ON HERMITIAN AND SKEW-HERMITIAN MATRIX ALGEBRAS OVER OCTONIONS

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ABSTRACT. We prove simplicity of algebras in the title, and compute their δ-derivations and symmetric associative forms.

INTRODUCTION

We consider algebras of Hermitian and skew-Hermitian matrices over octonions. While such algebras of matrices of low order are well researched and well understood (the algebra of 3×3 Hermitian matrices being the famous exceptional simple Jordan algebra), this is not so for higher orders; the case of Hermitian matrices of order 4×4 appears in modern physics (string theory, M-theory).

Derivation algebras of algebras of Hermitian and skew-Hermitian matrices over octonions were recently computed in [P], and here we continue to study these algebras. After the preliminary §1, where we set notation and remind basic facts about algebras with involution, we prove simplicity of the algebras in question (§2), and compute their δ-derivations (§3) and symmetric associative forms (§4). The last §5 contains some further questions.

1. NOTATION, CONVENTIONS, PRELIMINARY REMARKS

1.1. The ground field K of characteristic $\neq 2,3$ is assumed to be arbitrary, unless stated otherwise; $\overline{K}$ and K^q denote the algebraic and the quadratic closure of K, respectively. "Algebra" means an arbitrary algebra over K, not necessary associative, or Lie, or Jordan, or satisfying any other distinguished identity, unless specified otherwise. If a is an element of an algebra A, then R_a denotes the linear operator of the right multiplication by a. All unadorned tensor products and Hom's are over the ground field K. The symbol $\dotplus$ denotes the direct sum of vector spaces, while $\oplus$ denotes the direct sum of algebras or modules.

1.2. Algebras with involution. An *involution* on a vector space V is a linear map $j: V \to V$ such that $j^2 = \mathrm{id}_V$. If j is an involution on V, define

$$\mathrm{S}^+(V,j) = \{x \in V \mid j(x) = x\}$$

and

$$\mathrm{S}^-(V,j) = \{x \in V \mid j(x) = -x\},$$

the subspaces of j-symmetric and j-skew-symmetric elements of V, respectively.

For an arbitrary vector space with involution j, we have the direct sum decomposition:

$$V = \mathrm{S}^+(V,j) \dotplus \mathrm{S}^-(V,j).$$

An *involution* on an algebra A is a linear map $j: A \to A$ which is an involution on A as a vector space, and, additionally, is an antiautomorphism of A, i.e., $j(xy) = j(y)j(x)$ for any $x,y \in A$.

For an arbitrary algebra A with involution j, the subspace $\mathrm{S}^+(A,j)$ is closed with respect to the half of the anticommutator $x \circ y = \frac{1}{2}(xy+yx)$, and thus forms a (commutative) algebra with respect to $\circ$. The operation $\circ$ will be also frequently referred as the *Jordan product*, despite that the ensuing algebras

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are, generally, not Jordan. Similarly, the subspace $\mathrm{S}^-(A,j)$ is closed with respect to the commutator $[x,y] = xy - yx$, and thus forms an (anticommutative) algebra with respect to $[\cdot,\cdot]$.

We have the following obvious inclusions:

$$
\begin{aligned}
&\mathrm{S}^+(A,j) \circ \mathrm{S}^+(A,j) \subseteq \mathrm{S}^+(A,j)\\
&\mathrm{S}^+(A,j) \circ \mathrm{S}^-(A,j) \subseteq \mathrm{S}^-(A,j)\\
&\mathrm{S}^-(A,j) \circ \mathrm{S}^-(A,j) \subseteq \mathrm{S}^+(A,j)
\end{aligned} \tag{1}
$$

and

$$
\begin{aligned}
&[\mathrm{S}^+(A,j), \mathrm{S}^+(A,j)] \subseteq \mathrm{S}^-(A,j)\\
&[\mathrm{S}^+(A,j), \mathrm{S}^-(A,j)] \subseteq \mathrm{S}^+(A,j)\\
&[\mathrm{S}^-(A,j), \mathrm{S}^-(A,j)] \subseteq \mathrm{S}^-(A,j).
\end{aligned} \tag{2}
$$

If (A,j) and (B,k) are two vector spaces, respectively algebras, with involution, then their tensor product $(A \otimes B, j \otimes k)$, is a vector space, respectively algebra, with involution. Here $j \otimes k$ acts on $A \otimes B$ in an obvious way:

$$(j \otimes k)(a \otimes b) = j(a) \otimes k(b)$$

for any $a \in A$, $b \in B$.

