

A CONVERSE TO THE WHITEHEAD THEOREM

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ABSTRACT. In this note, which is a postscript to the earlier note [Z], we show that finite-dimensional Lie algebras over a field of characteristic zero such that their high-degree cohomology in any finite-dimensional non-trivial irreducible module vanishes, are, essentially, direct sums of semisimple and nilpotent algebras.

The classical First and Second Whitehead Lemmata state that the first, respectively second, cohomology group of a finite-dimensional semisimple Lie algebra over a field of characteristic zero with coefficients in any finite-dimensional module vanishes. This pattern breaks down, however, at the third cohomology: it is well-known that the third cohomology of a simple Lie algebra with coefficients in the trivial one-dimensional module is one-dimensional, the basic cocycle being constructed from the Killing form. There is, however, a related and not less classical result holding for all higher cohomology groups which is called sometimes the Whitehead Theorem: for any finite-dimensional semisimple Lie algebra L and any non-trivial finite-dimensional irreducible L -module V , $H^n(L, V) = 0$ for any $n \geq 3$.

It is very natural to ask whether a converse to these statements holds. A converse to the Second Whitehead Lemma was established in [Z]. The aim of this note is to observe that a converse to the Whitehead Theorem readily follows from the results already established in the literature.

In what follows, all algebras and modules assumed to be finite-dimensional, and the base field is of characteristic zero. Finite-dimensionality is obviously crucial in most of the places. Some reasonings below are valid in any characteristic, but the case of positive characteristic is trivial modulo existing results, as noted in [Z].

Our notations and terminology are standard: for a Lie algebra L and an L -module V , $H^n(L, V)$ denotes the n th cohomology of L with coefficients in V , V^L denotes the submodule of L -invariant points, $Z(L)$ and $Rad(L)$ denotes the center and the solvable radical of L respectively, V^* denotes an L -module adjoint to V . When considered as an L -module, the base field K is understood as the trivial one-dimensional module. L is called unimodular if $Tr(ad x) = 0$ for any $x \in L$, where Tr denotes the trace of a linear operator.

Theorem. *For a Lie algebra L , the following are equivalent:*

- (i) L is the direct sum of a semisimple algebra and a nilpotent algebra.
- (ii) $H^n(L, V) = 0$ for any n and any non-trivial irreducible L -module V .
- (iii) $H^{\dim L - 1}(L, V) = 0$ for any non-trivial irreducible L -module V .
- (iv) $H^1(L, V) = 0$ for any non-trivial irreducible L -module V .

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Proof. (i) \Rightarrow (ii): For a nilpotent Lie algebra even a more general statement is true: by [D, Théorème 1], cohomology of a nilpotent Lie algebra with coefficients in a module without trivial submodules, vanishes. The same is true for a semisimple Lie algebra: this follows from the classical results mentioned above, complete reducibility of representations of a semisimple Lie algebra, and the additivity of cohomology (cohomology with coefficients in the direct sum of modules decomposes as the direct sum of cohomologies with coefficients in direct summands).

Now let $L = S \oplus N$, the direct sum of a semisimple Lie algebra S and a nilpotent Lie algebra N , and V be a non-trivial irreducible L -module. Suppose that V , as an S -module, contains the trivial submodule K . As the actions of S and N on V commute, $S(N^i K) = N^i(SK) = 0$ for any $i \in \mathbb{N}$. Thus $\sum_i N^i K$ is an L -submodule of V and hence coincides with V , so V is a trivial S -module. Then any trivial N -submodule of V will be also a trivial L -submodule of V , hence V does not contain trivial submodules as an N -module.

Interchanging in this reasoning S and N (we have not used any properties of S and N , and it is valid for the direct sum of any Lie algebras), we see that at least for one of algebras S , N , the module V over that algebra does not contain trivial submodules.

By the Künneth formula, $H^n(S \oplus N, V) \simeq \bigoplus_{i+j=n} H^i(S, V) \otimes H^j(N, V)$, and by above, at least one of the tensor factors in each summand vanishes.

(ii) \Rightarrow (iii), as well as (ii) \Rightarrow (iv), is obvious.

(iii) \Rightarrow (iv): By the result of Hazewinkel [H], which is a generalization of the well-known Poincaré duality for the cohomology of unimodular Lie algebras ([F, Chapter 1, §1.6B]) to all Lie algebras, $H^n(L, (V^{tw})^*) \simeq H^{\dim L - n}(L, V)^*$ for any $0 \leq n \leq \dim L$ and any L -module V , where V^{tw} is a “twisted” L -module defined on the vector space V as follows: $x \mapsto \rho_V(x) - \text{Tr}(ad x)1_V$ for $x \in L$, where $x \mapsto \rho_V(x)$ is the representation corresponding to the module V , and 1_V is the identity map on V .

Let V be a non-trivial irreducible L -module. By the Hazewinkel’s duality,

$$(1) \quad H^1(L, (V^{tw})^*) \simeq H^{\dim L - 1}(L, V)^*$$

and

$$(2) \quad H^1(L, V) \simeq H^{\dim L - 1}(L, (V^*)^{-tw}),$$

where the “minus-twisted” module W^{-tw} is obtained from the module W by the formula $x \mapsto \rho_W(x) + \text{Tr}(ad x)1_W$. It is well-known (see, for example, [Bo, Chapter 1, §3.3]) that a module V is irreducible if and only if V^* is irreducible, and it is obvious that V is irreducible if and only if V^{-tw} is irreducible. Hence $(V^*)^{-tw}$ is an irreducible L -module.

