

ON δ -DERIVATIONS OF LIE ALGEBRAS AND SUPERALGEBRAS

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ABSTRACT. We study δ -derivations – a construction simultaneously generalizing derivations and centroid. First, we compute δ -derivations of current Lie algebras and of modular Zassenhaus algebra. This enables us to provide examples of Lie algebras having $\frac{1}{2}$ -derivations which are divisors of zero, thus answering negatively a question of Filippov. Second, we note that δ -derivations allow, in some circumstances, to construct examples of non-semigroup gradings of Lie algebras, in addition to the recent ones discovered by Elduque. Third, we note that utilizing the construction of the Grassmann envelope allows to obtain results about δ -(super)derivations of Lie superalgebras from the corresponding results about Lie algebras. In this way, we prove that prime Lie superalgebras do not possess nontrivial δ -(super)derivations, generalizing the recent result of Kaygorodov.

INTRODUCTION

Let L be a Lie algebra and δ is an element of the ground field. A linear map $D : L \rightarrow L$ is called a δ -derivation of L if

$$(1) \quad D([x, y]) = \delta[D(x), y] + \delta[x, D(y)]$$

for any $x, y \in L$. It is clear that ordinary derivations are 1-derivations, the elements of the centroid of the algebra are $\frac{1}{2}$ -derivations, and the maps vanishing on the commutant of the algebra are 0-derivations. Thus, δ -derivations appear to be a natural simultaneous generalization of the notions of derivations and centroid. This alone justifies their study, but there is more to that: (-1) -derivations arise in the study of some noncommutative Jordan algebras (see [BH] and references therein), as well as are closely related to commutative 2-cocycles (the same way as ordinary derivations are related to the second cohomology group; see [DZ]), and δ -derivations arise in description of (ordinary) derivations of certain current Lie algebras ([Z2, §3]). (-1) -derivations (under the name of *antiderivations*) are mentioned also in the classical book of Jacobson ([J, p. 179]).

δ -derivations were studied by Filippov in a series of papers ([F1], [F2] and [F3]). In particular, he proved that prime Lie algebras, as a rule, do not have nontrivial (i.e., different from the above three examples for $\delta = 0, 1, \frac{1}{2}$) δ -derivations. He also noted that $\frac{1}{2}$ -derivations of a prime Lie algebra form a commutative ring which contains the centroid of the algebra, and posed the question: is it true that this ring does not contain divisors of zero?

In the recent papers [Kay1, Kay2], Kaygorodov initiated a study of a similar notion for nonassociative superalgebras. In particular, he proved in [Kay2] that finite-dimensional classical Lie superalgebras over a field of characteristic zero do not have nontrivial δ -derivations.

A more general concept – the so-called *quasiderivations* – was studied by Leger and Luks in [LL]. These are the linear maps $D : L \rightarrow L$ of a Lie algebra L such that there is a linear map $F(D) : L \rightarrow L$ such that

$$F(D)([x, y]) = [D(x), y] + [x, D(y)]$$

holds for any $x, y \in L$. Clearly, δ -derivations with $\delta \neq 0$ are quasiderivations with $F(D) = \frac{1}{\delta}D$. While Filippov's methods are essentially ring-theoretic and consist of sophisticated manipulation with identities, Leger and Luks confined themselves to the case of finite-dimensional algebras

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over an algebraically closed field and used a Lie-algebraic technique of the root space decomposition with respect to a torus. In this way, they obtained some useful criteria for quasiderivations of a Lie algebra to be reduced to the ordinary derivations and centroid.

We hope that the present paper reveals further interesting interconnections and provides additional motivation to study δ -derivations. In §1 we give a simple formula expressing δ -derivations of current Lie algebras in terms of its tensor factors, similar to the known formula for the ordinary derivations. In §2 we compute δ -derivations of the modular Zassenhaus algebra. It turns out that Zassenhaus algebra has $\frac{1}{2}$ -derivations which are divisors of zero, what provides a negative answer to the Filippov's question. Further examples can be obtained by tensoring Zassenhaus algebra with the reduced polynomial algebra and adding a "tail" of derivations – a construction typical in the structure theory of modular Lie algebras. Note that the property to possess nontrivial δ -derivations distinguishes Zassenhaus algebra, together with $sl(2)$, among all simple finite-dimensional Lie algebras.

In §3 we note an elementary, albeit somewhat unexpected connection with a different topic – non-semigroup gradings. The relevant story is quite intriguing: in the influential paper [PZ], Patera and Zassenhaus claimed that each grading of a finite-dimensional Lie algebra is a semigroup grading. This was believed to be true for almost two decades, until Elduque found a flaw in the proof and constructed counterexamples to this statement ([E1] and [E2]). It turns out that, under certain additional conditions, δ -derivations lead to non-semigroup gradings of Lie algebras, what allows to construct further counterexamples systematically. We also note a restriction on δ which should satisfy a non-nilpotent δ -derivation of a complex perfect finite-dimensional Lie algebra.

Finally, in §4, we show that prime Lie superalgebras do not have nontrivial δ -derivations and δ -superderivations. This extends the result of [Kay2] mentioned above. Kaygorodov achieves this by utilizing the known classification of simple classical Lie superalgebras and case-by-case calculations, while our approach based on passing from superalgebras to algebras via the venerable Grassmann envelope, and utilizing Filippov's results.

In the last section we take the liberty to suggest some directions for further investigations.

NOTATION AND CONVENTIONS

Throughout the paper, δ denotes an element of the ground field K , which is assumed to be of characteristic $\neq 2, 3$ (though some of the intermediate results are valid also in characteristic 3).

It is obvious that for a fixed δ , the set of all δ -derivations of a given Lie algebra L forms a linear space, which will be denoted by $Der_\delta(L)$. For $\delta = 1$, the space (actually, a Lie algebra) of the ordinary derivations will retain the usual notation $Der(L)$.

In what follows, we will need also a straightforward extension of this notion by considering δ -derivations with values in modules. Let L be a Lie algebra and M is an L -module. A linear map $D : L \rightarrow M$ will be called a δ -derivation of L with values in M if

$$D([x, y]) = \delta y \bullet D(x) - \delta x \bullet D(y)$$

for any $x, y \in L$. The linear space of all such maps will be denoted as $Der_\delta(L, M)$. The same way as for δ -derivations with values in the adjoint module, in the case $\delta = 1$ the subscript δ will be omitted.

Other standard notions and notation we use: $Hom(V, W)$ is the space of all linear maps between vector spaces V and W . The commutant and the center of a Lie algebra L are denoted as $[L, L]$ and $Z(L)$, respectively. A Lie algebra is called *perfect* if it coincides with its commutant. An algebra is called *prime* if the product of any two its nonzero ideals is nonzero. The *centroid* of an algebra A is a set of linear maps $\chi \in Hom(A, A)$ commuting with all left and right multiplications in A : $\chi(ab) = \chi(a)b = a\chi(b)$ for any $a, b \in A$. For an associative commutative algebra A , the operator of multiplication by an element $a \in A$ is denoted as R_a .