1.3. Matrix algebras. $M_n(K)$ denotes the (associative) algebra of $n \times n$ matrices with entries in K. The matrix transposition, denoted by ${}^\top$, is an involution on $M_n(K)$. $\mathrm{Tr}(X)$ denotes the trace of a matrix X, and E denotes the unit matrix. We use the shorthand notation $M_n^+(K) = \mathrm{S}^+(M_n(K), {}^\top)$ and $M_n^-(K) = \mathrm{S}^-(M_n(K), {}^\top)$ for the spaces of symmetric and skew-symmetric $n \times n$ matrices, respectively.

The algebra $M_n^+(K)$ with respect to the Jordan product is a simple Jordan algebra. The space $M_n^-(K)$ is an irreducible Jordan module over $M_n^+(K)$ (see, for example, [Ja, Chapter VII, §3, Theorem 7]). In particular, $M_n^+(K) \circ M_n^-(K) = M_n^-(K)$.

The algebra $M_n^-(K)$ with respect to the commutator is the orthogonal Lie algebra, customarily denoted by $\mathfrak{so}_n(K)$. We have $\mathfrak{so}_1(K) = 0$, and $\mathfrak{so}_2(K) \simeq K$, the one-dimensional (abelian) Lie algebra. If $n = 3$ or $n \geq 5$, the Lie algebra $\mathfrak{so}_n(K)$ is simple; if $n = 4$, $\mathfrak{so}_4(K)$ is isomorphic to the direct sum of two copies of the 3-dimensional simple Lie algebra with the basis $\{e_1, e_2, e_3\}$ and the multiplication table $[e_1,e_2] = e_3$, $[e_2,e_3] = e_1$, $[e_3,e_1] = e_2$, denoted by us as $\mathfrak{su}_2(K)$ (of course, isomorphic to $\mathfrak{sl}_2(K)$ if K is algebraically closed). If $n \geq 3$, the $\mathfrak{so}_n(K)$-module $M_n^+(K)$, being isomorphic to the symmetric square of the tautological module, decomposes as the direct sum $KE \oplus SM_n(K)$, where KE is the trivial 1-dimensional module spanned by the unit matrix, and the vector space

$$SM_n(K) = \{X \in M_n^+(K) \mid \mathrm{Tr}(X) = 0\}$$

forms the $\frac{n^2+n-2}{2}$-dimensional irreducible module. In the case $n = 4$, the latter $\mathfrak{su}_2(K) \oplus \mathfrak{su}_2(K)$-module is isomorphic to the tensor product $\mathfrak{su}_2(K) \otimes \mathfrak{su}_2(K)$ of two irreducible adjoint modules over two copies of $\mathfrak{su}_2(K)$. (See, for example, [BBM, Lemma 3.1].) In particular, $[M_n^-(K), M_n^+(K)] = SM_n(K)$.

Lemma 1. *If $x \in M_n^-(K)$ is such that $x \circ M_n^-(K) = 0$, then $x = 0$.*

Proof. For $n = 1$ the statement is vacuous, so assume $n \geq 2$. Considering this on the Lie algebra level, we have $xy + yx = 0$ for any $y \in \mathfrak{so}_n(K)$. Taking the trace of the both sides of this equality, we have $\mathrm{Tr}(xy) = 0$. But the trace form $(x,y) \mapsto \mathrm{Tr}(xy)$ is nondegenerate on $\mathfrak{so}_n(K)$ (this can be verified directly, or see, for example, [Kap, p. 66]), and, consequently, $x = 0$. □

Lemma 2. *If $m \in M_n^+(K)$ is such that $[m, M_n^-(K)] = 0$ or $[m, M_n^+(K)] = 0$, then m is a multiple of E.*

Proof. Case of $[m, M_n^-(K)] = 0$ for $n = 1,2$ is verified immediately, and for $n \geq 3$ the proof follows from the above description of $M_n^+(K)$ as an $\mathfrak{so}_n(K)$-module.

