristic 2. Note that, due to these peculiarities, the definition of Massey brackets adopted here deviates from the standard one: it accounts for only a “half” of the usual terms and is not symmetric.)

As in the case of positive cohomology, when speaking about filtered deformations of algebras, we always mean the standard grading.

1.6. Cyclic cohomology. One of the beauties of cohomology of current Lie algebras $L \otimes A$ is that it intertwines various cohomology theories of the underlying algebras L and A . We will need two such theories – or, rather, merely their low-degree incarnations – associated with the algebra A .

Recall that a map f of two variables is *symmetric* if it satisfies the condition $f(a, b) = f(b, a)$ for any its two arguments a, b , and is *alternating* if $f(a, a) = 0$. Obviously, in the class of bilinear maps, alternating maps are symmetric, but not vice versa.

Consider bilinear maps $\alpha : A \times A \rightarrow K$ satisfying the cocycle equation

$$(1.5) \quad \alpha(ab, c) + \alpha(ca, b) + \alpha(bc, a) = 0$$

for any $a, b, c \in A$. Such *symmetric* maps form the first cyclic cohomology $\mathrm{HC}^1(A)$ (see, for example, [L, Proposition 2.1.14] for equivalence of this definition and the usual definition in terms of the double complex; note also that normally we would say here “skew-symmetric”, but in characteristic 2 skew-symmetry and symmetry are the same). Note an obvious but useful fact: if A contains a unit 1, then $\alpha(1, A) = 0$ for any $\alpha \in \mathrm{HC}^1(A)$.

We will consider a variation of the first cyclic cohomology: the space of *alternating* maps satisfying the same cocycle equation (1.5) will be denoted by $\widehat{\mathrm{HC}}^1(A)$. We have an obvious inclusion $\widehat{\mathrm{HC}}^1(A) \subseteq \mathrm{HC}^1(A)$.

1.7. Harrison cohomology. $\mathrm{Har}^n(A, A)$ denotes the n th degree Harrison cohomology of A . See [H] where this cohomology is introduced, and [GS] for a more modern treatment, but all what we will need is an interpretation of Harrison cohomology in low degrees: Harrison 1-coboundaries vanish, Harrison 1-cocycles are the same as Hochschild 1-cocycles, i.e., derivations, so $\mathrm{Har}^1(A, A)$ coincides with $\mathrm{Der}(A)$, the space (actually, a Lie algebra) of derivations of A ; Harrison 2-cochains form the space of symmetric bilinear maps $A \times A \rightarrow A$, denoted by $S^2(A, A)$; Harrison 2-coboundaries are the same as Hochschild 2-coboundaries, and Harrison 2-cocycles are symmetric Hochschild 2-cocycles. An easy but useful observation is that every Harrison 2-cochain α is cohomologous to one with $\alpha(1, A) = 0$ ([H, p. 194]).

Another elementary observation which, nevertheless, will play role in the sequel: any Harrison 1-cocycle, i.e., derivation, of an algebra A , vanishes on $A^{[2]}$, where $A^{[2]}$ is the subalgebra of A linearly spanned by squares of all elements (obviously, already the *set* of all squares forms a subring in A , and is a subalgebra if the ground field is perfect).

1.8. Toral ranks, Skryabin’s theorem. Recall that a (relative) *toral rank* of a subalgebra S of a Lie algebra L is the maximal dimension of tori in the p -subalgebra generated by S in the p -envelope of L . An *absolute toral rank* $\mathrm{TR}(L)$ of a Lie algebra L is the toral rank of L as a subalgebra of itself (in other words, the maximal dimension of tori in the p -envelope of L). In the classification scheme of simple Lie algebras, both in small and big characteristics, a significant role is played by intermediate classifications of simple Lie algebras having Cartan subalgebras of “small” toral rank.

We quote the relevant part of [Sk1, Theorem 6.3]:

Theorem 1.1 (Skryabin). *A simple finite-dimensional Lie algebra over an algebraically closed field, having a Cartan subalgebra of toral rank 1, is isomorphic either to the Zassenhaus algebra $W_1'(m)$ ($m > 1$), or to the Hamiltonian algebra in two variables $H_2''(m_1, m_2)$ ($m_1, m_2 > 1$), or to a filtered deformation of the graded semisimple Lie algebra L such that*

$$(1.6) \quad S \otimes \mathcal{O}_1(n) \subset L \subseteq \text{Der}(S) \otimes \mathcal{O}_1(n) + K\partial,$$

where either $n = 2$ and $S \simeq W_1'(m)$, or $n = 1$ and $S \simeq H_2''(m_1, m_2)$. The grading on L is of depth 1 and the induced grading on the socle $S \otimes \mathcal{O}_1(n)$ is standard.

Note that the condition of the grading to be standard, implicit in the formulation of the theorem in [Sk1], follows from Theorems 1.1 and 1.2 of that paper.

Concrete realization of Hamiltonian algebras is immaterial for our purpose here. All what we need to know is the absolute toral rank of the algebras involved: $\text{TR}(W_1'(m)) = m$ and $\text{TR}(H_2''(m_1, m_2)) = m_1 + m_2 - 1$.

We are going to describe filtered deformations appearing in Theorem 1.1 in the simplest case $S = \mathfrak{S}$.

2. SECOND COHOMOLOGY OF $\mathfrak{S} \otimes A$ WITH TRIVIAL COEFFICIENTS

As explained in the introduction, we are interested in low-degree cohomology of $\mathfrak{S} \otimes A$. Our first goal is to compute $H^2(\mathfrak{S} \otimes A, K)$. (In what follows, A denotes an arbitrary associative commutative algebra with unit.)

There are general formulas for $H^2(L \otimes A, K)$ for an arbitrary Lie algebra L (see [Z3, Theorem 1]), but they are valid in characteristic $\neq 2, 3$. To extend these results to the case of characteristic 2, new notions and techniques will be needed. We defer their development to a subsequent paper, and treat here the case $L = \mathfrak{S}$ via straightforward computations.

Proposition 2.1.

$$H^2(\mathfrak{S} \otimes A, K) \simeq (A/A^{[2]})^* \oplus (A/A^{[2]})^* \oplus \widehat{\text{HC}}^1(A).$$

The basic cocycles can be chosen as follows:

- (i) $(f \otimes a) \wedge (f \otimes b) \mapsto \xi(ab)$;
- (ii) $(e \otimes a) \wedge (e \otimes b) \mapsto \xi(ab)$, where, in both (i) and (ii), $\xi : A \rightarrow K$ is a linear map such that $\xi(A^{[2]}) = 0$;
- (iii) $(e \otimes a) \wedge (f \otimes b) \mapsto \alpha(a, b)$
 $(h \otimes a) \wedge (h \otimes b) \mapsto \alpha(a, b)$, where $\alpha \in \widehat{\text{HC}}^1(A)$.

Proof. Consider the standard \mathbb{Z} -grading of $\mathfrak{S} \otimes A$, and the induced decomposition (1.4) of its cohomology:

$$H^2(\mathfrak{S} \otimes A, K) = \bigoplus_{k \in \mathbb{Z}} H_k^2(\mathfrak{S} \otimes A, K).$$

Obviously, the nonzero terms here are possible only for $-2 \leq k \leq 2$.

Further, the root space decomposition of $\mathfrak{S} \otimes A$ with respect to the toral element $h \otimes 1$ has the form

$$(2.1) \quad \mathfrak{S} \otimes A = L_{\bar{0}} \oplus L_{\bar{1}},$$

where $L_{\bar{0}} = L_0 = h \otimes A$, and $L_{\bar{1}} = L_{-1} \oplus L_1 = \langle e, f \rangle \otimes A$. Hence, by the theorem of invariance of cohomology with respect to the torus action (see, for example, [F, Chapter 1, §5.2]), we may restrict our attention to the subcomplex generated by cocycles respecting this decomposition. Being coupled with decomposition (2.1), this shows that it will be enough to consider cocycles of weight $-2, 0$ and 2 .

The proof of Proposition 2.1 now follows from the two Lemmas below, and noting that the case of cohomology of weight 2 is, by symmetry interchanging e and f , completely similar to those of weight -2 described in Lemma 2.2. \square

Lemma 2.2. $H_{-2}^2(\mathfrak{S} \otimes A, K) \simeq (A/A^{[2]})^*$. The basic cocycles can be chosen as in part (i) of Proposition 2.1.

Proof. Let Φ be a 2-cocycle with trivial coefficients on $S \otimes A$ of weight -2 . We may write

$$\Phi(f \otimes a, f \otimes b) = \alpha(a, b)$$

for any $a, b \in A$, and for some bilinear alternating map $\alpha : A \times A \rightarrow K$. Writing the cocycle equation for the triple $h \otimes a, f \otimes b, f \otimes c$, we get $\alpha(ab, c) + \alpha(ac, b) = 0$, which implies $\alpha(a, b) = \xi(ab)$ for some linear map $\xi : A \rightarrow K$. Since α is alternating, we have $\xi(A^{[2]}) = 0$. As there are no nonzero linear maps $\Omega : S \otimes A \rightarrow K$, and, hence, coboundaries, of weight -2 , the linear independence of cocycles of weight -2 implies their cohomological independence. \square

Lemma 2.3. $H_0^2(S \otimes A, K) \simeq \widehat{HC}^1(A)$. *The basic cocycles can be chosen as in part (iii) of Proposition 2.1.*

Proof. Let Φ be a 2-cocycle with trivial coefficients on $S \otimes A$ of weight 0. We may write

$$\begin{aligned} \Phi(e \otimes a, f \otimes b) &= \alpha(a, b) \\ \Phi(h \otimes a, h \otimes b) &= \beta(a, b) \end{aligned}$$

for any $a, b \in A$, and for some bilinear maps $\alpha, \beta : A \times A \rightarrow K$. Since Φ is alternating, β is alternating too.