For convenience, we record Filippov's results which we will cite frequently (and generalize the first two of them to superalgebras in §4):

Theorem FA ([F2], Theorem 2). *A prime Lie algebra does not have nonzero δ -derivations if $\delta \neq -1, 0, \frac{1}{2}, 1$.*

Theorem FB ([F1], Corollary 3). *The space of $\frac{1}{2}$ -derivations of a central prime Lie algebra having a nondegenerate symmetric bilinear invariant form, coincides with K .*

Theorem FC ([F2], Corollary 1). *A central simple Lie algebra having nonzero (-1) -derivations is a form of $sl(2)$.*

Filippov's results overlap with those of Leger–Luks:

Theorem LL (Leger–Luks). *The space of quasiderivations of a simple finite-dimensional Lie algebra of rank > 1 coincides with the direct sum of derivations and the centroid of the algebra.*

Proof. This is a direct consequence of [LL, Corollary 4.16]. Note that the condition about “special weight spaces” stated in [LL, Theorem 4.12], assumes that the rank of the algebra > 1 (see [LL, Lemma 4.8(1)]), and automatically follows from the simplicity. \square

1. δ -DERIVATIONS OF CURRENT LIE ALGEBRAS

Recall that the *current Lie algebra* is a Lie algebra of the form $L \otimes A$ obtained by tensoring a Lie algebra L with an associative commutative algebra A , equipped with the Lie multiplication

$$(2) \quad [x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for $x, y \in L$, $a, b \in A$. If M is an L -module, and V is an A -module, consider also the “current module” $M \otimes V$ with action

$$(x \otimes a) \bullet (m \otimes v) = (x \bullet m) \otimes (a \bullet v)$$

for $x \in L$, $a \in A$, $m \in M$, $v \in V$ (here, by abuse of notation, the same bullet sign denotes the respective actions of L , A and $L \otimes A$).

Theorem 1.1. *Let L be a Lie algebra, A an associative commutative algebra with unit, M an L -module, V a unital A -module, and either both L and M , or both A and V are finite-dimensional. Then there is an isomorphism of vector spaces:*

$$\begin{aligned} & Der_{\delta}(L \otimes A, M \otimes V) \\ & \simeq Der_{\delta}(L, M) \otimes V \\ & \oplus \{\varphi \in Hom(L, M) \mid \varphi([x, y]) = \delta x \bullet \varphi(y) \text{ for any } x, y \in L\} \otimes Der(A, V) \\ & \oplus Hom(L/[L, L], M^L) \otimes Hom(A, V)/(V + Der(A, V)). \end{aligned}$$

Proof. Verbatim repetition of the proof of [Z1, Theorem 2.1][†]. \square

When taking $M = L$ and $V = A$, this generalizes the corresponding statement for ordinary derivations in [Z1, Corollary 2.2], and numerous results about the centroid of some particular current Lie algebras scattered over the literature ([BN, Corollary 2.23], [G, Remark 2.19(1)], [Kr, Lemma 5.1], [M, §3], etc.).

Corollary 1.2. *Let L be a Lie algebra which is either perfect or centerless, A is a commutative algebra with unit, one of L , A is finite-dimensional, and $\delta \neq 0, 1$. Then $Der_{\delta}(L \otimes A) \simeq Der_{\delta}(L) \otimes A$.*

[†]There is one minor inaccuracy in that proof, which does not affect the general flow of reasoning and the final answer: namely, after substituting $b = 1$ in the cocycle equation (2.2) (page 76 in the published version, lines 14-13 from the bottom), the conclusion about vanishing of φ_i 's with $i \in I_1$ and under assumption of the vanishing of the corresponding first tensor factor $x \bullet \varphi_i(y) - \varphi_i([x, y])$, is wrong. The correct conclusion is that corresponding φ_i 's are just “very degenerate”: $\varphi_i([L, L]) = 0$ and $\varphi_i(L) \subseteq M^L$, what gives rise to cocycles of type (iii) in the statement of the theorem. This is rectified in the arXiv version of that paper.

Proof. By assumption, the third direct summand in the isomorphism of Theorem 1.1 vanishes. Let us consider elements from the tensor factor of the second direct summand, i.e., linear maps $\varphi : L \rightarrow L$ satisfying the condition $\varphi([x, y]) = \delta[x, \varphi(y)]$ (and hence, by anti-commutativity, also $\varphi([x, y]) = \delta[\varphi(x), y]$) for any $x, y \in L$. For such maps we have, for any $x, y, z \in L$:

$$\begin{aligned} \varphi([[x, y], z]) &= -\varphi([[z, x], y]) - \varphi([[y, z], x]) \\ &= -\delta[\varphi([z, x]), y] - \delta[\varphi([y, z]), x] \\ &= -\delta^2[[\varphi(z), x], y] - \delta^2[[y, \varphi(z)], x] \\ &= \delta^2[[x, y], \varphi(z)] \\ &= \delta\varphi([[x, y], z]) \end{aligned}$$

(first and fourth equalities hold by the Jacobi identity). As $\delta \neq 1$, this implies $\varphi([[L, L], L]) = 0$. Hence if L is perfect, all such φ 's vanish. If L is centerless, the equality

$$0 = \varphi([[x, y], z]) = \delta[\varphi([x, y]), z] = \delta^2[[\varphi(x), y], z]$$

for any $x, y, z \in L$ implies $[\varphi(L), L] \subseteq Z(L) = 0$, and hence $\varphi(L) \subseteq Z(L) = 0$. Thus the second direct summand in the isomorphism of Theorem 1.1 vanishes too, and we are left with the first one. \square

2. $\frac{1}{2}$ -DERIVATIONS OF THE ZASSENHAUS ALGEBRA

Naturally, when one encounters a new Lie-algebraic invariant, one of the first examples one wishes to compute this invariant for, is finite-dimensional simple algebras. By Theorem LL, nontrivial δ -derivations are possible only for algebras of rank 1, i.e., for $sl(2)$ and, in the case of the ground field of positive characteristic, for the Zassenhaus algebra $W_1(n)$ and the Hamiltonian algebra $H_2(m_1, m_2)$. On the other hand, by Theorem FA, the only cases we may encounter are $\delta = -1, \frac{1}{2}, 0, 1$. The case $\delta = 1$ corresponds to ordinary derivations which were studied exhaustively. The case $\delta = 0$ is not interesting at all: 0-derivations are merely the maps vanishing on the commutant of an algebra (and thus, vanishing on the whole algebra in the simple case). By Theorem FC, the only central simple Lie algebras having nonzero (-1) -derivations are forms of $sl(2)$. What happens in the remaining case $\delta = \frac{1}{2}$?

According to [F2, Theorem 5], the space of $\frac{1}{2}$ -derivations of a prime Lie algebra L forms an associative commutative algebra which contains the centroid of L . Moreover, by Theorem FB, for all simple Lie algebras having a nonzero symmetric bilinear invariant form – in particular, for all classical Lie algebras, as well as for all Hamiltonian Lie algebras (see, for example, [SF, §4.6, Theorem 6.5]) – this commutative algebra coincides with the ground field (interpreted as operations of multiplication by the field element).

Therefore, it remains only to consider $\frac{1}{2}$ -derivations of the Zassenhaus algebra.

When computing outer derivations (and, more generally, cohomology) of a Lie algebra, we are greatly helped with the triviality of the Lie algebra (co)homology under the torus action – a very simple, yet one of the most useful facts about (co)homology of Lie algebras. Thus, when a Lie algebra in question possesses, for example, a grading compatible with the action of some torus, all outer derivations could be assumed to preserve that grading.

This nicety is, generally, no longer true in the case of δ -derivations. Moreover, δ -derivations for $\delta \neq 1$ do not form a cohomology group: there are no notions of inner and outer δ -derivations, and we cannot take a quotient by an appropriate space of 1-coboundaries. Nevertheless, we can still benefit from considering the torus action, as was observed, for the case of quasiderivations, by Leger and Luks in [LL, §4]. Indeed, it is easy to check (as noted, for example, in [F2, §1]) that for $\delta, \delta' \in K$,

$$[Der_\delta(L), Der_{\delta'}(L)] \subseteq Der_{\delta\delta'}(L),$$

where both $Der_\delta(L)$, $Der_{\delta'}(L)$ are understood here as subspaces of the Lie algebra $Hom(L, L) \simeq gl(L)$ of all linear maps of the Lie algebra L . In particular, $Der_\delta(L)$ is invariant under inner derivations of L . Let T be a torus in L such that L decomposes into the direct sum

of the root spaces with respect to the action of T . This action induces, in the standard way, the action of T on $\text{Hom}(L, L) \simeq L \otimes L^*$ [†], and the latter, as well as any its T -invariant subspace, can be decomposed into the direct sum of the root spaces with respect to this action as well.