Case of $[m, M_n^+(K)] = 0$. It is easy to check that this condition implies

$$(m,s,t) = (s,m,t) = (s,t,m) = 0$$

for any $s,t \in M_n^+(K)$, where $(x,y,z) = (x \circ y) \circ z - x \circ (y \circ z)$ is the Jordan associator, i.e., m lies in the center of the simple Jordan algebra $(M_n^+(K), \circ)$, which coincides with KE. $\square$

1.4. Octonion algebras. Octonion algebras over an arbitrary field K form the 3-parametric family $\mathbb{O}_\mu(K)$, where $\mu = (\mu_1, \mu_2, \mu_3)$ is a triple of nonzero elements of K. Let us recall its multiplication table in the standard basis $\{1, e_1, \dots, e_7\}$ (by abuse of notation, the basis element 1 is the unit of the algebra):

	e_1	e_2	e_3	e_4	e_5	e_6	e_7
e_1	$\mu_1 1$	$-e_3$	$-\mu_1 e_2$	$-e_5$	$-\mu_1 e_4$	e_7	$\mu_1 e_6$
e_2	e_3	$\mu_2 1$	$\mu_2 e_1$	$-e_6$	$-e_7$	$-\mu_2 e_4$	$-\mu_2 e_5$
e_3	$\mu_1 e_2$	$-\mu_2 e_1$	$-\mu_1\mu_2 1$	$-e_7$	$-\mu_1 e_6$	$\mu_2 e_5$	$\mu_1\mu_2 e_4$
e_4	e_5	e_6	e_7	$\mu_3 1$	$\mu_3 e_1$	$\mu_3 e_2$	$\mu_3 e_3$
e_5	$\mu_1 e_4$	e_7	$\mu_1 e_6$	$-\mu_3 e_1$	$-\mu_1\mu_3 1$	$-\mu_3 e_3$	$-\mu_1\mu_3 e_2$
e_6	$-e_7$	$\mu_2 e_4$	$-\mu_2 e_5$	$-\mu_3 e_2$	$\mu_3 e_3$	$-\mu_2\mu_3 1$	$\mu_2\mu_3 e_1$
e_7	$-\mu_1 e_6$	$\mu_2 e_5$	$-\mu_1\mu_2 e_4$	$-\mu_3 e_3$	$\mu_1\mu_3 e_2$	$-\mu_2\mu_3 e_1$	$\mu_1\mu_2\mu_3 1$

(the table, up to obvious notational changes, is reproduced from [Sch, p. 5]). Over some fields, there are isomorphisms within this family; for example, if the field is algebraically closed or finite, all octonion algebras are isomorphic to each other. As explained below, in the proofs of our main results we may assume the ground field to be algebraically closed, so we are free to choose any form of an octonion algebra we wish. The two most natural candidates would be $\mathbb{O}_{(-1,-1,-1)}(K)$ (for example, over $\mathbb{R}$ this is the single octonion division algebra), or the split octonion algebra $\mathbb{O}_{(-1,-1,1)}(K)$.

We have decided that for our calculations the most convenient will be the algebra $\mathbb{O}_{(-1,-1,-1)}(K)$, denoted just by $\mathbb{O}(K)$ in the sequel[†]. A quick glance at the multiplication table reveals the following properties of the basis elements we will need: $e_i^2 = -1$, $e_i e_j = -e_j e_i$, and, denoting by B_i the 6-dimensional linear span of all the basis elements except for 1 and e_i, we have $e_i B_i = B_i e_i = B_i$, for any $i = 1, \dots, 7$. By

$$* : \{1,\dots,7\} \times \{1,\dots,7\} \to \{1,\dots,7\}$$

we denote the partial binary operation such that $e_i e_j = -e_j e_i = \pm e_{i*j}$ for any $i \neq j$.

Extending the base field K to its algebraic closure $\overline{K}$, we have an isomorphism of $\overline{K}$-algebras

$$\mathbb{O}_\mu(K) \otimes_K \overline{K} \simeq \mathbb{O}(\overline{K}). \tag{3}$$

The standard conjugation in $\mathbb{O}_\mu(K)$, denoted by $^-$, and defined by $\overline{1} = 1$, $\overline{e}_i = -e_i$, turns $\mathbb{O}_\mu(K)$ into an algebra with involution. We have $\mathrm{S}^+(\mathbb{O}_\mu(K), ^-) = K1$, and $\mathrm{S}^-(\mathbb{O}_\mu(K), ^-)$ is the 7-dimensional subspace of imaginary octonions, linearly spanned by $e_1, \dots, e_7$. The latter subspace forms a 7-dimensional simple Malcev algebra with respect to the commutator. We will use the shorthand notation $\mathbb{O}_\mu^-(K) = \mathrm{S}^-(\mathbb{O}_\mu(K), ^-)$ and $\mathbb{O}^-(K) = \mathrm{S}^-(\mathbb{O}(K), ^-)$.