Any coboundary of weight 0 is generated by a linear map $\Omega : S \otimes A \rightarrow K$ defined by $\Omega(e \otimes a) = \Omega(f \otimes a) = 0$ and $\Omega(h \otimes a) = \omega(a)$ for some linear map $\omega : A \rightarrow K$, and hence has the form $d\Phi(e \otimes a, f \otimes b) = \omega(ab)$, with zero values on all other pairs of basic elements. Setting $\omega(a) = \alpha(a, 1)$, and modifying the cocycle Φ by the respective coboundary, we may assume $\Phi(e \otimes a, f \otimes 1) = 0$, i.e., $\alpha(a, 1) = 0$.

Writing the cocycle equation for the triple $e \otimes a, h \otimes b, f \otimes c$, we get

$$\alpha(ab, c) + \beta(ac, b) + \alpha(bc, a) = 0$$

for any $a, b, c \in A$, which implies $\beta = \alpha$ and $\alpha \in \widehat{HC}^1(A)$, thus arriving at cocycles of type (iii) in Proposition 2.1.

The equality $\Phi = d\Omega$ for some linear map $\Omega : S \otimes A \rightarrow K$ of weight 0, yields $\alpha = 0$, which shows that these cocycles are cohomologically dependent if and only if the corresponding α 's are linearly dependent. \square

Corollary 2.4. $H_+^2(S \otimes A, K) \simeq (A/A^{[2]})^*$. *The basic cocycles can be chosen as in part (ii) of Proposition 2.1.*

Proof. We have $H_+^2(S \otimes A, K) = H_2^2(S \otimes A, K)$. \square

3. LOW-DEGREE COHOMOLOGY OF $S \otimes A$ WITH ADJOINT COEFFICIENTS

Now let us turn to cohomology of $S \otimes A$ with coefficients in the adjoint module. A few general remarks are in order.

As noted in [Z4, Lemma 1.1], for any Lie algebra L and non-negative integer k , the space $H^k(L \otimes A, L \otimes A)$ contains $H^k(L, L) \otimes A$. On the level of cocycles, this embedding is given by

$$(3.1) \quad (x_1 \otimes a_1) \wedge \cdots \wedge (x_k \otimes a_k) \mapsto \varphi(x_1, \dots, x_k) \otimes a_1 \cdots a_k u$$

for any $x_1, \dots, x_k \in L$, $a_1, \dots, a_k \in A$, where φ is some k -cocycle φ on L , and $u \in A$.

Similarly, one can show that $H^k(L \otimes A, L \otimes A)$ always contains $\text{Cent}(L) \otimes \text{Har}^k(A, A)$. Here $\text{Cent}(L)$ is the centroid of L , i.e., the space of linear maps $\omega : L \rightarrow L$ such that $\omega([x, y]) = [x, \omega(y)]$ for any $x, y \in L$. The latter embedding is, however, more involved – due to the fact that higher Harrison cohomology is determined in terms of complicated permutations and is sensitive to characteristic of the ground field – and we restrict ourselves to the cases of low cohomology degree (as that is all what we need here anyway): on the level of 1- and 2-cocycles, the asserted embedding is given by

$$(3.2) \quad x \otimes a \mapsto \omega(x) \otimes d(a)$$

and

$$(3.3) \quad (x \otimes a) \wedge (y \otimes b) \mapsto \omega([x, y]) \otimes \alpha(a, b)$$

respectively, for any $x, y \in L$, $a, b \in A$, and where $\omega \in \text{Cent}(L)$, $d \in \text{Der}(A) = \text{Har}^1(A, A)$, $\alpha \in \text{Har}^2(A, A)$.

Moreover, the cocycles of types (3.1) and (3.2)-(3.3) are cohomologically independent, so $H^k(L \otimes A, L \otimes A)$ always contains the direct sum

$$(3.4) \quad \left(H^k(L, L) \otimes A \right) \oplus \left(\text{Cent}(L) \otimes \text{Har}^k(A, A) \right).$$

For an (easy straightforward) proof, see [Z2, Theorem 2.1] ($k = 1$) and [Z1, §2] ($k = 2$).

We will prove that in the case of interest that is all what we have:

Proposition 3.1. *For $n = 1, 2$, we have $H^n(S \otimes A, S \otimes A) \simeq \left(H^n(S, S) \otimes A \right) \oplus \text{Har}^n(A, A)$.*

Remark 3.2.

1) As S is, obviously, central, i.e., its centroid coincides with the ground field, the second summand in (3.4) is reduced to a mere $\text{Har}^n(A, A)$.

2) This is similar with the cases of characteristics 0 and > 3 . In these cases, for $n = 1, 2$,

$$(3.5) \quad H^n(\mathfrak{sl}(2) \otimes A, \mathfrak{sl}(2) \otimes A) \simeq \text{Har}^n(A, A).$$

The case $n = 1$ is a particular case of many previous results in the literature, of which [Z2, Theorem 2.1] is, perhaps, the most general one (see the proof of Proposition 3.3 below). The case $n = 2$ is a particular case of [C] (where similar formulas are obtained for all finite-dimensional simple Lie algebras over an algebraically closed field of characteristic zero), and of [B] (where cohomology up to degree 4 is computed). The proofs there utilize complete reducibility of representations of $\mathfrak{sl}(2)$ and other facts peculiar to characteristic zero case.

In the case of characteristic > 3 , the formula (3.5) is established in [Z1, Proposition 2.8]. Our computations are organized similar to those in [Z1], but characteristic 2 case turns out to be more cumbersome. The reason for this is twofold. First, unlike the cases of big characteristic, $H^2(S, S)$ is not zero (see Lemma 3.4 below). Second, the root space decomposition (2.1) with respect to the toral element $h \otimes 1$ is more coarse than in the big characteristic case (the roots -1 and 1 coalesce).

The case $n = 1$ of Proposition 3.1 is a particular case of the general statement:

Proposition 3.3. *If L is perfect and central Lie algebra, then*

$$H^1(L \otimes A, L \otimes A) \simeq \left(H^1(L, L) \otimes A \right) \oplus \text{Der}(A).$$

Proof. This is a direct consequence of [Z2, Corollary 2.2]. The standing assumption of [Z2] is characteristic $\neq 2, 3$ of the ground field, but an easy inspection of the proof of Theorem 2.1 from that paper, on which Corollary 2.2 is based, shows that it is characteristic-free. As L is perfect and central, the term in the general formula for $H^1(L \otimes A, L \otimes A)$ involving commutant vanishes, and the centroid of L coincides with the ground field, so we are left with the desired isomorphism. \square

We start the proof of the $n = 2$ case with recording necessary facts about cohomology of S .

Lemma 3.4.

- (i) $H^1(S, S)$ is 2-dimensional, with basic cocycles (outer derivations) given by $(\text{ad } e)^2$ and $(\text{ad } f)^2$.
- (ii) $H^2(S, S)$ is 2-dimensional, with basic cocycles given by

$$(3.6) \quad f \wedge h \mapsto e$$

and

$$(3.7) \quad e \wedge h \mapsto f.$$

Proof. Direct computations, similar to those performed below, but simpler. \square

Remark 3.5.

1) This, essentially, provides information on the whole cohomology of S with coefficients in the adjoint module, as $H^0(S, S)$ coincides with the center of S and hence is zero, and $H^3(S, S) \simeq H_0(S, S) = 0$ by the Poincaré duality. (Here H_0 denotes the 0th homology, not to be confused with the subscript denoting the cohomology weight. As this is the only place in the paper where homology makes – a rather superficial – appearance, this should not lead to confusion.)

2) Part (i) is contained implicitly in [P], where the 5-dimensional algebra $\text{Der}(S)$ is used to exhibit some phenomenon peculiar to characteristic 2. Part (ii) is stated in [BGL, Lemma 4.1].

3) The algebra linearly spanned by S and $(\text{ad } f)^2$, is isomorphic to $W_1(2)$ (with g being identified with $(\text{ad } f)^2$).

4) Any linear combination of the nontrivial 2-cocycles exhibited in (ii) has trivial Massey bracket, and hence prolonged trivially to a (nontrivial) deformation of S . However, any such deformation is isomorphic to S . Such phenomenon, under the name of *semitrivial deformations*, is thoroughly investigated in [BLLS].

Proof of Proposition 3.1, case $n = 2$. Employing again the standard grading and root space decomposition, like in the proof of Proposition 2.1, we obtain

$$H^2(S \otimes A, S \otimes A) = \bigoplus_{k \in \mathbb{Z}} H_k^2(S \otimes A, S \otimes A).$$

with nonzero terms for $k = -2, 0, 2$.

The next lemma is formulated in a setting slightly more general than it is needed for computation of cohomology of $S \otimes A$ of weight 2. This will be used later in Proposition 4.1, where we will compute the positive cohomology of an extended algebra of the form (1.3).