Let G be an additively written abelian group, and R a subset of G which contains 0 and such that the sum of any two distinct elements of R lies in R . By the *Lie algebra of Witt type* W_R we will understand a Lie algebra having the basis $\{e_\alpha \mid \alpha \in R\}$ with multiplication

$$[e_\alpha, e_\beta] = (\beta - \alpha)e_{\alpha+\beta},$$

where $\alpha, \beta \in R$.

Specializing to $R = G = \mathbb{Z}_p^n$, for a prime p and $n \in \mathbb{N}$, we get the famous *Zassenhaus algebra* $W_1(n)$ (which is a p^n -dimensional algebra defined over a field of positive characteristic p). The case $n = 1$ deserves the special name of *Witt algebra*. Specializing to $G = \mathbb{Z}$, and to $R = \mathbb{Z}$ and $R = \{i \in \mathbb{Z} \mid i \geq -1\}$, we get the *two-sided* and *one-sided* (infinite-dimensional) *Witt algebras*, respectively.

Theorem 2.1. *For a Lie algebra of Witt type W_R , there is isomorphism of commutative algebras:*

$$\text{Der}_{\frac{1}{2}}(W_R) \simeq K[\frac{1}{2}R],$$

where $\frac{1}{2}R = \{\gamma \in R \mid 2\gamma \in R\}$.

Remark. It is easy to see that the set $\frac{1}{2}R$ forms a semigroup, so $K[\frac{1}{2}R]$ is a semigroup algebra. Of course, when $R = G$, then $\frac{1}{2}R = G$ and it is a group algebra.

Proof. W_R decomposes into the direct sum of one-dimensional root spaces Ke_α , $\alpha \in R$, with respect to the action of the semisimple element e_0 . By considerations above, the space of $\frac{1}{2}$ -derivations of W_R is decomposed as the direct sum of one-dimensional root spaces with respect to the induced e_0 -action. Any root of the latter action is the sum of two roots from R . If an $\frac{1}{2}$ -derivation D of W_R lies in the root space with the root $\gamma \in R + R$, then

$$[e_0, D(e_\alpha)] - D([e_0, e_\alpha]) = \gamma D(e_\alpha),$$

for any $\alpha \in R$, what implies that $D(e_\alpha)$ lies in the root space with the root $\alpha + \gamma$, i.e., belongs to $Ke_{\alpha+\gamma}$ if $\alpha + \gamma \in R$, and vanishes otherwise. Assuming

$$D(e_\alpha) = \begin{cases} \lambda_\alpha e_{\alpha+\gamma}, & \alpha + \gamma \in R \\ 0, & \alpha + \gamma \notin R, \end{cases}$$

for some $\lambda_\alpha \in K$, the condition that D is an $\frac{1}{2}$ -derivation, written for the generic basis pair e_α, e_β , is equivalent to:

$$(3) \quad \lambda_{\alpha+\beta} = \begin{cases} \frac{\lambda_\alpha}{2}(1 - \frac{\gamma}{\beta-\alpha}) + \frac{\lambda_\beta}{2}(1 + \frac{\gamma}{\beta-\alpha}), & \alpha + \gamma, \beta + \gamma \in R \\ \frac{\lambda_\alpha}{2}(1 - \frac{\gamma}{\beta-\alpha}), & \alpha + \gamma \in R, \beta + \gamma \notin R \\ 0, & \alpha + \gamma, \beta + \gamma \notin R. \end{cases}$$

for any $\alpha, \beta \in R$ such that $\alpha \neq \beta$, $\alpha + \beta + \gamma \in R$

Suppose $\gamma \notin R$. Substituting into the second case of (3) $\beta = 0$, we get $\lambda_\alpha = 0$ for any $\alpha \in R$ such that $\alpha + \gamma \in R$. Hence $D = 0$.

Suppose $\gamma \in R$. Substituting into the first case of (3) $\alpha = 0$, we get

$$(4) \quad \lambda_\beta = \lambda_0$$

for any $\beta \in R$ such that $\beta \neq \gamma$. If $\gamma = 0$, this implies that D is proportional to the identity map. Let $2\gamma \notin R$. Note that if R consists of only two roots 0 and γ , then W_R is isomorphic to the two-dimensional nonabelian Lie algebra. For this algebra, the statement of the Theorem is verified

[†]Added December 22, 2021: The isomorphism $\text{Hom}(L, L) \simeq L \otimes L^*$ is valid in the finite-dimensional case only, but this does not affect the reasonings. I am grateful to Viktor Lopatkin for this remark.

by elementary computations, so we can exclude this possibility from consideration. Thus, substituting into the second case of (3) any $\alpha \neq 0, \gamma$ and $\beta = \gamma$ and taking into account (4), we get $\lambda_0 = 0$ and $D = 0$.

If $\gamma \neq 0$ and $2\gamma \in R$, then substituting into the first case of (3) $\alpha = \gamma$ and $\beta = 3\gamma$ (at this point, the assumption that the characteristic of the ground field $\neq 2, 3$ is essential) and taking into account (4), yields $\lambda_\gamma = \lambda_0$, and $D = \lambda_0 D_\gamma$, where $D_\gamma(e_\alpha) = e_{\alpha+\gamma}$ for each $\alpha \in R$.

We have proved that each $\frac{1}{2}$ -derivation of W_R is a linear combination of D_γ 's. On the other hand, it is easy to see that for each $\gamma \in R$ such that $2\gamma \in R$, D_γ is an $\frac{1}{2}$ -derivation of W_R , and they are linearly independent for different γ 's. \square

Passing in Theorem 2.1 to the different specializations of R , as specified above, we get:

$$(5) \quad \begin{aligned} Der_{\frac{1}{2}}(W_1(n)) &\simeq K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p) \\ Der_{\frac{1}{2}}(W_1) &\simeq K[t, t^{-1}] \\ Der_{\frac{1}{2}}(W_1^+) &\simeq K[t]. \end{aligned}$$

All these three isomorphisms could be considered also as a manifestation of another general observation. Let A be a commutative associative algebra and ∂ its derivation. Then we may consider a Lie algebra, denoted as $A\partial$, of derivations of A of the form $a\partial$, $a \in A$. The Lie bracket is determined by the formula

$$[a\partial, b\partial] = (a\partial(b) - b\partial(a))\partial$$

for $a, b \in A$.

Proposition 2.2. *$Der_{\frac{1}{2}}(A\partial)$ contains a subalgebra isomorphic to A .*

Proof. To any $u \in A$ we may associate a linear map $D_u : A\partial \rightarrow A\partial$ by the rule $D_u(a\partial) = ua\partial$. A straightforward computation shows that D_u is an $\frac{1}{2}$ -derivation of $A\partial$ for any $u \in A$. \square

The 3-dimensional algebra $sl(2)$ and the Zassenhaus algebra are characterized among simple finite-dimensional Lie algebras in various interesting ways: these are algebras having a subalgebra of codimension 1, algebras having a maximal solvable subalgebra, algebras with given properties of the lattice of subalgebras, etc. Isomorphism (5), together with the results of Filippov and Leger–Luks cited above, adds another characterization to this list: these are the only finite-dimensional simple Lie algebras having nontrivial δ -derivations (for $\delta = -1$ in the case of $sl(2)$ and for $\delta = \frac{1}{2}$ in the case of the Zassenhaus algebra).