Since for any $a \in \mathbb{O}_\mu(K)$, the elements $a + \overline{a}$ and $a\overline{a}$ belong to $K1$, we can define the linear map $T : \mathbb{O}_\mu(K) \to K$ and the quadratic map $N : \mathbb{O}_\mu(K) \to K$ by $T(a) = a + \overline{a}$ and $N(a) = a\overline{a}$, called the *trace* and *norm*, respectively. Any element $a \in \mathbb{O}_\mu(K)$ satisfies the quadratic equality

$$a^2 - T(a)a + N(a)1 = 0 \tag{4}$$

(see, for example, [Sch, Chapter III, §4] or [Ja, p. 233, Exercise 1]).

For any two elements $a, b \in \mathbb{O}_\mu^-(K)$, writing the equality (4) for the element $a + b$, subtracting from it the same equalities for a and for b, and taking into account that $T(a) = T(b) = 0$, yields

$$ab + ba = -N(a,b)1, \tag{5}$$

where

$$N(a,b) = N(a+b) - N(a) - N(b).$$

[†]Of course, it is also possible to perform all our calculations in the case of generic 3-parametric octonion algebra $\mathbb{O}_\mu(K)$, but then they will be somewhat more cumbersome.

1.5. Algebras of Hermitian and skew-Hermitian matrices over octonions. Our main characters, the algebras of Hermitian and skew-Hermitian matrices over octonions, are defined as $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ and $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$ respectively, where $M_n(\mathbb{O}_\mu(K))$ is the algebra of $n\times n$ matrices with entries in $\mathbb{O}_\mu(K)$. The involution on $M_n(\mathbb{O}_\mu(K))$ is defined as $J : (a_{ij}) \mapsto (\overline{a_{ji}})$, i.e., the matrix is transposed and each entry is conjugated, simultaneously.

The algebras $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ contain the unit matrix, so they are unital. These algebras for small n's are Jordan algebras, well-known from the literature: for $n=1$, this is nothing but the ground field K; for $n=2$, they are 10-dimensional simple Jordan algebras of symmetric nondegenerate bilinear form (see, for example, [KMRT, Chapter IX, Exercise 4] and [R, §6]); and for $n=3$, they are the famous 27-dimensional exceptional simple Jordan algebras. For $n\geq 4$, they are no longer Jordan algebras.

Interestingly enough, the algebras $\mathrm{S}^+(M_4(\mathbb{O}_\mu(K)),J)$ were considered already in a little-known dissertation [R] (for a more accessible exposition, see [LRH, §5]), under the direction of Hel Braun and Pascual Jordan. More recently, the algebra $\mathrm{S}^+(M_4(\mathbb{O}(\mathbb{R})),J)$ appeared in [LT, §4] under the name "octonionic M-algebra", where it was suggested as an alternative to the standard M-algebra (a sort of generalization of the Poincaré algebra of spacetime symmetries). This algebra features some M-theory numerology (lesser number of real bosonic generators, equivalence between supermembrane and super-five-brane sectors) which, as suggested in [LT], could make this algebra a better alternative.

The algebras $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K),J)$ are less prominent: for $n=1$ these are the 7-dimensional simple Malcev algebras $\mathbb{O}^-_\mu(K)$; it seems that the only place where they appeared in the literature in the case of (small) $n>1$ is [BH], where identities of these algebras were studied.

Due to the isomorphism of algebras

$$M_n(\mathbb{O}_\mu(K)) \simeq M_n(K)\otimes \mathbb{O}_\mu(K),$$

the algebra with involution $(M_n(\mathbb{O}_\mu(K)),J)$ can be represented as the tensor product of two algebras with involution: $(M_n(K),^\top)$, the associative algebra of $n\times n$ matrices over K with involution defined by the matrix transposition, and $(\mathbb{O}_\mu(K),^-)$.