If $(V^*)^{-tw}$ is a trivial module, then V^* is an one-dimensional module Kv defined by $xv = -\text{Tr}(ad x)v$ for $x \in L$, so V is an one-dimensional module Kv defined by $xv = \text{Tr}(ad x)v$, and V^{tw} , as well as $(V^{tw})^*$, is the one-dimensional trivial L -module. Since $H^{\dim L - 1}(L, V) = 0$, (1) implies $H^1(L, K) = 0$, what is equivalent to $[L, L] = L$. The latter, in its turn, implies that $\text{Tr}(ad x) = 0$ for any $x \in L$ (i.e. L is unimodular), so V is trivial, a contradiction.

Thus $(V^*)^{-tw}$ is a non-trivial module, hence $H^{\dim L - 1}(L, (V^*)^{-tw}) = 0$, and by (2), $H^1(L, V) = 0$.

Of course, very similar reasonings could be utilized to prove the reverse implication (iv) \Rightarrow (iii).

(iv) \Rightarrow (i): In [Ba, Theorem 3] a very close statement is proved: a Lie algebra L is the direct sum of a semisimple algebra and a supersolvable algebra if and only if $H^1(L, V) = 0$ for any irreducible L -module V of dimension > 1 . Note a subtle but important difference between two conditions on L -modules: solvable Lie algebras (and, more generally, Lie algebras with a nonzero radical) may have (and necessarily have if the ground field is algebraically closed, by the Lie Theorem) one-dimensional (and, hence, irreducible) non-trivial module. What follows is, essentially, modified for our needs proof from [Ba].

First note that condition (iv) is closed under quotients. Indeed, let a Lie algebra L satisfies (iv), I be an ideal of L , and V be a non-trivial irreducible L/I -module. We can lift V to an L -module by letting I act on it trivially. Obviously, the such lifted L -module is also non-trivial and irreducible. The beginning of the standard 5-term long exact sequence which follows from the Hochschild–Serre spectral sequence, reads: $0 \rightarrow H^1(L/I, V^I) \rightarrow H^1(L, V)$. As $V^I = V$ and $H^1(L, V) = 0$, we have $H^1(L/I, V) = 0$.

Now we argue by induction on the dimension of L . The case of dimension 1 is obvious. Suppose L is not semisimple and satisfies (iv). Then L has a minimal abelian ideal I , L/I also satisfies (iv), and by induction assumption satisfies (i). Let $L = S \ltimes \text{Rad}(L)$ be the Levi–Malcev decomposition of L . Obviously, $I \subseteq \text{Rad}(L)$, and $L/I \simeq S \ltimes \text{Rad}(L)/I$ is the Levi–Malcev decomposition of L/I . As (i) amounts to saying that the Levi–Malcev decomposition of a Lie algebra degenerates to the direct sum and its radical is nilpotent, it follows that $L/I \simeq S \oplus \text{Rad}(L)/I$ and $\text{Rad}(L)/I$ is nilpotent. Note also that since I is minimal, it is an irreducible L/I -module.

If I is a trivial L/I -module, then $I \subseteq Z(L) \subseteq Z(\text{Rad}(L))$. Since extension of a nilpotent Lie algebra by an ideal lying in its center is obviously nilpotent, $\text{Rad}(L)$ is nilpotent. Also, since S acts on $\text{Rad}(L)/I$ trivially, $[S, \text{Rad}(L)] \subseteq I$, and hence S acts on $\text{Rad}(L)$ nilpotently. But since S is semisimple, this means $[S, \text{Rad}(L)] = 0$.

Suppose I is not a trivial L/I -module. By (i) \Rightarrow (ii), $H^2(L/I, I) = 0$, and hence the extension $0 \rightarrow I \rightarrow L \rightarrow L/I \rightarrow 0$ splits, i.e. $L \simeq S \oplus \text{Rad}(L)/I \oplus I$, thus L is isomorphic to the direct sum of a semisimple algebra S and a nilpotent algebra $\text{Rad}(L)/I \oplus I$. \square

Corollary. *For a Lie algebra L , the following are equivalent:*

- (i) $H^n(L, V) = 0$ for any $n \geq 3$ and any non-trivial irreducible L -module V .
- (ii) L is either the direct sum of a semisimple and a nilpotent algebra, or a 2-dimensional algebra, or a 3-dimensional unimodular algebra.

Proof. For $\dim L > 3$, in one direction, put $n = \dim L - 1$ and invoke implication (iii) \Rightarrow (i) of the Theorem, and in another direction this is exactly implication (i) \Rightarrow (ii) of the Theorem.

For $\dim L = 2$, (i) is satisfied vacuously.

Let $\dim L = 3$. (i) is non-vacuous only for $n = 3$. By the Hazewinkel’s duality, $H^3(L, V) \simeq H^0(L, (V^*)^{-tw})^* \simeq (((V^*)^{-tw})^L)^*$. By the same arguments as in the proof of implication (iii) \Rightarrow (iv) of the Theorem, $(V^*)^{-tw}$ is irreducible if and only if V is irreducible, and is trivial if and only if V is isomorphic to K^{-tw} . Consequently,

(i) in this case is equivalent to the triviality of the module K^{-tw} , what, in its turn, is equivalent to unimodularity of L . \square

The implication (i) \Rightarrow (ii) of the Corollary is the converse to the Whitehead Theorem.

Using the well-known classification of 3-dimensional Lie algebras (see, for example, [Bo, Chapter 1, §6, Exercise 23]), it is easy to list 3-dimensional unimodular Lie algebras. The list consists of the abelian algebra, the nilpotent (Heisenberg) algebra, simple algebras, and solvable algebras with basis $\{x, y, z\}$ and multiplication table $[x, y] = 0$, $[x, z] = ax + by$ $[y, z] = cx - ay$ for certain $a, b, c \in K$ with $a^2 + bc \neq 0$.

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