The Zassenhaus algebra admits many different realizations, and writing its $\frac{1}{2}$ -derivations in these realizations may be beneficial for elucidating some interesting points. Recall that the *divided powers algebra* $O_1(n)$ is a p^n -dimensional associative commutative algebra with basis $\{x^i \mid 0 \leq i < p^n\}$ and multiplication

$$x^i x^j = \binom{i+j}{j} x^{i+j}.$$

It is isomorphic to the reduced polynomial algebra $O_n = K[x_1, \dots, x_n]/(x_1^p, \dots, x_n^p)$. $W_1(n)$ can be viewed as a Lie algebra of derivations $O_1(n)\partial$, where $\partial(x^i) = x^{i-1}$. In such realization, it possesses a basis $\{e_i = x^{i+1}\partial \mid -1 \leq i \leq p^n - 2\}$ with multiplication

$$[e_i, e_j] = \left(\binom{i+j+1}{j} - \binom{i+j+1}{i} \right) e_{i+j}.$$

Though in the general case the formulas expressing the concrete $\frac{1}{2}$ -derivation in terms of this basis appear to be cumbersome, in the case $n = 1$ the basis of $Der_{\frac{1}{2}}(W_1(1))$ could be written as

$\{1, D, \dots, D^{p-1}\}$, where D is the nilpotent map of the following neat form:

$$(6) \quad D(e_i) = \begin{cases} (i+2)e_{i+1}, & -1 \leq i < p-2 \\ 0, & i = p-2. \end{cases}$$

As was noted by Kuznetsov (Remark at the very end of [Ku]), $W_1(n)$ for $n > 1$ also can be realized as deformation of the current Lie algebra $W_1(1) \otimes O_1(n-1)$: $\{x, y\} = [x, y] + \Phi(x, y)$ for $x, y \in W_1(1) \otimes O_1(n-1)$, where the 2-cocycle Φ is defined as

$$\Phi(e_i \otimes a, e_j \otimes b) = \begin{cases} e_{p-2} \otimes (a\partial(b) - b\partial(a)), & i = j = -1 \\ 0 & \text{otherwise} \end{cases}$$

for $-1 \leq i, j \leq p-2$ and $a, b \in O_1(n-1)$. By Corollary 1.2,

$$Der_{\frac{1}{2}}(W_1(1) \otimes O_1(n-1)) \simeq Der_{\frac{1}{2}}(W_1(1)) \otimes O_1(n-1),$$

each $\frac{1}{2}$ -derivation of $W_1(1) \otimes O(n-1)$ being the linear span of elements of the form

$$(7) \quad D \otimes R_u$$

for $D \in Der_{\frac{1}{2}}(W_1(1))$ and $u \in O_1(n-1)$. According to the general principle, invariants of algebras can only decrease under deformation (this easily could be made more precise in the case of δ -derivations following the general format of the Gerstenhaber's deformation theory, but we will not delve into it), so $Der_{\frac{1}{2}}(W_1(n))$ is "less or equal" than $Der_{\frac{1}{2}}(W_1(1)) \otimes O_1(n-1)$. The comparison of dimensions ensures that they are, in fact, equal. This could also be explained by the simple fact that for any $D \in Der_{\frac{1}{2}}(W_1(1))$ and $u \in O_1(n-1)$,

$$\Phi((D \otimes R_u)(W_1(1) \otimes O_1(n-1), W_1(1) \otimes O_1(n-1))) = 0$$

and

$$(D \otimes R_u)(\Phi(W_1(1) \otimes O_1(n-1), W_1(1) \otimes O_1(n-1))) = 0,$$

so each $\frac{1}{2}$ -derivation of $W_1(1) \otimes O_1(n-1)$ can be lifted, in a trivial way, to those of $W_1(n)$.

In [F2, §3], Filippov asked the question: is it true that the commutative algebra of $\frac{1}{2}$ -derivations of a prime Lie algebra does not contain divisors of zero? According to (5), Zassenhaus algebra provides a negative answer to this question. In fact, for Zassenhaus algebra the situation is, in a sense, opposite to the situation without divisors of zero: the algebra of $\frac{1}{2}$ -derivations is a local one, i.e., its radical, consisting of nilpotent elements, is of codimension 1. The concrete examples of nilpotent $\frac{1}{2}$ -derivations are provided by the map (6) and its powers in the case of the Witt algebra, and, more generally, by the maps of the form (7) in the case of the Zassenhaus algebra.

Further examples providing negative answer to the Filippov's question are semisimple Lie algebras having $W_1(n) \otimes O_m$ as the socle. Indeed, semisimple Lie algebras of the form $W_1(n) \otimes O_m + 1 \otimes \mathcal{D}$, where \mathcal{D} is a subalgebra of $Der(O_m)$ such that O_m does not have \mathcal{D} -invariant ideals, are prime. It is possible to prove that under some conditions imposed on L , A (like, for example, in Corollary 1.2), and on $\mathcal{D} \subseteq Der(A)$,

$$Der_{\delta}(L \otimes A + 1 \otimes \mathcal{D}) \simeq Der_{\delta}(L) \otimes A^{\mathcal{D}}.$$

In particular, as the absence of \mathcal{D} -invariant ideals in O_m implies $O_m^{\mathcal{D}} = K1$, we have:

$$Der_{\frac{1}{2}}(W_1(n) \otimes O_m + 1 \otimes \mathcal{D}) \simeq Der_{\frac{1}{2}}(W_1(n)) \simeq O_n.$$

According to the classical Block's theorem about the structure of modular finite-dimensional semisimple Lie algebras ([B, Theorem 9.3]), this, essentially, exhausts all possible examples of finite-dimensional prime Lie algebras whose $\frac{1}{2}$ -derivations have divisors of zero.

The same constructions provide also infinite-dimensional examples of prime Lie algebras whose $\frac{1}{2}$ -derivations have divisors of zero. For example, the Lie algebra $W_{\mathbb{Q}}$, where \mathbb{Q} is the additive group of rationals, is simple (see, for example, [AS, Chapter 10, Theorem 3.1]). As \mathbb{Q} is a group with torsion, the group algebra $K[\mathbb{Q}]$ contains divisors of zero. Another example: the

Lie algebra $W_1(1) \otimes K[x] + 1 \otimes \frac{d}{dx}$ is prime, and it is easy to see, by the same reasoning as for finite-dimensional modular semisimple Lie algebras above, that its algebra of $\frac{1}{2}$ -derivations is isomorphic to $K[x]/(x^p)$.

3. NON-SEMIGROUP GRADINGS

A *grading* of a Lie algebra L is the direct sum decomposition $L = \bigoplus_{\alpha \in G} L_\alpha$ into subspaces indexed by a set G such that for each $\alpha, \beta \in G$ either $[L_\alpha, L_\beta] = 0$ or there is $\gamma(\alpha, \beta) \in G$ such that $[L_\alpha, L_\beta] \subseteq L_{\gamma(\alpha, \beta)}$. All “interesting” gradings appearing on practice (like Cartan decompositions, gradings induced by automorphisms of finite order, \mathbb{Z} -gradings associated with filtrations, etc.) are *group (or semigroup) gradings*, i.e., G could be embedded into an (additively written) abelian group (or semigroup) such that $\gamma(\alpha, \beta) = \alpha + \beta$ for any pair $\alpha, \beta \in G$ for which $\gamma(\alpha, \beta)$ exists.

However, a priori the existence of such embedding is not obvious, and the natural question arises whether each grading of a finite-dimensional Lie algebra is a semigroup grading. In [E1] Elduque constructed a quite unexpected example, refuting a two-decades old claim by Patera and Zassenhaus [PZ, Theorem 1(a)]. Further examples constructed in [E2] allowed to think that non-semigroup gradings are quite common. However, they seem to be produced by ad-hoc trial and error, and no systematic method of constructing such examples was known till now.

Suppose \mathcal{D} is a set of commuting δ -derivations of a Lie algebra L over an algebraically closed field K . Consider the standard root space decomposition with respect to the set \mathcal{D} :

$$(8) \quad L = \bigoplus_{\lambda \in \mathcal{D}^*} L_\lambda.$$

The same simple inductive reasoning based on the binomial formula and used in the proof of the usual multiplicative properties of root spaces with respect to ordinary derivations (see, for example, [J, Chapter III, §2]), can be used to prove that

$$(9) \quad [L_\lambda, L_\mu] \subseteq L_{\delta(\lambda + \mu)}$$

for any $\lambda, \mu \in \mathcal{D}^*$.