Finally, due to isomorphism (3), we have an isomorphism of $\overline{K}$-algebras:

$$\mathrm{S}^\pm(M_n(\mathbb{O}_\mu(K),J))\otimes_K \overline{K} \simeq \mathrm{S}^\pm(M_n(\mathbb{O}(\overline{K})),J). \tag{6}$$

2. SIMPLICITY

We start with rewriting our matrix algebras as the vector space direct sums of certain tensor products, which appears to be more convenient for computations. For this, we need the following simple lemma of linear algebra.

Lemma 3 ([Z, Lemma 1.1])**.** *Let V,W be two vector spaces, $\varphi,\varphi' \in \mathrm{Hom}(V,\cdot)$, $\psi,\psi'\in \mathrm{Hom}(W,\cdot)$. Then*

$$\begin{aligned}&\mathrm{Ker}(\varphi\otimes\psi)\cap \mathrm{Ker}(\varphi'\otimes\psi')\\ &\qquad\simeq (\mathrm{Ker}\,\varphi\cap\mathrm{Ker}\,\varphi')\otimes W + \mathrm{Ker}\,\varphi\otimes\mathrm{Ker}\,\psi' + \mathrm{Ker}\,\varphi'\otimes \mathrm{Ker}\,\psi + V\otimes(\mathrm{Ker}\,\psi\cap\mathrm{Ker}\,\psi').\end{aligned}$$

Proposition 4. *For any two vector spaces with involution (V,j) and (W,k), there are isomorphisms of vector spaces*

$$\begin{aligned}\mathrm{S}^+(V\otimes W, j\otimes k) &\simeq \mathrm{S}^+(V,j)\otimes \mathrm{S}^+(W,k) \dotplus \mathrm{S}^-(V,j)\otimes\mathrm{S}^-(W,k)\\ \mathrm{S}^-(V\otimes W, j\otimes k) &\simeq \mathrm{S}^+(V,j)\otimes \mathrm{S}^-(W,k) \dotplus \mathrm{S}^-(V,j)\otimes\mathrm{S}^+(W,k).\end{aligned}$$

Proof. Let us prove the first isomorphism, the proof of the second one is completely similar. By definition, an element $\sum_{i\in\mathbb{I}} v_i\otimes w_i$ of $V\otimes W$, where $\mathbb{I}$ is a set of indices, belongs to $\mathrm{S}^+(V\otimes W, j\otimes k)$, if and only if

$$\sum_{i\in\mathbb{I}}\Big(j(v_i)\otimes k(w_i) - v_i\otimes w_i\Big) = 0.$$

Applying to this equality the linear maps $(\mathrm{id}_V + j) \otimes \mathrm{id}_W$ and $(\mathrm{id}_V - j) \otimes \mathrm{id}_W$, we get respectively:

$$\sum_{i \in \mathbb{I}} (j(v_i) + v_i) \otimes (k(w_i) - w_i) = 0$$

and

$$\sum_{i \in \mathbb{I}} (j(v_i) - v_i) \otimes (k(w_i) + w_i) = 0.$$

Applying Lemma 3 to the last two equalities, we can replace v_i's and w_i's by their linear combinations in such a way that the index set splits into the disjoint union $\mathbb{I} = \mathbb{I}_{11} \cup \mathbb{I}_{12} \cup \mathbb{I}_{21} \cup \mathbb{I}_{22}$, where

$$\begin{aligned} &v_i \in \mathrm{S}^-(V,j),\ v_i \in \mathrm{S}^+(V,j) && \text{for } i \in \mathbb{I}_{11}, \\ &v_i \in \mathrm{S}^-(V,j),\ w_i \in \mathrm{S}^-(W,k) && \text{for } i \in \mathbb{I}_{12}, \\ &v_i \in \mathrm{S}^+(V,j),\ w_i \in \mathrm{S}^+(W,k) && \text{for } i \in \mathbb{I}_{21}, \\ &w_i \in \mathrm{S}^+(W,k), w_i \in \mathrm{S}^-(W,k) && \text{for } i \in \mathbb{I}_{22}. \end{aligned}$$

All elements with indices from $\mathbb{I}_{11}$ and $\mathbb{I}_{22}$ vanish, and we are done. □