Lemma 3.6. *Let $\xi : A \rightarrow K$ and $D : A \rightarrow A$ be linear maps such that $\xi(1) = 0$ and $D(1) = 0$, and $\Lambda : A \times A \rightarrow A$ a bilinear alternating map such that $\Lambda(1, A) = 0$. Then there exists a cochain $\Phi \in C_2^2(S \otimes A, S \otimes A)$ such that its coboundary $d\Phi$ has the form:*

$$(3.8) \quad \begin{aligned} (e \otimes a) \wedge (e \otimes b) \wedge (e \otimes c) &\mapsto e \otimes \left(\xi(ab)D(c) + \xi(ca)D(b) + \xi(bc)D(a) \right) \\ (e \otimes a) \wedge (e \otimes b) \wedge (h \otimes c) &\mapsto h \otimes \xi(ab)D(c) \\ (e \otimes a) \wedge (e \otimes b) \wedge (f \otimes c) &\mapsto f \otimes \left(\xi(ab)D(c) + a\Lambda(b, c) + b\Lambda(a, c) \right) \\ (e \otimes a) \wedge (h \otimes b) \wedge (h \otimes c) &\mapsto f \otimes a\Lambda(b, c) \end{aligned}$$

if and only if

$$(3.9) \quad \Lambda(a, b) = \xi(a)D(b) + \xi(b)D(a)$$

and

$$(3.10) \quad \begin{aligned} \left(\xi(a)b + \xi(b)a + \xi(ab)1 \right) D(c) + \left(\xi(c)a + \xi(a)c + \xi(ca)1 \right) D(b) + \left(\xi(b)c + \xi(c)b + \xi(bc)1 \right) D(a) \\ = 0 \end{aligned}$$

for any $a, b, c \in A$.

Moreover, in this case Φ is equal to the sum of a 2-coboundary and a map of the form

$$\begin{aligned} (e \otimes a) \wedge (e \otimes b) &\mapsto h \otimes \left(\xi(a)D(b) + \xi(b)D(a) \right) \\ (e \otimes a) \wedge (h \otimes b) &\mapsto f \otimes \left(av + \xi(b)D(a) \right) \end{aligned}$$

for some $v \in A$.

Proof. We may write

$$\begin{aligned}\Phi(e \otimes a, e \otimes b) &= h \otimes \beta(a, b) \\ \Phi(e \otimes a, h \otimes b) &= f \otimes \alpha(a, b)\end{aligned}$$

for any $a, b \in A$, and for some bilinear maps $\alpha, \beta : A \times A \rightarrow A$, where β is alternating.

Modifying Φ by the coboundary $d\Omega$, where the cochain $\Omega : S \otimes A \rightarrow S \otimes A$ is defined as follows: $\Omega(e \otimes a) = f \otimes \beta(1, a)$, and $\Omega(f \otimes A) = \Omega(h \otimes A) = 0$, we may assume that, up to coboundaries, $\Phi(e \otimes 1, e \otimes a) = 0$, i.e., $\beta(1, a) = 0$.

Evaluating $d\Phi$ for all 4 triples present in the condition (3.8), we get respectively:

$$(3.11) \quad \beta(a, b)c + \beta(c, a)b + \beta(b, c)a = \xi(ab)D(c) + \xi(ca)D(b) + \xi(bc)D(a)$$

$$(3.12) \quad \beta(ac, b) + \beta(bc, a) + b\alpha(a, c) + a\alpha(b, c) = \xi(ab)D(c)$$

$$(3.13) \quad \alpha(b, ac) + \alpha(a, bc) + c\beta(a, b) = \xi(ab)D(c) + a\Lambda(b, c) + b\Lambda(a, c)$$

$$(3.14) \quad \alpha(ab, c) + \alpha(ac, b) + c\alpha(a, b) + b\alpha(a, c) = a\Lambda(b, c)$$

for any $a, b, c \in A$. (Since Φ is of weight 2, the cocycle equation $d\Phi = 0$ for other basic triples is satisfied automatically.)

Substituting $c = 1$ in (3.11), we get

$$(3.15) \quad \beta(a, b) = \xi(a)D(b) + \xi(b)D(a)$$

for any $a, b \in A$. Substituting $a = 1$ in (3.13), we get

$$\alpha(b, c) = \alpha(1, bc) + \xi(b)D(c) + \Lambda(b, c)$$

for any $b, c \in A$. Substituting the latter equality in (3.14) with $a = 1$, we get

$$(3.16) \quad b\alpha(1, c) + c\alpha(1, b) = \xi(b)D(c) + \xi(c)D(b) + \Lambda(b, c)$$

for any $b, c \in A$. Setting in the latter equality $c = 1$, we get

$$(3.17) \quad \alpha(1, b) = b\alpha(1, 1)$$

for any $b \in A$. Substituting this back in (3.16), we infer that the left-hand side of that equality vanish, and hence (3.9) holds. Due to (3.15), $\beta = \Lambda$.

Substituting $b = 1$ in (3.12), and taking into account (3.15) and (3.17), we get

$$\alpha(a, c) = ac\alpha(1, 1) + \xi(c)D(a)$$

for any $a, c \in A$.

Now each of (3.11), (3.12), and (3.13) is equivalent to (3.10), and (3.14) is satisfied automatically. \square

Corollary 3.7. $H_2^2(S \otimes A, S \otimes A) \simeq A$. The basic cocycles can be chosen as

$$(3.18) \quad (e \otimes a) \wedge (h \otimes b) \mapsto f \otimes av$$

for some $v \in A$.

Proof. Apply Lemma 3.6 with $\xi = 0$, $D = 0$, and $\Lambda = 0$. It is straightforward to check that the cocycles of the form (3.18) are cohomologically independent for linearly independent values of v . (This also follows from Lemma 3.4(ii) and observation that the resulting map is a 2-cocycle of kind (3.1), where φ coincides with (3.7).) \square

Lemma 3.8. $H_0^2(S \otimes A, S \otimes A) \simeq \text{Har}^2(A, A)$. The basic cocycles can be chosen as

$$(3.19) \quad (x \otimes a) \wedge (y \otimes b) \mapsto [x, y] \otimes \beta(a, b),$$

where $x, y \in S$, $a, b \in A$, and β is a Harrison 2-cocycle on A .

Proof. Let Φ be a 2-cocycle on $S \otimes A$ respecting the standard grading. We may write

$$\begin{aligned}\Phi(h \otimes a, h \otimes b) &= h \otimes \alpha(a, b) \\ \Phi(e \otimes a, h \otimes b) &= e \otimes \beta(a, b) \\ \Phi(f \otimes a, h \otimes b) &= f \otimes \gamma(a, b) \\ \Phi(e \otimes a, f \otimes b) &= h \otimes \delta(a, b)\end{aligned}$$

for any $a, b \in A$, and some bilinear maps $\alpha, \beta, \gamma, \delta : A \times A \rightarrow A$. Due to the fact that Φ is alternating, α is alternating too.

Evaluating $d\Phi$ for the following 4 triples:

$$\begin{aligned}h \otimes a, h \otimes b, e \otimes c \\ e \otimes a, e \otimes b, f \otimes c \\ f \otimes a, f \otimes b, e \otimes c \\ h \otimes a, e \otimes b, f \otimes c\end{aligned}$$

we get respectively:

$$(3.20) \quad \beta(ac, b) + \beta(bc, a) + b\beta(c, a) + a\beta(c, b) + c\alpha(a, b) = 0$$

$$(3.21) \quad \beta(b, ac) + \beta(a, bc) + b\delta(a, c) + a\delta(b, c) = 0$$

$$(3.22) \quad \gamma(b, ac) + \gamma(a, bc) + b\delta(c, a) + a\delta(c, b) = 0$$

$$(3.23) \quad \delta(ab, c) + \delta(b, ac) + \alpha(bc, a) + c\beta(b, a) + b\gamma(c, a) = 0$$

for any $a, b, c \in A$.

Modifying the cocycle Φ by the coboundary $d\Omega$, where the cochain $\Omega : S \otimes A \rightarrow S \otimes A$ is defined as follows: $\Omega(e \otimes A) = \Omega(f \otimes A) = 0$, and $\Omega(h \otimes a) = h \otimes \beta(1, a)$, we may assume that $\Phi(e \otimes 1, h \otimes a) = 0$, i.e., $\beta(1, a) = 0$.

Substituting $c = 1$ in (3.20) and in (3.21), we get respectively:

$$(3.24) \quad \begin{aligned}\alpha(a, b) &= \beta(a, b) + \beta(b, a) \\ \beta(a, b) + \beta(b, a) &= b\delta(a, 1) + a\delta(b, 1)\end{aligned}$$

and, consequently,

$$(3.25) \quad \alpha(a, b) = b\delta(a, 1) + a\delta(b, 1)$$

for any $a, b \in A$.

Similarly, substituting $c = 1$ in (3.22), we get:

$$(3.26) \quad \gamma(a, b) + \gamma(b, a) = b\delta(1, a) + a\delta(1, b)$$

for any $a, b \in A$.

Substituting $b = 1$ in (3.23), we get:

$$(3.27) \quad \delta(a, c) + \delta(1, ac) + \alpha(c, a) + \gamma(c, a) = 0$$

for any $a, c \in A$.

Symmetrizing the last equality and using (3.26), we get:

$$\delta(a, c) + \delta(c, a) + a\delta(1, c) + c\delta(1, a) = 0$$

for any $a, c \in A$.

Setting here $a = 1$ and $c = 1$ respectively, we get $\delta(a, 1) = \delta(1, a) = a\delta(1, 1)$, and hence δ is symmetric. Together with (3.25) this implies $\alpha = 0$, and, according to (3.24), that β is symmetric. Moreover, (3.27) now yields

$$\gamma(a, b) = \delta(a, b) + ab\delta(1, 1)$$

for any $a, b \in A$, and hence γ is symmetric.

Substituting now $b = 1$ in (3.21), we get

$$\beta(a, c) = \delta(a, c) + ac\delta(1, 1)$$

for any $a, c \in A$, and hence $\beta = \gamma$.