When (8) will be a semigroup grading and when it will not? Suppose for a moment that all products between root spaces are nonzero. Then the set of elements $\lambda \in \mathcal{D}^*$ such that $L_\lambda \neq 0$, with operation

$$(10) \quad \lambda \circ \mu = \delta(\lambda + \mu)$$

should be a semigroup. Associativity of \circ , due to (9), is equivalent to

$$(\delta^2 - \delta)(\lambda - \mu) = 0$$

for any λ, μ . Clearly, for $\delta \neq 0, 1$ this implies that any root space decomposition (8) with more than one root space would be a non-semigroup grading.

The problem is that some products of root spaces may vanish, so if the ratio of the number of root spaces to the number of pairs of root spaces with zero product is “small”, the operation (10) on the remaining pairs could be not enough to violate associativity. Accordingly, under the natural assumption of having “enough” root spaces with nonzero product, we are guaranteed to get a non-semigroup grading, as the following elementary proposition shows.

Proposition 3.1. *Let L be a finite-dimensional Lie algebra over an algebraically closed field, $\delta \neq 0, 1$, and \mathcal{D} is a set of commuting δ -derivations of L . Then the root space decomposition (8) with respect to \mathcal{D} is a non-semigroup grading of L if one of the following holds:*

- (i) *there are three pairwise distinct roots $\lambda, \mu, \eta \in \mathcal{D}^*$ such that $[[L_\lambda, L_\mu], L_\eta] \neq 0$;*
- (ii) *there are two distinct roots $\lambda, \mu \in \mathcal{D}^*$ such that $[[L_\lambda, L_\lambda], L_\mu] \neq 0$.*

Proof. (i) Suppose that (8) is a semigroup grading. The Jacobi identity and the condition of Proposition imply that at least one of the triple products $[[L_\eta, L_\lambda], L_\mu]$ and $[[L_\mu, L_\eta], L_\lambda]$ does not vanish

too. Suppose, without loss of generality, that $[L_\lambda, [L_\mu, L_\eta]] \neq 0$. Then both expressions $(\lambda \circ \mu) \circ \eta$ and $\lambda \circ (\mu \circ \eta)$ exist, hence they are equal, what implies, as noted above, $\lambda = \eta$.

(ii) Similarly: the Jacobi identity implies $[[L_\lambda, L_\mu], L_\lambda] \neq 0$ and the same reasoning applies to derive $\lambda = \mu$, a contradiction. \square

On the other hand, it is obvious that if there is only one root space, for example, if all δ -derivations in \mathcal{D} are nilpotent, then the grading collapses to one summand and we get nothing interesting in this way.

Proposition 3.1 provides a way to construct non-semigroup gradings of Lie algebras in a systematic way. Some of the examples constructed by Elduque could also be recovered using this scheme. For example, consider the 4-dimensional Lie algebra from [E2, §1]. This is an algebra with basis $\{a, u, v, w\}$ and multiplication table:

$$[a, u] = u, \quad [a, v] = w, \quad [a, w] = v,$$

all other products between basic elements being zero. It has the following (-1) -derivation:

$$a \mapsto 0, \quad u \mapsto 0, \quad v \mapsto -w, \quad w \mapsto v.$$

The root space decomposition with respect to this (-1) -derivation is:

$$(Ka \oplus Ku) \oplus K(v + iw) \oplus K(v - iw),$$

the direct summands correspond to eigenvalues $0, i$ and $-i$. We have:

$$0 \circ 0 = 0, \quad 0 \circ i = -i, \quad 0 \circ (-i) = i.$$

This is exactly the operation on the set of 3 elements obtained in [E2] (though the grading itself is a bit different), which is shown there not to be embeddable in any semigroup.

On the other hand, not all examples of non-semigroup gradings could be obtained in this way. For example, for each $\delta \neq 0$, the space of δ -derivations of the 16-dimensional nilpotent Lie algebra from [E1], represented as matrices in the given basis of the algebra, has a basis which could be divided into 2 or 3 non-intersecting sets: all matrices in the first set are strictly upper-triangular; for each pair X, Y of matrices from the second set, $XY = 0$ and $XY = 0$ holds for any matrix B from the basis; and, in the case $\delta = \frac{1}{2}$, the identity matrix. This is verified on computer (see Appendix). Consequently, each δ -derivation of that algebra can be represented as the sum of a (possibly zero) multiplication by a field element and a nilpotent δ -derivation.

Note also that, unfortunately, we cannot in this way get a (negative) answer to the question from [E1]: is every grading of a *simple* finite-dimensional Lie algebra over an algebraically closed field of characteristic zero a semigroup grading? For, as mentioned in the previous section, either by combined results of Theorems FA, FB and FC, or by Theorem LL, among such algebras only $sl(2)$ has nontrivial δ -derivations, and, obviously, every grading of $sl(2)$ is a semigroup grading.

In conclusion of this section, let us note another fact which, though, not directly related to non-semigroup gradings, also follows from elementary considerations related to the root space decomposition with respect to δ -derivations:

Proposition 3.2. *For any non-nilpotent δ -derivation of a finite-dimensional perfect Lie algebra over the field of complex numbers, $\frac{1}{2} \leq |\delta| \leq 1$ and δ is algebraic.*

Proof. Let L be such a Lie algebra and D is its δ -derivation. From (8) (where \mathcal{D} is assumed to consist of a single element D and the set of corresponding roots is denoted as $R \subset \mathbb{C}$), it follows that

$$L = [L, L] = \sum_{\lambda, \mu \in R} [L_\lambda, L_\mu].$$

Hence, in view of (9), for each root $\eta \in R$, $L_\eta \subseteq \sum_{\eta = \delta(\lambda + \mu)} L_{\delta(\lambda + \mu)}$. In particular, for each $\eta \in R$ the set over which the last summation is performed is not empty, i.e., there are $\lambda, \mu \in R$ such that

$$(11) \quad \eta = \delta(\lambda + \mu).$$

By induction, for any $\eta \in R$ and $n \in \mathbb{N}$, there are $\lambda_1, \dots, \lambda_{2^{n+1}} \in R$ such that $\eta = \delta^n(\lambda_1 + \dots + \lambda_{2^{n+1}})$. Then, denoting $M = \max_{\lambda \in R} |\lambda|$, we have $M \leq |\delta|^n 2^{n+1} M$ for any $n \in \mathbb{N}$, and either

$$|\delta| \geq \lim_{n \rightarrow \infty} \frac{1}{\sqrt[n]{2^{n+1}}} = \frac{1}{2},$$

or $M = 0$. In the latter case the set of roots consists of a single root 0, whence D is nilpotent.

Similarly, denoting

$$N = \min_{\substack{k_1 \lambda_1 + \dots + k_m \lambda_m \neq 0 \\ \lambda_i \in R, k_i \in \mathbb{Z}}} |k_1 \lambda_1 + \dots + k_m \lambda_m|$$

(as R is finite and does not consist of a single element 0, this minimum exists), we have $|\eta| \geq |\delta|^n N$ for any nonzero $\eta \in R$ and $n \in \mathbb{N}$, whence $|\delta| \leq 1$.