Hence, Φ may be written in the form (3.19), plus the map sending $(e \otimes a) \wedge (f \otimes b)$ to $h \otimes ab\delta(1, 1)$. The latter map is obviously equal to the coboundary $d\Omega$ for $\Omega(e \otimes a) = e \otimes a\delta(1, 1)$ and $\Omega(f \otimes A) = \Omega(h \otimes A) = 0$. \square

Conclusion of the proof of Proposition 3.1. According to Lemma 3.4(ii),

$$H^2(S, S) = H_{-2}^2(S, S) \oplus H_2^2(S, S),$$

with basic cocycles (3.6) and (3.7) of weight -2 and 2 , respectively. This, coupled with Corollary 3.7 (together with its negative grading counterpart), implies that

$$H_{-2}^2(S \otimes A, S \otimes A) \oplus H_2^2(S \otimes A, S \otimes A) \simeq H^2(S, S) \otimes A.$$

The rest follows from Lemma 3.8. \square

4. SECOND COHOMOLOGY AND DEFORMATIONS OF $S \otimes A$ EXTENDED BY DERIVATIONS

Now, having the low-degree cohomology of the current Lie algebra $S \otimes A$ at hand, we glue its components together, via the standard Hochschild–Serre spectral sequence, to compute the second positive cohomology of an extension of $S \otimes A$ of the form (1.3), where both \mathfrak{D} and \mathfrak{E} are 1-dimensional (in fact, \mathfrak{D} is linearly spanned by $(\text{ad } f)^2$), i.e., of Lie algebras of the form

$$S \otimes A + g \otimes U + KD,$$

where D is a nonzero derivation of A , and U is a D -invariant subspace of A . According to Remark 3.5, part 3, those are exactly Lie algebras lying between $S \otimes A + KD$ and $W_1(2) \otimes A + KD$.

Following §1.3, we consider the grading of such algebras of length 2 (length 1 in the case $U = 0$):

$$(4.1) \quad \begin{array}{ll} \text{weight } -1 : & e \otimes A \\ \text{weight } 0 : & h \otimes A + KD \\ \text{weight } 1 : & f \otimes A \\ \text{weight } 2 : & g \otimes U. \end{array}$$

It is possible to compute the whole second cohomology, as well as to consider cases of more general algebras \mathfrak{D} and \mathfrak{E} , but this will lead to numerous trivial but very cumbersome technicalities. Also, the resulting formulas are not very aesthetically appealing. That is why we restrict ourselves just to the case necessary for our present goal. The computations are similar to those performed in [Z1, §5].

Proposition 4.1. *Assume $\dim A > 2$ and $U \neq 0$. Then*

$$H_+^2(S \otimes A + g \otimes U + KD, S \otimes A + g \otimes U + KD) \simeq A^D \oplus \frac{A}{D(A) + U} \oplus \mathfrak{E}_{D,U},$$

where $\mathfrak{E}_{D,U}$ consists of linear maps $\xi : A \rightarrow K$ vanishing on $A^{[2]}$, $D(A)$, and U , and satisfying the conditions

$$\xi(a)D(b) + \xi(b)D(a) \in U$$

and

$$(4.2) \quad \begin{aligned} & \left(\xi(a)b + \xi(b)a + \xi(ab)1 \right) D(c) + \left(\xi(c)a + \xi(a)c + \xi(ca)1 \right) D(b) + \left(\xi(b)c + \xi(c)b + \xi(bc)1 \right) D(a) \\ & \hspace{20em} = 0 \end{aligned}$$

for any $a, b, c \in A$.

The basic cocycles can be chosen as:

(i)

$$(e \otimes a) \wedge (h \otimes b) \mapsto f \otimes abv$$

 for some $v \in A^D$;

(ii)

$$(e \otimes a) \wedge D \mapsto f \otimes av$$

 for some $v \in A$;

(iii)

$$(e \otimes a) \wedge (e \otimes b) \mapsto h \otimes \left(\xi(a)D(b) + \xi(b)D(a) \right) + \xi(ab)D$$

$$(e \otimes a) \wedge (h \otimes b) \mapsto f \otimes \xi(b)D(a)$$

$$(e \otimes a) \wedge (f \otimes b) \mapsto g \otimes \left(\xi(a)D(b) + \xi(b)D(a) \right)$$

$$(h \otimes a) \wedge (h \otimes b) \mapsto g \otimes \left(\xi(a)D(b) + \xi(b)D(a) \right),$$

 where $\xi \in \Xi_{D,U}$.

 The case $U = 0$ is similar, if not slightly simpler.

Proposition 4.2.

$$H_+^2(S \otimes A + KD, S \otimes A + KD) \simeq A^D \oplus \frac{A}{D(A)} \oplus \Xi_{D,0}.$$

The basic cocycles can be chosen as follows: types (i) and (ii) as in Proposition 4.1, and

(iii)

$$(e \otimes a) \wedge (e \otimes b) \mapsto \xi(ab)D$$

$$(e \otimes a) \wedge (h \otimes b) \mapsto f \otimes \xi(b)D(a),$$

 where $\xi \in \Xi_{D,0}$.

Remark 4.3. By definition, $\Xi_{D,0}$ consists of linear maps $\xi : A \rightarrow K$ vanishing on $A^{[2]}$ and $D(A)$, and such that $\xi(a)D(b) + \xi(b)D(a) = 0$ for any $a, b \in A$.

Proof of Proposition 4.1. By [Z1, Proposition 1.1(i)], $Z(S \otimes A) \simeq Z(S) \otimes A = 0$ (where $Z(L)$ denotes the center of a Lie algebra L), so $S \otimes A$ is centerless and, obviously, perfect. Further, $g \otimes U + KD$ consists, obviously, of outer derivations of the algebra $S \otimes A$ (cf. [Z1, Proposition 1.1(ii)]), so [Z1, Lemma 5.1] is applicable. By this Lemma, the relevant terms in the Hochschild–Serre spectral sequence abutting to the second cohomology in question, relative to the ideal $S \otimes A$, are:

$$E_\infty^{20} = 0$$

$$(4.3) \quad E_\infty^{11} = E_2^{11} \simeq H^1 \left(g \otimes U + KD, \frac{H^1(S \otimes A, S \otimes A)}{g \otimes U + KD} \right)$$

$$(4.4) \quad E_2^{21} \simeq H^2 \left(g \otimes U + KD, \frac{H^1(S \otimes A, S \otimes A)}{g \otimes U + KD} \right)$$

$$E_2^{02} \simeq H^2(S \otimes A, S \otimes A)^{(g \otimes U + KD)} \oplus (\text{Ker } F)^{(g \otimes U + KD)}$$

$$E_\infty^{02} = E_3^{02} = \text{Ker } d_2^{02}.$$

(The formula for the E_2^{21} term is not specified in [Z1, Lemma 5.1], but it is implicit in its proof, and follows from the same standard homological considerations as for the E_2^{11} term.) The linear map

$$F : H^2(S \otimes A, K) \otimes (U + KD) \rightarrow H^3(S \otimes A, S \otimes A)$$

sends the elements $[\Phi] \otimes u$ and $[\Phi] \otimes D$, where $[\Phi]$ is the class of a 2-cocycle Φ , and $u \in U$, to the class of a 3-cocycle defined, respectively, as

$$(x \otimes a) \wedge (y \otimes b) \wedge (z \otimes c) \mapsto \Phi(x \otimes a, y \otimes b)[z, g] \otimes cu + \Phi(z \otimes c, x \otimes a)[y, g] \otimes bu + \Phi(y \otimes b, z \otimes c)[x, g] \otimes au$$

and

$$(x \otimes a) \wedge (y \otimes b) \wedge (z \otimes c) \mapsto \Phi(x \otimes a, y \otimes b)z \otimes D(c) + \Phi(z \otimes c, x \otimes a)y \otimes D(b) + \Phi(y \otimes b, z \otimes c)x \otimes D(a)$$

for any $x, y, z \in S$, $a, b, c \in A$ (this map arises as a connection homomorphism in the appropriate cohomological long exact sequence, see proof of [Z1, Lemma 5.1] for details).

Assuming that the element

$$(4.5) \quad \sum_i [\Phi_i] \otimes u_i + [\Phi_D] \otimes D$$

lies in $\text{Ker } F$, the corresponding element of E_2^{02} is generated by a cochain of the form $\sum_i \Phi'_i + \Phi'_D + \tilde{\Phi}$, where Φ'_i, Φ'_D are $(g \otimes U + KD)$ -valued 2-cochains built from Φ_i, Φ_D :

$$\begin{aligned} \Phi'_i(x \otimes a, y \otimes b) &= \Phi_i(x \otimes a, y \otimes b)g \otimes u_i \\ \Phi'_D(x \otimes a, y \otimes b) &= \Phi_D(x \otimes a, y \otimes b)D, \end{aligned}$$

and $\tilde{\Phi}$ is a 2-cochain on $S \otimes A$ such that $F\tilde{\Phi} = d\tilde{\Phi}$. Note that when choosing the basic cocycles induced by this part, we should exclude those corresponding to the case $F\tilde{\Phi} = 0$, as such cocycles are already accounted for by the $H^2(S \otimes A, S \otimes A)^{(g \otimes U + KD)}$ summand of E_2^{02} .