Finally, writing the condition (11) for each root $\eta \in R$, we get a homogeneous system of $|R|$ equations with coefficients depending linearly, over \mathbb{Z} , on δ , and in $|R|$ unknowns. This system has a nonzero solution, hence its determinant equal to zero, what provides a nonzero polynomial with integer coefficients vanishing on δ . \square

4. δ -DERIVATIONS AND δ -SUPERDERIVATIONS OF PRIME LIE SUPERALGEBRAS

δ -derivations of superalgebras are defined, as in the case of ordinary algebras, by the condition (1). One may argue that the more natural approach would be to generalize the corresponding super notion. Recall that a homogeneous linear map $D : A \rightarrow A$ is called a *superderivation of a superalgebra* $A = A_0 \oplus A_1$ if

$$D(ab) = D(a)b + (-1)^{\deg(D)\deg(a)} aD(b)$$

for any two homogeneous elements $a, b \in A$. The *supercentroid* of A is the set of all homogeneous linear maps $\chi : A \rightarrow A$ such that

$$\chi(ab) = \chi(a)b = (-1)^{\deg(\chi)\deg(a)} a\chi(b)$$

for any two homogeneous elements $a, b \in A$. Accordingly, let us call a homogeneous linear map D a δ -*superderivation* of A if

$$D(ab) = \delta D(a)b + \delta (-1)^{\deg(D)\deg(a)} aD(b)$$

for any two homogeneous elements $a, b \in A$. Like in the ordinary case, this generalizes superderivations (for $\delta = 1$) and elements of supercentroid (for $\delta = \frac{1}{2}$).

Here we will consider δ -derivations of Lie superalgebras, like in [Kay2], as well as δ -superderivations. These two notions overlap: even δ -superderivations are the same as homogeneous (i.e., preserving the \mathbb{Z}_2 -grading) δ -derivations.

Though the mathematics below is pretty much elementary (modulo existing results), we call the reader to distinct carefully between ordinary, \mathbb{Z}_2 -graded and super variants of different constructions, as we pass freely back and forth between them.

Let A and B be two superalgebras. Their tensor product $A \otimes B$ as of ordinary algebras (i.e., with multiplication defined similarly to (2), as for the current Lie algebras), will, naturally, carry a superalgebra structure, with the zero component

$$(A \otimes B)_0 = (A_0 \otimes B_0) \oplus (A_1 \otimes B_1).$$

If $G = G_0 \oplus G_1$, the Grassmann superalgebra in the countable number of odd variables, $(A \otimes G)_0$ is nothing but the *Grassmann envelope of A* , denoted as $G(A)$ – a construction proved to be very useful in many questions pertained to varieties of (ordinary) algebras. It is well known (and easy to see) that for an arbitrary superalgebra A , $G(A)$ satisfies some identity w if and only if A satisfies the corresponding superidentity (obtained from w by appropriately injecting signs). In particular, L is a Lie superalgebra if and only if $G(L)$ is a Lie algebra (on the other hand, the whole $L \otimes G$ is, in general, not a Lie superalgebra).

Though it is possible to develop a super or \mathbb{Z}_2 -graded modifications of the technique from [Z1], in general it seems to be fairly difficult to say something general about δ -derivations of $G(L)$ in terms of L . However, the following elementary observations take place:

Lemma 4.1. *Let A and B be two superalgebras.*

(i) *If D is a δ -derivation of A , then the map $\widehat{D} : (A \otimes B)_0 \rightarrow A \otimes B$ defined as*

$$a_0 \otimes b_0 + a_1 \otimes b_1 \mapsto D(a_0) \otimes b_0 + D(a_1) \otimes b_1$$

for $a_0 \in A_0, a_1 \in A_1, b_0 \in B_0, b_1 \in B_1$, is a δ -derivation of $(A \otimes B)_0$ with values in $A \otimes B$.

(ii) *If D is a δ -superderivation of A , and χ is an element of the supercentroid of B such that $\deg(D) = \deg(\chi)$, then the map $\widehat{D} : (A \otimes B)_0 \rightarrow (A \otimes B)_0$ defined as*

$$a_0 \otimes b_0 + a_1 \otimes b_1 \mapsto D(a_0) \otimes \chi(b_0) + D(a_1) \otimes \chi(b_1)$$

for $a_0 \in A_0, a_1 \in A_1, b_0 \in B_0, b_1 \in B_1$, is a δ -derivation of $(A \otimes B)_0$.

Proof. Direct verification. □

And similarly:

Lemma 4.2. *Let $L = L_0 \oplus L_1$ be a Lie superalgebra, (\cdot, \cdot) a nondegenerate supersymmetric invariant bilinear form on L , and $f : G \rightarrow K$ a nonzero linear form. Then the bilinear map $(\cdot, \cdot) : G(L) \times G(L) \rightarrow K$ defined as*

$$\begin{aligned} &(x_0 \otimes g_0 + x_1 \otimes g_1, x'_0 \otimes g'_0 + x'_1 \otimes g'_1) \\ &= (x_0, x'_0) \otimes f(g_0 g'_0) + (x_0, x'_1) \otimes f(g_0 g'_1) + (x_1, x'_0) \otimes f(g_1 g'_0) + (x_1, x'_1) \otimes f(g_1 g'_1) \end{aligned}$$

for $x_0, x'_0 \in L_0, x_1, x'_1 \in L_1, g_0, g'_0 \in G_0, g_1, g'_1 \in G_1$, is a nondegenerate symmetric invariant bilinear form on $G(L)$.

Proof. Direct verification. □

A thorough analysis, sometimes with slight modifications, of some of the Filippov's arguments allows to establish a bit more than actually was stated in his papers.

Lemma 4.3. *If I is a nonzero ideal of a prime Lie superalgebra L , $\delta \neq 0$, and D is a nonzero δ -derivation or δ -superderivation of L , then $D(I) \neq 0$.*

Proof. This was proved in [F1, Lemma 3] in the case of Lie algebras, and exactly the same proof is valid in the case of superalgebras. □

Recall that the *standard identity of degree 5* is the Lie identity of the form

$$\sum_{\sigma \in S_4} (-1)^\sigma [[[[y, x_{\sigma(1)}], x_{\sigma(2)}], x_{\sigma(3)}], x_{\sigma(4)}] = 0$$

(summation is performed over all elements of the symmetric group S_4). Its super analog (with appropriately injected signs) will be called the *standard superidentity of degree 5*. By $s_4(L)$ for a Lie (super)algebra L we will understand the ideal of L consisting of the linear spans of the left hand side of the standard (super)identity of degree 5 for all $x_1, x_2, x_3, x_4, y \in L$.

Lemma 4.4. *Let L be a Lie algebra, M an L -module, and D a δ -derivation of L with values in M , $\delta \neq 0, 1, \frac{1}{2}$. Then $s_4(L) \subseteq \text{Ker } D$.*

Proof. Consider the semidirect sum $\mathcal{L} = L \oplus M$, with the Lie bracket defined in the usual way: $[m, x] = x \bullet m$ for $x \in L, m \in M$, and $[M, M] = 0$. Extend D to a linear map $\mathcal{D} : \mathcal{L} \rightarrow \mathcal{L}$ by putting $\mathcal{D}(M) = 0$. It is obvious that the so extended map is a δ -derivation of \mathcal{L} . By [F2, Theorem 1], $s_4(\mathcal{L}) \subseteq \text{Ker } \mathcal{D}$. It is obvious that $s_4(\mathcal{L}) = s_4(L) \oplus M'$ for some subspace $M' \subseteq M$. On the other hand, $\text{Ker } \mathcal{D} = \text{Ker } D \oplus M$, and the desired statement follows. □

Lemma 4.5. *A Lie algebra satisfying the standard identity of degree 5 and having a nonzero δ -derivation, $\delta \neq -1, 0, \frac{1}{2}, 1$, is solvable.*

Proof. In the proof of Theorem FA in [F2], the condition of primeness of an algebra is used only twice: first to make use of [F2, Lemma 3] claiming that a prime Lie algebra having a nonzero δ -derivation, $\delta \neq 0, \frac{1}{2}, 1$, satisfies the standard identity of degree 5 and a certain its consequence, and then at the very end, to obtain contradiction with solvability. Consequently, that proof, almost verbatim, will serve as the proof of the Lemma. \square