The action of D and of $g \otimes U$ on these 2-cochains is determined, respectively, by:

$$(4.6) \quad \begin{aligned} (D\tilde{\Phi})(x \otimes a, y \otimes b) &= \tilde{\Phi}(x \otimes D(a), y \otimes b) + \tilde{\Phi}(x \otimes a, y \otimes D(b)) + D(\tilde{\Phi}(x \otimes a, y \otimes b)) \\ (D\Phi'_i)(x \otimes a, y \otimes b) &= g \otimes \left(\left(\Phi_i(x \otimes D(a), y \otimes b) + \Phi_i(x \otimes a, y \otimes D(b)) \right) u_i + \Phi_i(x \otimes a, y \otimes b)D(u_i) \right) \\ (D\Phi'_D)(x \otimes a, y \otimes b) &= \left(\Phi_D(x \otimes D(a), y \otimes b) + \Phi_D(x \otimes a, y \otimes D(b)) \right) D \end{aligned}$$

and

$$(4.7) \quad \begin{aligned} ((g \otimes u)\tilde{\Phi})(x \otimes a, y \otimes b) &= \tilde{\Phi}([x, g] \otimes au, y \otimes b) + \tilde{\Phi}(x \otimes a, [y, g] \otimes bu) + [\tilde{\Phi}(x \otimes a, y \otimes b), g \otimes u] \\ ((g \otimes u)\Phi'_i)(x \otimes a, y \otimes b) &= g \otimes \left(\Phi_i([x, g] \otimes au, y \otimes b) + \Phi_i(x \otimes a, [y, g] \otimes bu) \right) u_i \\ ((g \otimes u)\Phi'_D)(x \otimes a, y \otimes b) &= \Phi_D(x \otimes a, y \otimes b)g \otimes D(u) \\ &\quad + \left(\Phi_D([x, g] \otimes au, y \otimes b) + \Phi_D(x \otimes a, [y, g] \otimes bu) \right) D \end{aligned}$$

for any $x, y \in S$, $a, b \in A$.

The grading (4.1) of the underlying Lie algebra induces a \mathbb{Z} -grading on each term of the spectral sequence (see, for example, [F, p. 44 of the English edition]), and we will concentrate on the positive part which abuts to the second positive cohomology in question.

By Proposition 3.1 (for $n = 1$), the module in (4.3) and (4.4) is isomorphic to

$$(4.8) \quad \frac{H^1(S, S) \otimes A}{g \otimes U} \oplus \frac{\text{Der}(A)}{KD},$$

and $g \otimes U$ acts on it trivially.

The algebra $g \otimes U + KD$ is \mathbb{Z} -graded with components of weight 0 and 2, and its module (4.8) is \mathbb{Z} -graded with components $-2, 0$ and 2 . Hence, in degree 1, the corresponding cochain complex has components of weight $-4, -2, 0, 2$, and in degree 2 it has components of weight $-6, -4, -2, 0, 2$.

In degree 2, the positive part of the second degree cohomology in (4.4) is generated by the cochains mapping a pair of elements of weight 0 to elements of weight 2, but since the zero component, KD , is 1-dimensional, this positive cohomology vanishes. Hence, the positive part of our Hochschild–Serre spectral sequence stabilizes at the E_2 term, and

$$H_+^2(S \otimes A + g \otimes U + KD, S \otimes A + g \otimes U + KD) \simeq (E_2^{11})_+ \oplus (E_2^{02})_+.$$

In degree 1, elementary reasonings based on the Hochschild–Serre spectral sequence abutting to the first degree cohomology in (4.3) with respect to ideal $g \otimes U$, yield

$$E_2^{11} \simeq \frac{H^1(S, S) \otimes A}{H^1(S, S) \otimes D(A) + g \otimes U} \oplus \frac{\text{Der}(A)}{[D, \text{Der}(A)] + KD} \oplus \text{Hom} \left(g \otimes U, \frac{H^1(S, S) \otimes A}{g \otimes U} \oplus \frac{\text{Der}(A)}{KD} \right)^D.$$

The positive component here, of weight 2, is

$$(E_2^{11})_+ \simeq H_+^1(S, S) \otimes \frac{A}{D(A) + U} \simeq \frac{A}{D(A) + U},$$

with the corresponding basic cocycles of type (ii).

Using Proposition 3.1 (for $n = 2$), we infer that the first summand in the E_2^{02} term is isomorphic to

$$(4.9) \quad \left(\left(H^2(S, S) \otimes A^D \right) \oplus \text{Har}^2(A, A)^D \right)^{g \otimes U}.$$

The corresponding 2-cocycles on $S \otimes A + g \otimes U + D$ are constructed as the maps $(S \otimes A) \wedge (S \otimes A) \rightarrow S \otimes A$, and the positive part of (4.9) is

$$(4.10) \quad \left(H_+^2(S, S) \otimes A^D \right)^{g \otimes U} \simeq H_+^2(S, S)^g \otimes A^D + H_+^2(S, S) \otimes (A^D \cap \{a \in A \mid aU = 0\}).$$

It is readily verified that the only basic cocycle (3.7) in $H_+^2(S, S)$ is g -invariant, so (4.10) is isomorphic to A^D . The corresponding basic cocycles are of type (i).

The $(\text{Ker } F)^{(g \otimes U + KD)}$ part of E_2^{02} , being a subspace of the tensor product of \mathbb{Z} -graded spaces $H^2(S \otimes A, K)$ (with weights $-2, 0, 2$, see §2), and $g \otimes U + KD$ (with weights 0 and 2), is \mathbb{Z} -graded itself, with graded components of weight $-2, 0, 2, 4$, which can be evaluated separately. As we are interested in the positive part, we deal with weights 2 and 4 only.

A general element Φ of $\text{Ker } F$ of the form (4.5) (where u_i 's may be assumed to be linearly independent) is of weight 2, if the cocycles Φ_i are of weight 0, and Φ_D is of weight 2. According to Lemma 2.3 and Corollary 2.4, we have

$$\begin{aligned} \Phi_i(e \otimes a, f \otimes b) &= \alpha_i(a, b) \\ \Phi_i(h \otimes a, h \otimes b) &= \alpha_i(a, b) \\ \Phi_D(e \otimes a, e \otimes b) &= \xi(ab) \end{aligned}$$

for any $a, b \in A$, and some $\alpha_i \in \widehat{\text{HC}}^1(A)$ and a linear map $\xi : A \rightarrow K$ vanishing on $A^{[2]}$. Then $F\Phi$ has the following form:

$$\begin{aligned} F\Phi(e \otimes a, e \otimes b, e \otimes c) &= e \otimes \left(\xi(ab)D(c) + \xi(ca)D(b) + \xi(bc)D(a) \right) \\ F\Phi(e \otimes a, e \otimes b, h \otimes c) &= h \otimes \xi(ab)D(c) \\ F\Phi(e \otimes a, e \otimes b, f \otimes c) &= f \otimes \left(\xi(ab)D(c) + \sum_i \left(a\alpha_i(b, c) + b\alpha_i(a, c) \right) u_i \right) \\ F\Phi(e \otimes a, h \otimes b, h \otimes c) &= f \otimes a \sum_i \alpha_i(b, c) u_i. \end{aligned}$$

for any $a, b, c \in A$.

Denoting $\Lambda(a, b) = \sum_i \alpha_i(a, b) u_i$, we are in situation of Lemma 3.6. That Lemma tells the necessary and sufficient conditions for $F\Phi$ to vanish in $H^3(S \otimes A, S \otimes A)$, i.e., to be a 3-coboundary (of weight 2). In particular,

$$(4.11) \quad \Lambda(a, b) = \xi(a)D(b) + \xi(b)D(a)$$

belongs to U for any $a, b \in A$. From the same Lemma it follows that $\tilde{\Phi}$ has the form

$$\begin{aligned}\tilde{\Phi}(e \otimes a, e \otimes b) &= h \otimes \left(\xi(a)D(b) + \xi(b)D(a) \right) \\ \tilde{\Phi}(e \otimes a, h \otimes b) &= f \otimes \left(av + \xi(b)D(a) \right)\end{aligned}$$

for some $v \in A$.

The condition of invariance of $\sum_i \Phi'_i + \Phi'_D + \tilde{\Phi}$ with respect to the D - and $g \otimes U$ -actions (4.7) and (4.6) yields, respectively, $\xi(D(A)) = 0$ and $D(v) = 0$, and

$$\left(\xi(b)u + \xi(bu)1 \right) D(a) + \left(\xi(a)u + \xi(au)1 \right) D(b) + \left(\xi(a)b + \xi(b)a + \xi(ab)1 \right) D(u) = 0$$

for any $a, b \in A$, $u \in U$. Coupled with (3.10), and the fact that D is nonzero, the latter equality is equivalent to $\xi(U) = 0$. Excluding the cocycles corresponding to the case $F\Phi = 0$, i.e., where $\xi = 0$ and $D = 0$ (those are exactly cocycles of type (i)), we arrive at basic cocycles of type (iii).

A general element Φ of $\text{Ker } F$ of the form (4.5) is of weight 4, if Φ_i are of weight 2, and $\Phi_D = 0$. According to Corollary 2.4, we have:

$$\Phi_i(e \otimes a, e \otimes b) = \xi_i(ab)$$

for any $a, b \in A$, where $\xi_i : A \rightarrow K$ are linear maps vanishing on $A^{[2]}$. Then $F\Phi$ has the following form:

$$F\Phi(e \otimes a, e \otimes b, e \otimes c) = f \otimes \sum_i \left(\xi_i(ab)c + \xi_i(ca)b + \xi_i(bc)a \right) u_i.$$

If $F\Phi = d\tilde{\Phi}$ for some 2-cochain $\tilde{\Phi} \in C^2(S \otimes A, S \otimes A)$, then the latter cochain should also have weight 4, and hence vanishes. Thus $F\Phi = 0$, and the linear independence of u_i 's implies

$$\xi_i(ab)c + \xi_i(ca)b + \xi_i(bc)a = 0$$

for any i and $a, b, c \in A$. Taking in the last equality a, b, c to be linearly independent, we get $\xi_i(ab) = 0$ for any two linearly independent $a, b \in A$. Coupled with the condition $\xi_i(A^{[2]}) = 0$, this implies $\xi_i = 0$. \square

Proof of Proposition 4.2. All the reasonings above are valid verbatim (and are somewhat simplified) in the case $U = 0$, except the part concerned with evaluation of $(\text{Ker } F)^D$ which leads to cocycles of type (iii) as in Proposition 4.1. In that case $\Lambda = 0$, so (4.11) implies $\xi(a)D(b) + \xi(b)D(a) = 0$, from which the condition (4.2) follows, and we arrive at the cocycles of type (iii) as in Proposition 4.2. \square

Proposition 4.4. *If $\dim A > 2$ and $U \neq 0$, any filtered deformation of the algebra $S \otimes A + g \otimes U + KD$ is isomorphic to a Lie algebra with the following bracket (assuming a, b are arbitrary elements of A , and*

u, t are arbitrary elements of U):

$$\begin{aligned}
 \{e \otimes a, e \otimes b\} &= h \otimes (\xi(a)D(b) + \xi(b)D(a)) + g \otimes \lambda(ab) + \xi(ab)D \\
 \{e \otimes a, h \otimes b\} &= e \otimes ab + f \otimes (abv + \xi(b)D(a)) \\
 \{e \otimes a, f \otimes b\} &= h \otimes ab + g \otimes (\xi(a)D(b) + \xi(b)D(a)) \\
 \{e \otimes a, g \otimes u\} &= f \otimes au \\
 \{h \otimes a, h \otimes b\} &= g \otimes (\xi(a)D(b) + \xi(b)D(a)) \\
 \{h \otimes a, f \otimes b\} &= f \otimes ab \\
 \{h \otimes a, g \otimes u\} &= 0 \\
 \{f \otimes a, f \otimes b\} &= 0 \\
 \{f \otimes a, g \otimes u\} &= 0 \\
 \{g \otimes u, g \otimes t\} &= 0 \\
 \{e \otimes a, D\} &= e \otimes D(a) + f \otimes aw \\
 \{h \otimes a, D\} &= h \otimes D(a) \\
 \{f \otimes a, D\} &= f \otimes D(a) \\
 \{g \otimes u, D\} &= g \otimes D(u)
 \end{aligned}$$

for some $v, w \in A$ such that $D(v) = 0$, $\xi \in \Xi_{D,U}$, and a linear map $\lambda : A \rightarrow U$ such that $\lambda(A^{[2]}) = 0$, and

$$\begin{aligned}
 (4.12) \quad & (\xi(ab)D(c) + \xi(ca)D(b) + \xi(bc)D(a))v + (\xi(ab)c + \xi(ca)b + \xi(bc)a)w \\
 & = c\lambda(ab) + b\lambda(ca) + a\lambda(bc)
 \end{aligned}$$

$$(4.13) \quad \xi(a)D(w) + \xi(w)D(a) = \lambda(D(a)) + D(\lambda(a)).$$

for any $a, b, c \in A$.

Proof. By Proposition 4.1, any such infinitesimal filtered deformation is a linear combination of the cocycles of type (i) for some $v \in A^D$, type (ii) for some $w \in A$, and type (iii) for some $\xi \in \Xi_{D,U}$, denoted by Φ_v , Ψ_w , and Υ_ξ , respectively. As all these cocycles are of weight 2, all the possible Massey brackets between them are of weight 4. Thus nonzero values of these brackets are possible only when either all the 3 arguments are of weight -1 , or two arguments are of weight -1 , and one of weight 0, i.e., for triples of the form

$$\begin{aligned}
 & e \otimes a, e \otimes b, e \otimes c \\
 & e \otimes a, e \otimes b, h \otimes c \\
 & e \otimes a, e \otimes b, D.
 \end{aligned}$$

Direct computation shows that the only possibly nonvanishing Massey brackets on these triples are:

$$\begin{aligned}
 [[\Phi_v, \Upsilon_\xi]](e \otimes a, e \otimes b, e \otimes c) &= f \otimes (\xi(ab)D(c) + \xi(ca)D(b) + \xi(bc)D(a))v \\
 [[\Psi_w, \Upsilon_\xi]](e \otimes a, e \otimes b, e \otimes c) &= f \otimes (\xi(ab)c + \xi(ca)b + \xi(bc)a)w \\
 [[\Upsilon_\xi, \Psi_w]](e \otimes a, e \otimes b, D) &= g \otimes (\xi(ab)D(w) + \xi(w)D(ab)).
 \end{aligned}$$

Each 2-cochain Ω on the algebra in question of weight 4 has the form

$$\Omega(e \otimes a, e \otimes b) = g \otimes \alpha(a, b)$$

for some alternating bilinear map $\alpha : A \times A \rightarrow U$, and the corresponding coboundary $d\Omega$ has the form

$$\begin{aligned} d\Omega(e \otimes a, e \otimes b, e \otimes c) &= f \otimes \left(a\alpha(b, c) + c\alpha(a, b) + b\alpha(c, a) \right) \\ d\Omega(e \otimes a, e \otimes b, h \otimes c) &= g \otimes \left(\alpha(ac, b) + \alpha(bc, a) \right) \\ d\Omega(e \otimes a, e \otimes b, D) &= g \otimes \left(\alpha(D(a), b) + \alpha(a, D(b)) + D(\alpha(a, b)) \right). \end{aligned}$$

Hence, the necessary and sufficient condition for the second order prolongability of the infinitesimal deformation $\Phi_v + \Psi_w + \Upsilon_\xi$, is the existence of α such that

$$\begin{aligned} &\left(\xi(ab)D(c) + \xi(ca)D(b) + \xi(bc)D(a) \right)v + \left(\xi(ab)c + \xi(ca)b + \xi(bc)a \right)w \\ &= a\alpha(b, c) + c\alpha(a, b) + b\alpha(c, a) \\ (4.14) \quad &\alpha(ac, b) + \alpha(bc, a) = 0 \\ &\xi(ab)D(w) + \xi(w)D(ab) = \alpha(D(a), b) + \alpha(a, D(b)) + D(\alpha(a, b)) \end{aligned}$$

for any $a, b, c \in A$. Among these 3 conditions, (4.14) is equivalent to $\alpha(a, b) = \lambda(ab)$ for some linear map $\lambda : A \rightarrow U$, and the other two conditions are equivalent then to (4.12) and (4.13), respectively.

The Massey brackets between 2-cochains $\Phi_v, \Psi_w, \Upsilon_\xi$ of weight 2, and 2-cochain α of weight 4, as well as between α and α , have weight 6 and 8, respectively, and hence vanish. Consequently, the second order deformation, if exists, is prolonged trivially to a global one. \square

The case $U = 0$ is easier:

Proposition 4.5. *Any filtered deformation of the algebra $S \otimes A + KD$ is isomorphic to the Lie algebra with the following bracket (assuming $a, b \in A$):*

$$\begin{aligned} \{e \otimes a, e \otimes b\} &= \xi(ab)D \\ \{e \otimes a, h \otimes b\} &= e \otimes ab + f \otimes (abv + \xi(b)D(a)) \\ \{e \otimes a, f \otimes b\} &= h \otimes ab \\ \{h \otimes a, h \otimes b\} &= 0 \\ \{h \otimes a, f \otimes b\} &= f \otimes ab \\ \{f \otimes a, f \otimes b\} &= 0 \\ \{e \otimes a, D\} &= e \otimes D(a) + f \otimes aw \\ \{h \otimes a, D\} &= h \otimes D(a) \\ \{f \otimes a, D\} &= f \otimes D(a) \end{aligned}$$

for some $v, w \in A$ such that $D(v) = 0$, and $\xi \in \Xi_D$ such that

$$(4.15) \quad \left(\xi(ab)c + \xi(ca)b + \xi(bc)a \right)w = 0$$

for any $a, b, c \in A$.

Proof. Similar to Proposition 4.4. Since, in this case, there are no 2-cochains of weight 4, the necessary and sufficient condition for prolongability of infinitesimal deformation is

$$\left(\xi(ab)D(c) + \xi(ca)D(b) + \xi(bc)D(a) \right)v + \left(\xi(ab)c + \xi(ca)b + \xi(bc)a \right)w = 0$$

for any $a, b, c \in A$. The first summand here vanishes, and we are left with (4.15). \square

To determine the isomorphism classes of algebras appearing in Propositions 4.4 and 4.5, yet alone to determine which of the corresponding 2-cocycles are semitrivial, in the parlance of [BLLS] (see Remark 3.5, part 4), seems to be a technically difficult task. A very partial solution of this task – with inconclusive answer, but enough to achieve our main goal – is presented in the next section.

5. SIMPLE LIE ALGEBRAS OF ABSOLUTE TORAL RANK 2 HAVING A CARTAN SUBALGEBRA OF TORAL RANK 1

We arrive, finally, at our main goal:

Theorem 5.1. *Any finite-dimensional simple Lie algebra over an algebraically closed field, of absolute toral rank 2, and having a Cartan subalgebra of toral rank 1, is isomorphic to S .*

Proof. Let \mathcal{L} be a Lie algebra satisfying the conditions of the theorem. Apply Theorem 1.1. If \mathcal{L} is isomorphic to Zassenhaus or Hamiltonian algebra, the absolute toral rank values of these algebras exclude all the cases except $\mathcal{L} \simeq S$.

Suppose \mathcal{L} is a filtered deformation of a Lie algebra L satisfying (1.6). Then we have:

$$\mathrm{TR}(S) \leq \mathrm{TR}(S \otimes \mathcal{O}_1(n)) \leq \mathrm{TR}(L) \leq \mathrm{TR}(\mathcal{L}) = 2.$$

The first two inequalities here follow from the obvious fact that the absolute toral rank of an algebra is not less than the absolute toral rank of its subalgebra, and the third one follows from [Sk1, Theorem 5.1] (see also [St, Vol. I, Theorem 1.4.6]). Therefore, $\mathrm{TR}(S) = 1$ or 2 . The first possibility is ruled out by [Sk1, Theorem 6.5] (an alternative, and shorter, proof of nonexistence of simple Lie algebras of absolute toral rank 1 is given in [Gr, Theorem 2]), and the second one implies, again, $S \simeq S$. Consequently, L is a graded semisimple Lie algebra lying between $S \otimes \mathcal{O}_1(2)$ and $\mathrm{Der}(S) \otimes \mathcal{O}_1(2) + K\partial$.