Lemma 4.6. *The space of $\frac{1}{2}$ -derivations of a perfect centerless Lie algebra having a nondegenerate symmetric invariant bilinear form, coincides with the centroid of the algebra.*

Proof. In the proof of [F1, Theorem 6], the condition of primeness of an algebra L is used only at the very end, to deduce that

$$\{x \in L \mid [[L, L], x] = 0\} = 0.$$

But for a perfect and centerless Lie algebra, the latter condition will be trivially satisfied as well. \square

All this together allows to establish super analogs of Theorems FA and FB:

Theorem 4.7. *A prime Lie superalgebra does not have nonzero δ -derivations and nonzero δ -superderivations if $\delta \neq -1, 0, \frac{1}{2}, 1$.*

Proof. Consider first the case of δ -derivations. Let D be a nonzero δ -derivation of a prime Lie superalgebra L . By Lemma 4.1(i), \widehat{D} is a nonzero δ -derivation of $G(L)$ with values in $L \otimes G$. By Lemma 4.4,

$$(12) \quad s_4(G(L)) \subseteq \text{Ker}(\widehat{D}),$$

Obviously,

$$(13) \quad s_4(G(L)) = G(s_4(L)).$$

On the other hand,

$$(14) \quad \text{Ker}(\widehat{D}) = (\text{Ker}(D|_{L_0}) \otimes G_0) \oplus (\text{Ker}(D|_{L_1}) \otimes G_1).$$

It follows from (12), (13) and (14) that

$$s_4(L) = s_4(L)_0 \oplus s_4(L)_1 \subseteq \text{Ker}(D|_{L_0}) \oplus \text{Ker}(D|_{L_1}) \subseteq \text{Ker} D,$$

where $s_4(L) = s_4(L)_0 \oplus s_4(L)_1$ is decomposition of the Lie superalgebra $s_4(L)$ into the even and odd parts. By Lemma 4.3, $s_4(L) = 0$, in other words, L satisfies the standard superidentity of degree 5, and $G(L)$ satisfies the standard identity of degree 5. By Lemma 4.5, $G(L)$ is solvable and hence L is solvable, a contradiction.

Now let D be a δ -superderivation of L . Take as χ the left multiplication by any homogeneous element of G whose parity coincides with the parity of D and apply Lemma 4.1(ii). \widehat{D} is a nonzero δ -derivation of $G(L)$, and the rest of reasoning is the same as in the case of δ -derivations. \square

Theorem 4.8. *The space of $\frac{1}{2}$ -derivations (respectively, $\frac{1}{2}$ -superderivations) of a perfect centerless Lie superalgebra having a nondegenerate supersymmetric invariant bilinear form, coincides with the centroid (respectively, supercentroid) of the superalgebra.*

Proof. Again, consider first the case of $\frac{1}{2}$ -derivations. Let L be such a superalgebra, and D is its $\frac{1}{2}$ -derivation. By Lemma 4.1(i), \widehat{D} is an $\frac{1}{2}$ -derivation of $G(L)$ with values in $L \otimes G$. We employ the same elementary trick as in the proof of Lemma 4.4 and consider the semidirect sum $\mathcal{L} = G(L) \oplus (L \otimes G)$, where the Lie bracket between $G(L)$ and $L \otimes G$ is determined by action of the former on the latter, and the space $L \otimes G$ is considered as an abelian subalgebra. It is obvious that this semidirect sum is perfect and centerless. We can also furnish it with the symmetric

bilinear form: on $G(L)$ it is defined as in Lemma 4.2, between elements of $G(L)$ and $L \otimes G$ it is defined, essentially, in the same way:

$$(x \otimes g, x' \otimes g') = (x, x')f(gg')$$

for homogeneous elements $x, x' \in L$, $g, g' \in G$, and on $L \otimes G$ the form vanishes. The same elementary calculation that verifies the validity of Lemma 4.2, verifies that this form is invariant and nondegenerate.

We also extend \widehat{D} from $L(G)$ to \mathcal{L} as in the proof of Lemma 4.4, and apply to the resulting $\frac{1}{2}$ -derivation of \mathcal{L} Lemma 4.6 to get:

$$D(x_0) \otimes g_0 + D(x_1) \otimes g_1 = \widehat{\chi}(x_0 \otimes g_0 + x_1 \otimes g_1)$$

for any $x_0 \in L_0$, $x_1 \in L_1$, $g_0 \in G_0$, $g_1 \in G_1$, and for a certain $\widehat{\chi}$ belonging to the centroid of \mathcal{L} . This immediately implies that D belongs to the centroid of L .

The case of $\frac{1}{2}$ -superderivations is similar, but simpler: we appeal to Lemma 4.1(ii) (where χ is taken, as in the respective part of the proof of Theorem 4.7, as the left multiplication by a homogeneous element $g \in G$ of the appropriate parity), directly to Lemma 4.2, and to Lemma 4.6, to deduce that for every $\frac{1}{2}$ -superderivation D of L ,

$$D(x_0) \otimes gg_0 + D(x_1) \otimes gg_1 = \widehat{\chi}(x_0 \otimes g_0 + x_1 \otimes g_1)$$

holds for any $x_0 \in L_0$, $x_1 \in L_1$, $g_0 \in G_0$, $g_1 \in G_1$, and for a certain $\widehat{\chi}$ belonging to the centroid of $G(L)$. This immediately implies that D belongs to the supercentroid of L . \square

Theorem 4.7 (together with its proof) and Theorem 4.8 allow to streamline most of the proofs from [Kay2] of the absence of nontrivial δ -derivations of classical Lie superalgebras. For example, when dealing with (-1) -derivations, the condition that a Lie superalgebra satisfies the standard superidentity of degree 5 allows to exclude most of the cases, and when dealing with $\frac{1}{2}$ -derivations, Theorem 4.8 allows to settle immediately the cases of superalgebras $A(m, n)$ with $m \neq n$, $B(m, n)$, $C(n)$, $D(m, n)$ with $m - n \neq 1$, $F(4)$ and $G(3)$, as follows from [Kac, §2.3.4].

Note that a similar approach can be used to obtain a super analog of the Filippov's result about absence of nonzero δ -derivations of prime associative algebras for $\delta \neq 0, \frac{1}{2}, 1$ ([F3, Theorem 1]). In that case, instead of taking the Grassmann envelope, we may take a similar tensor product construction with any centrally closed associative superalgebra (for example, a free associative superalgebra). Then one may extend in the straightforward way notions and results from [EMO] pertaining generalized (Martindale) centroid and primeness of the tensor product of prime algebras to the super case, and utilize Lemma 4.1 to reduce the super case to the ordinary one.

5. FURTHER QUESTIONS

5.1. The isomorphisms (5) in §2, as well as Proposition 2.2, suggest that δ -derivations of a Lie algebra of derivations of a commutative algebra are strongly related to the underlying commutative algebra. In this connection, it would be interesting to try to obtain δ -derivational counterparts of the very general results [S] of Skryabin about the first cohomology of Lie algebras of derivations of commutative rings.

5.2. It was suggested by Dimitry Leites and his collaborators (see, for example, [BGL, §1] and [LLD] and references therein), that in some circumstances the cohomology theory of modular Lie (super)algebras, especially in characteristic 2, is inadequate to describe the structural phenomena, and they posed a question to devise a more “ideologically correct” cohomology. Leites suggested further that δ -derivations may have something to do with that. Modular Lie superalgebras or not, it appears to be interesting to devise a cohomology theory which will encompass δ -derivations. All “naive” attempts to do that seemingly fail.