According to Lemma 3.4(i), the algebra of outer derivations of S is 2-dimensional abelian, spanned by $(\mathrm{ad} e)^2$ and $(\mathrm{ad} f)^2$. Accordingly, the spaces $(\mathrm{ad} e)^2 \otimes A$ and $(\mathrm{ad} f)^2 \otimes A$ of derivations of $S \otimes A$, consist of derivations of weight -2 and 2 , respectively. On the other hand, ∂ , considered as a derivation of $S \otimes A$, has weight 0 . As the grading of L has depth 1, this rules out derivations belonging to $(\mathrm{ad} e)^2 \otimes A$, and since the induced grading on $S \otimes \mathcal{O}_1(2)$ is standard, the algebra L necessary has the form

$$L = S \otimes \mathcal{O}_1(2) + (\mathrm{ad} f)^2 \otimes U + K\partial,$$

where U is a ∂ -invariant subspace of $\mathcal{O}_1(2)$. In other words (see discussion at the beginning of §4),

$$(5.1) \quad S \otimes \mathcal{O}_1(2) + K\partial \subseteq L \subseteq W_1(2) \otimes \mathcal{O}_1(2) + K\partial.$$

In particular, $13 \leq \dim L \leq 17$.

We are in situation of Propositions 4.4 and 4.5 with $A = \mathcal{O}_1(2)$ and $D = \partial$, and the rest of the proof consists of a somewhat boring elucidation of the structure of algebras appearing there.

Note that in both cases $U \neq 0$ and $U = 0$, the space $\Xi_{\partial, U}$ must be nonzero, otherwise by Propositions 4.4 and 4.5, \mathcal{L} contains an ideal isomorphic to a filtered deformation of $S \otimes \mathcal{O}_1(2) + g \otimes U$ (of $S \otimes \mathcal{O}_1(2)$ in the case $U = 0$), and hence is not simple. Since, by definition, any element $\xi \in \Xi_{\partial, U}$ vanishes on $\partial(\mathcal{O}_1(2)) = \langle 1, x, x^{(2)} \rangle$, we should have, up to a scalar, $\xi(x^{(3)}) = 1$. We have then $x = \xi(x^{(2)})\partial(x^{(3)}) + \xi(x^{(3)})\partial(x^{(2)}) \in U$, which excludes the case $U = 0$. On the other hand, in the case $U = \mathcal{O}_1(2)$, the space $\Xi_{\partial, \mathcal{O}_1(2)}$ is equal to zero, since its elements vanish on $\mathcal{O}_1(2)$.

Successive application of ∂ to elements of $\mathcal{O}_1(2)$ shows that any proper ∂ -invariant subspace of $\mathcal{O}_1(2)$ containing x coincides with either $\langle 1, x \rangle$ or $\langle 1, x, x^{(2)} \rangle$.

It is straightforward to check that the map $\xi : \mathcal{O}_1(2) \rightarrow K$ defined, as above, by

$$1 \mapsto 0, \quad x \mapsto 0, \quad x^{(2)} \mapsto 0, \quad x^{(3)} \mapsto 1,$$

satisfies (4.2). As $\mathcal{O}_1(2)^{[2]} = K$, and $U \subseteq \partial(\mathcal{O}_1(2))$, the rest of the defining conditions of $\Xi_{\partial, U}$ are satisfied automatically, so the latter space is 1-dimensional, linearly spanned by the map ξ .

Further, since $\mathrm{Ker} \partial = K1$, for the element $v \in A$ from Proposition 4.4 we have $v = \alpha 1$ for some $\alpha \in K$. The element $w \in A$ from the same Proposition originates from cocycles of type (ii) in Proposition 4.1. Up to coboundaries, we may assume that w lies in a subspace of $\mathcal{O}_1(2)$ complementary to $\partial(\mathcal{O}_1(2)) + U = \langle 1, x, x^{(2)} \rangle$, and hence $w = \beta x^{(3)}$ for some $\beta \in K$.

Denote, for a moment, the Lie algebra appearing in our case of Proposition 4.4 by $\mathcal{L}(\alpha, \beta, \xi, \lambda)$. For any nonzero $\gamma \in K$, the map

$$\begin{aligned} e \otimes a &\mapsto \frac{1}{\sqrt{\gamma}} e \otimes a \\ h \otimes a &\mapsto h \otimes a \\ f \otimes a &\mapsto \sqrt{\gamma} f \otimes a \\ g \otimes u &\mapsto \gamma g \otimes u \\ D &\mapsto D \end{aligned}$$

provides us with an isomorphism

$$\mathcal{L}(\alpha, \beta, \xi, \lambda) \simeq \mathcal{L}\left(\frac{\alpha}{\gamma}, \frac{\beta}{\gamma}, \frac{1}{\gamma}\xi, \frac{1}{\gamma^2}\lambda\right),$$

so we indeed can normalize ξ by setting $\xi(x^{(3)}) = 1$.

The equality (4.12) for the triple $1, x, x^{(3)}$ yields $x\lambda(x^{(3)}) + x^{(3)}\lambda(x) = \alpha 1$. The left-hand side here belongs to the ideal $\langle x, x^{(2)}, x^{(3)} \rangle$ of $\mathcal{O}_1(2)$, hence both sides vanish, and $\alpha = 0$. The remaining part of (4.12) can be rewritten as

$$(5.2) \quad c\Lambda(ab) + b\Lambda(ca) + a\Lambda(bc) = 0$$

for any $a, b, c \in A$, where $\Lambda(a) = \lambda(a) + \beta\xi(a)x^{(3)}$. The condition (5.2) is equivalent to Λ being a derivation of $\mathcal{O}_1(2)$. Consequently,

$$\lambda(a) + \beta\xi(a)x^{(3)} = f\partial(a) + g\partial^2(a)$$

for any $a \in A$, and some elements $f, g \in \mathcal{O}_1(2)$. The obvious computations then yield that this is possible if and only if

$$(5.3) \quad x^{(2)}\lambda(x) + x\lambda(x^{(2)}) + \lambda(x^{(3)}) = \beta x^{(3)}.$$

Writing the equality (4.13) for $a = x, x^{(2)}, x^{(3)}$, we get respectively:

$$\begin{aligned} \partial(\lambda(x)) &= \beta 1 \\ \partial(\lambda(x^{(2)})) &= \lambda(x) + \beta x \\ \partial(\lambda(x^{(3)})) &= \lambda(x^2). \end{aligned}$$

Coupled with (5.3), this gives

$$\begin{aligned} \lambda(1) &= 0 \\ \lambda(x) &= \gamma 1 + \beta x \\ \lambda(x^{(2)}) &= \delta 1 + \gamma x \\ \lambda(x^{(3)}) &= \delta x + \gamma x^{(2)} \end{aligned}$$

for some $\gamma, \delta \in K$.

Now let us look at the pairs in the multiplication table of the deformed algebra \mathcal{L} in Proposition 4.4, whose product contains elements from $g \otimes U$: $\{e \otimes a, e \otimes b\}$, $\{e \otimes a, f \otimes b\}$, $\{h \otimes a, h \otimes b\}$, and $\{g \otimes u, \partial\}$. Considered on the basic elements of $\mathcal{O}_1(2)$, the term $\xi(a)\partial(b) + \xi(b)\partial(a)$, occurring in $\{e \otimes a, f \otimes b\}$ and $\{h \otimes a, h \otimes b\}$, is nonzero only when one of a, b is equal to $x^{(3)}$, and the other one belongs to $\langle x, x^{(2)} \rangle$, and hence the values of that term lie in $\langle 1, x \rangle$. The elements of $\partial(U)$, occurring in $\{g \otimes u, \partial\}$, also lie in $\langle 1, x \rangle$. Thus the only occurrences of elements of the form $g \otimes u$, where $u \notin \langle 1, x \rangle$, is possible in the product $\{e \otimes a, e \otimes b\}$, and all the relevant values are:

$$\{e \otimes 1, e \otimes x^{(3)}\} = \{e \otimes x, e \otimes x^{(2)}\} = g \otimes (\delta x + \gamma x^{(2)}) + \partial.$$

Consequently, the commutant $[\mathcal{L}, \mathcal{L}]$ lies in

$$S \otimes \mathcal{O}_1(2) + g \otimes \langle 1, x \rangle + \langle g \otimes (\delta x + \gamma x^{(2)}) + \partial \rangle,$$

and hence is of dimension 15 at most (in fact, as it is easy to see from considerations below, it is of dimension 15). This excludes the case $U = \langle 1, x, x^{(2)} \rangle$, where $\dim \mathcal{L} = 16$.

In the remaining case we have $U = \langle 1, x \rangle$, $\dim \mathcal{L} = 15$, and $\gamma = 0$. Let us now write the multiplication table of the deformed algebra explicitly. We list only those multiplications between the basic elements whose product in the deformed algebra \mathcal{L} differs from the product in the graded algebra L .

$$(5.4) \quad \begin{aligned} \{e \otimes 1, \quad e \otimes x\} &= g \otimes \beta x \\ \{e \otimes 1, \quad e \otimes x^{(2)}\} &= g \otimes \delta 1 \\ \{e \otimes 1, \quad e \otimes x^{(3)}\} &= \{e \otimes x, \quad e \otimes x^{(2)}\} = g \otimes \delta x + \partial \\ \{e \otimes x, \quad e \otimes x^{(3)}\} &= h \otimes 1 \\ \{e \otimes x^{(2)}, e \otimes x^{(3)}\} &= h \otimes x \\ \{e \otimes x, \quad h \otimes x^{(3)}\} &= f \otimes 1 \\ \{e \otimes x^{(2)}, h \otimes x^{(3)}\} &= f \otimes x \\ \{e \otimes x^{(3)}, h \otimes x^{(3)}\} &= f \otimes x^{(2)} \\ \{e \otimes x, \quad f \otimes x^{(3)}\} &= \{e \otimes x^{(3)}, f \otimes x\} = g \otimes 1 \\ \{e \otimes x^{(2)}, f \otimes x^{(3)}\} &= \{e \otimes x^{(3)}, f \otimes x^{(2)}\} = g \otimes x \\ \{h \otimes x, \quad h \otimes x^{(3)}\} &= g \otimes 1 \\ \{h \otimes x^{(2)}, h \otimes x^{(3)}\} &= g \otimes x \\ \{e \otimes 1, \quad \partial\} &= f \otimes \beta x^{(3)} \end{aligned}$$

A tedious, but straightforward computation shows that the following 3 elements:

$$\begin{aligned} h \otimes 1 + e \otimes x + (e \otimes x)^{[2]}, \\ h \otimes (1 + x^{(2)}) + e \otimes x^{(3)}, \\ h \otimes 1 + (e \otimes x^{(2)})^{[2]} + \delta (h \otimes x^{(3)})^{[2]} \end{aligned}$$

(note that $h \otimes x^{(2)} = (e \otimes x^{(3)})^{[2]}$, so the first and the second lines here are transposed with respect to the substitution $x \leftrightarrow x^{(3)}$) linearly span a torus in the p -envelope of \mathcal{L} , so the absolute toral rank of \mathcal{L} is at least 3, a contradiction. \square

6. ON 15-DIMENSIONAL SIMPLE LIE ALGEBRAS

The algebras \mathcal{L} defined by multiplication table (5.4) are simple. Indeed, assume notation from the proof of Theorem 5.1, and let I be a nonzero ideal of \mathcal{L} . Any filtration on \mathcal{L} induces a filtration on I , whose associated graded algebra is an ideal in the semisimple algebra L . Hence, I contains $S \otimes \mathcal{O}_1(2)$, the socle of L , as a vector space. The multiplication table (5.4) reveals that then I contains $g \otimes \langle 1, x \rangle$ and the element $g \otimes \delta x + \partial$, and hence coincides with \mathcal{L} .

In [Sk1, Example at pp. 691–692], a series of simple Lie algebras satisfying the conditions of Theorem 1.1, is constructed. These algebras are filtered deformations of semisimple Lie algebras of the form

$$W'_1(n) \otimes \mathcal{O}_1(2) + e_{2n-2} \otimes \langle 1, x \rangle + K\partial.$$

The smallest algebra in the series, obtained when $n = 2$, coincides with algebra (5.4) for $\beta = \delta = 0$.

At present, we do not know whether algebras (5.4) are isomorphic or not for different values of parameters β and δ . Computer calculations suggest that all these algebras share many numerical invariants: they are of absolute toral rank 3^\dagger , the second cohomology $H^2(\mathcal{L}, K)$ vanishes, the cohomology $H^1(\mathcal{L}, \mathcal{L})$, $H^2(\mathcal{L}, \mathcal{L})$, and $H^3(\mathcal{L}, K)$ is of dimension 4, 13, and 15 respectively, they do not possess nontrivial symmetric invariant bilinear forms, the p -envelope coincides with the whole derivation algebra, and hence is of dimension 19, and the subalgebra generated by absolute zero divisors (i.e., elements $x \in \mathcal{L}$ such that $(\text{ad}x)^2 = 0$) has dimension 7.

In [E], a computer-generated list of simple Lie algebras over $\text{GF}(2)$ of small dimensions is presented. It seems that the algebras considered here are not isomorphic to any 15-dimensional algebra from this list, though, again, at present a rigorous proof of this is lacking. A thorough study of the known 15-dimensional simple Lie algebras is deferred to another paper.

ACKNOWLEDGEMENTS

Thanks are due to Askar Dzhumadil'daev, Dimitry Leites, Serge Skryabin, and the anonymous referee for useful remarks and clarifications. GAP [GA] was utilized to check some of the computations performed in this paper. Grishkov was supported by Russian Science Foundation (project 16-11-10002) (§§1–4), and by CNPq (grant 308221/2012-5) and FAPESP (grant 14/09310-5) (§§5–6). Zusmanovich was supported by FAPESP (grant 13/12050-2), by the Statutory City of Ostrava (grant 0924/2016/SaŠ), and by the Ministry of Education and Science of the Republic of Kazakhstan (grant 0828/GF4). Most of this work was carried out during Zusmanovich's visit to the University of São Paulo.

REFERENCES

- [B] C. Bennis, *Homologie de l'algèbre de Lie $sl_2(A)$* , Comptes Rendus Acad. Sci. Paris **310** (1990), 339–341.
- [BGL] S. Bouarroudj, P. Grozman, and D. Leites, *Infinitesimal deformations of symmetric simple modular Lie algebras and Lie superalgebras*; arXiv:0807.3054v1.
- [BLLS] ———, A. Lebedev, D. Leites, and I. Shchepochkina, *Lie algebra deformations in characteristic 2*, Math. Res. Lett. **22** (2015), 353–402; arXiv:1301.2781.
- [C] J.-L. Cathelineau, *Homologie de degré trois d'algèbres de Lie simple déployées étendues à une algèbre commutative*, Enseign. Math. **33** (1987), 159–173.
- [D] A. Dzhumadil'daev, *Special identity for Novikov-Jordan algebras*, Comm. Algebra **33** (2005), 1279–1287.
- [E] B. Eick, *Some new simple Lie algebras in characteristic 2*, J. Symb. Comput. **45** (2010), 943–951.
- [F] D.B. Fuchs, *Cohomology of Infinite-Dimensional Lie Algebras*, Nauka, Moscow, 1984 (in Russian); Consultants Bureau, N.Y., 1986 (English translation).
- [GA] The GAP Group, *GAP – Groups, Algorithms, and Programming*, Version 4.7.5, 2014; <http://gap-system.org/>
- [GS] M. Gerstenhaber and S.D. Schack, *Algebraic cohomology and deformation theory*, Deformations Theory of Algebras and Structures and Applications (ed. M. Hazewinkel and M. Gerstenhaber), Kluwer, 1988, 11–264.
- [Gr] A. Grishkov, *On simple Lie algebras over a field of characteristic 2*, J. Algebra **363** (2012), 14–18.
- [H] D.K. Harrison, *Commutative algebras and cohomology*, Trans. Amer. Math. Soc. **104** (1962), 191–204.
- [L] J.-L. Loday, *Cyclic Homology*, 2nd ed., Springer, 1998.
- [P] A.P. Petravchuk, *Lie algebras which can be decomposed into the sum of an abelian subalgebra and a nilpotent subalgebra*, Ukrain. Mat. Zh. **40** (1988), 385–388 (in Russian); Ukrain. Math. J. **40** (1988), 331–334 (English translation).
- [Sk1] S. Skryabin, *Toral rank one simple Lie algebras of low characteristics*, J. Algebra **200** (1998), 650–700.
- [Sk2] ———, *Open questions on modular Lie algebras*, Lie Algebras and Related Topics (ed. M. Avitabile, J. Feldvoss, and T. Weigel), Contemp. Math. **652** (2015), 237–239.
- [St] H. Strade, *Simple Lie Algebras over Fields of Positive Characteristic. Vol. I. Structure Theory. Vol. II. Classifying the Absolute Toral Rank Two Case. Vol. III. Completion of the Classification*, De Gruyter, 2004, 2009, 2013.
- [W] B. Weisfeiler, *On subalgebras of simple Lie algebras of characteristic $p > 0$* , Trans. Amer. Math. Soc. **286** (1984), 471–503.
- [Z1] P. Zusmanovich, *Deformations of $W_1(n) \otimes A$ and modular semisimple Lie algebras with a solvable maximal subalgebra*, J. Algebra **268** (2003), 603–635; arXiv:math/0204004.

[†]Added March 18, 2021: This is not true; the absolute toral rank of these algebras (actually, all algebras within the family are isomorphic) is equal to 4. See the manuscript by A. Grishkov, H. Guzzo Jr., M. Rasskazova, and P. Zusmanovich, *On simple 15-dimensional Lie algebras in characteristic 2*.

- [Z2] ———, *Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds*, *Lin. Algebra Appl.* **407** (2005), 71–104; arXiv:math/0302334.
- [Z3] ———, *Invariants of Lie algebras extended over commutative algebras without unit*, *J. Nonlin. Math. Phys.* **17** (2010), Suppl. 1 (special issue in memory of F.A. Berezin), 87–102; arXiv:0901.1395.
- [Z4] ———, *A compendium of Lie structures on tensor products*, *Zapiski Nauchnykh Seminarov POMI* **414** (2013) (N.A. Vavilov Festschrift), 40–81; reprinted in *J. Math. Sci.* **199** (2014), 266–288; arXiv:1303.3231.

(Alexander Grishkov) INSTITUTE OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SÃO PAULO, SÃO PAULO, BRAZIL AND OMSK F.M. DOSTOEVSKY STATE UNIVERSITY, OMSK, RUSSIA

Email address: shuragri@gmail.com

(Pasha Zusmanovich) DEPARTMENT OF MATHEMATICS, UNIVERSITY OF OSTRAVA, OSTRAVA, CZECH REPUBLIC

Email address: pasha.zusmanovich@gmail.com