5.3. Let D be a δ -derivation of a Lie algebra L . Suppose that one of the following holds: D is nilpotent with index of nilpotency less than the (positive) characteristic of the ground field; D is nilpotent and the ground field is of characteristic 0; L is finite-dimensional and the ground field is \mathbb{R} or \mathbb{C} . Then

$$\exp(\delta D) = 1 + \delta D + \frac{\delta^2 D^2}{2!} + \frac{\delta^3 D^3}{3!} + \cdots : L \rightarrow L$$

is a well-defined linear map, and

$$\exp(D)([x, y]) = [\exp(\delta D)(x), \exp(\delta D)(y)]$$

holds for any $x, y \in L$. This is proved exactly in the same way as in the classical case of derivations for $\delta = 1$ (see, for example, [J, Chapter 1, §2]).

More generally, if D is a quasiderivation of L such that one of the assumptions above holds both for D and $F(D)$, then

$$\exp(F(D))([x, y]) = [\exp(D)(x), \exp(D)(y)]$$

holds for any $x, y \in L$.

This suggests the following definition. Let us call a bijective linear map $\varphi : L \rightarrow L$ a *quasiautomorphism* of L if there exists a bijective linear map $\psi : L \rightarrow L$ such that

$$\psi([x, y]) = [\varphi(x), \varphi(y)]$$

holds for any $x, y \in L$. Obviously, the set of all quasiautomorphisms of a Lie algebra forms a group. To our knowledge, this notion was not studied yet. May be it is interesting to undertake such a study.

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Thanks are due to late Hans Zassenhaus for supplying me, long time ago, with a reprint of his paper [PZ]. Though nowadays, with all these electronic accesses, this does not seem to be a big deal, it was such at 1989, when I was a student in a remote part of the former Soviet Union. The letter from him containing the requested reprint with warm words of encouragement would always be remembered with gratitude and nostalgia.

APPENDIX

Here we describe a simple-minded GAP program, available as <http://justpasha.org/math/delta-der.gap>[†], for computation of δ -derivations of a Lie (and, more general, anticommutative) algebra. It was used to verify various claims made in this paper.

Let A be an anticommutative algebra with basis $\{e_1, \dots, e_n\}$ defined over a field K , and with multiplication table $e_i e_j = \sum_{k=1}^n C_{ij}^k e_k$, and let $\delta \in K$ and D be a δ -derivation of A . Writing $D(e_i) = \sum_{j=1}^n d_{ij} e_j$, for certain $d_{ij} \in K$, the condition (1), written for each pair of basic elements, is equivalent to the system of $\frac{n^2(n-1)}{2}$ linear homogeneous equations in d_{ij} :

$$(15) \quad \sum_{k=1}^n C_{ij}^k d_{kl} - \delta \sum_{k=1}^n C_{kj}^l d_{ik} + \delta \sum_{k=1}^n C_{ki}^l d_{jk} = 0$$

for $1 \leq i < j \leq n$, $1 \leq l \leq n$. So we just solve this system using the standard library routine.

When dealing with the 16-dimensional Lie algebra from [E1] (as mentioned in §3), we want to compute δ -derivations for *all* δ , i.e., to solve a linear system with parameter. As GAP (version 4.4.12 as of time of this writing) does not support transcendental field extensions – what would be a natural way to work with parameters – we are cheating by using cyclotomic fields instead. However, this cheating could be made rigorous by picking a cyclotomic extension of prime degree (of course, any other field extension by an irreducible polynomial will do) larger than the highest possible power of a parameter involved in computation. For example, if we deal with a

[†]December 26, 2020: Currently available at <https://web.osu.cz/~Zusmanovich/papers/delta-der.gap>

16-dimensional algebra, the system (15) will be of size 1920×256 . As all coefficients of that system depend on the parameter δ linearly, any power of δ occurring in its solution will not exceed 256, so the cyclotomic extension of degree 257 would be enough.

REFERENCES

- [AS] R.K. Amayo and I. Stewart, *Infinite-dimensional Lie Algebras*, Noordhoff, Leyden, 1974.
- [BN] G. Benkart and E. Neher, *The centroid of extended affine and root graded Lie algebras*, J. Pure Appl. Algebra **205** (2006), 117–145; arXiv:math/0502561.
- [B] R.E. Block, *Determination of the differentially simple rings with a minimal ideal*, Ann. Math. **90** (1969), 433–459.
- [BGL] S. Bouarroudj, P. Grozman and D. Leites, *Infinitesimal deformations of symmetric simple modular Lie algebras and Lie superalgebras*; arXiv:0807.3054v1.
- [BH] R.B. Brown and N.C. Hopkins, *Noncommutative matrix Jordan algebras, Cayley–Dickson algebras, and Schafer’s theorem*, Comm. Algebra **23** (1995), 373–397.
- [DZ] A. Dzhumadil’dav and P. Zusmanovich, *Commutative 2-cocycles on Lie algebras*, J. Algebra **324** (2010), 732–748; arXiv:0907.4780.
- [E1] A. Elduque, *A Lie grading which is not a semigroup grading*, Lin. Algebra Appl. **418** (2006), 312–314; arXiv:math/0512618.
- [E2] ———, *More non-semigroup Lie gradings*, Lin. Algebra Appl. **431** (2009), 1603–1606; arXiv:0809.4547.
- [EMO] T.S. Erickson, W.S. Martindale and J.M. Osborn, *Prime nonassociative algebras*, Pacific J. Math. **60** (1975), 49–63.
- [F1] V.T. Filippov, *On δ -derivations of Lie algebras*, Siber. Math. J. **39** (1998), 1218–1230.
- [F2] ———, *δ -derivations of prime Lie algebras*, Siber. Math. J. **40** (1999), 174–184.
- [F3] ———, *δ -derivations of prime alternative and Mal’tsev algebras*, Algebra and Logic **39** (2000), 354–358.
- [G] H. Gündoğan, *Lie algebras of smooth sections*, Diploma Thesis, Technische Univ. Darmstadt, 2007; arXiv:0803.2800v3.
- [J] N. Jacobson, *Lie Algebras*, Dover, 1979 (reprint of Interscience Publ., 1962).
- [Kac] V. Kac, *Lie superalgebras*, Adv. Math. **26** (1977), 8–96.
- [Kay1] I.B. Kaygorodov, *δ -derivations of simple finite-dimensional Jordan superalgebras*, Algebra and Logic **46** (2007), 318–329.
- [Kay2] ———, *δ -derivations of classical Lie superalgebras*, Siber. Math. J. **50** (2009), 434–449.
- [Kr] Ya.S. Krylyuk, *On automorphisms and isomorphisms of quasi-simple Lie algebras*, J. Math. Sci. **100** (2000), 1944–2002.
- [Ku] M.I. Kuznetsov, *Simple modular Lie algebras with a solvable maximal subalgebra*, Math. USSR Sbornik **30** (1976), 68–76.
- [LLD] A. Lebedev and D. Leites (with an appendix by P. Deligne), *A realization of the Steenrod algebra*, J. Prime Res. Math. **2** (2006), 1–13.
- [LL] G.F. Leger and E.M. Luks, *Generalized derivations of Lie algebras*, J. Algebra **228** (2000), 165–203.
- [M] D.J. Melville, *Centroids of nilpotent Lie algebras*, Comm. Algebra **20** (1992), 3649–3682.
- [PZ] J. Patera and H. Zassenhaus, *On Lie gradings. I*, Lin. Algebra Appl. **112** (1989), 87–159.
- [S] S. Skryabin, *Degree one cohomology for the Lie algebras of derivations*, Lobachevskii J. Math. **14** (2004), 69–107.
- [SF] H. Strade and R. Farnsteiner, *Modular Lie Algebras and Their Representations*, Marcel Dekker, 1988.
- [Z1] P. Zusmanovich, *Low-dimensional cohomology of current Lie algebras and analogs of the Riemann tensor for loop manifolds*, Lin. Algebra Appl. **407** (2005), 71–104; arXiv:math/0302334.
- [Z2] ———, *Invariants of Lie algebras extended over commutative algebras without unit*, J. Nonlin. Math. Phys. **17** (2010), Suppl. 1, to appear; arXiv:0901.1395v1.

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