In the particular case $(V,j) = (M_n(K), ^\top)$ and $(W,k) = (\mathbb{O}_\mu(K), ^-)$, denoting $J = ^\top \otimes ^-$, and taking into account that $\mathrm{S}^+(\mathbb{O}_\mu(K), ^-) = K1$, we get:

$$\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)), J) \simeq M_n^+(K) \otimes 1 \dotplus M_n^-(K) \otimes \mathbb{O}_\mu^-(K). \tag{7}$$

(In the case where $n = 3$ and K is algebraically closed and of characteristic zero, and so $\mathrm{S}^+(M_3(\mathbb{O}_\mu(K), J))$ is the 27-dimensional exceptional simple Jordan algebra, this decomposition was noted in [DM, §3.3].)

In particular,

$$\dim \mathrm{S}^+(M_n(\mathbb{O}_\mu(K), J)) = \frac{n(n+1)}{2} + 7 \cdot \frac{n(n-1)}{2} = 4n^2 - 3n.$$

For any $m, s \in M_n^+(K)$, we have

$$(m \otimes 1) \circ (s \otimes 1) = (m \circ s) \otimes 1,$$

what implies that $M_n^+(K) \otimes 1$ is a (Jordan) subalgebra of $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K), J))$. Moreover, for any $x, y \in M_n^-(K)$, and $a \in \mathbb{O}_\mu^-(K)$, we have:

$$\begin{aligned} (m \otimes 1) \circ (x \otimes a) &= (m \circ x) \otimes a, \\ (x \otimes a) \circ (y \otimes a) &= -N(a)\ (x \circ y) \otimes 1. \end{aligned}$$

It follows that $M_n^+(K) \otimes 1 \dotplus M_n^-(K) \otimes a$ is a subalgebra of $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K), J))$; let us denote this subalgebra by $\mathscr{L}^+(a)$. If $N(a) \neq 0$, we have an isomorphism of Jordan algebras $\mathscr{L}^+(a) \otimes_K K^q \simeq M_n(K^q)$; the isomorphism is provided by sending $m \otimes 1$ to m for $m \in M_n^+(K^q)$, and $x \otimes a$ to $\sqrt{-N(a)}x$ for $x \in M_n^-(K^q)$.

Further,

$$(M_n^+(K) \otimes 1) \circ (M_n^-(K) \otimes \mathbb{O}_\mu^-(K)) = M_n^-(K) \otimes \mathbb{O}_\mu^-(K).$$

On the other hand, the subspace $M_n^-(K) \otimes \mathbb{O}_\mu^-(K)$ is not a subalgebra. The formula for multiplication in this subspace in terms of the decomposition (7) is obtained using (5): for any $x, y \in M_n^-(K)$ and $a, b \in \mathbb{O}_\mu^-(K)$, we have

$$\begin{aligned} (x \otimes a) \circ (y \otimes b) = \frac{1}{2}(xy \otimes ab + yx \otimes ba) &= \frac{1}{4}(xy + yx) \otimes (ab + ba) + \frac{1}{4}(xy - yx) \otimes (ab - ba) \\ &= -\frac{N(a,b)}{2}\,(x \circ y) \otimes 1 + \frac{1}{4}[x,y] \otimes [a,b]. \end{aligned}$$

Similarly, we have

$$\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)), J) \simeq M_n^-(K) \otimes 1 \dotplus M_n^+(K) \otimes \mathbb{O}_\mu^-(K), \tag{8}$$

and
$$\dim \mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J) = \frac{n(n-1)}{2} + 7\cdot\frac{n(n+1)}{2} = 4n^2+3n.$$
For any $x,y \in M_n^-(K)$, $m,s \in M_n^+(K)$, and $a \in \mathbb{O}_\mu^-(K)$, we have:
$$\begin{aligned}
[x\otimes 1, y\otimes 1] &= [x,y]\otimes 1\\
[x\otimes 1, m\otimes a] &= [x,m]\otimes a\\
[m\otimes a, s\otimes a] &= N(a)[s,m]\otimes 1.
\end{aligned}$$

It follows that both $M_n^-(K)\otimes 1$ and
$$\mathscr{L}^-(a) = M_n^-(K)\otimes 1 \dotplus M_n^+(K)\otimes a$$
are Lie subalgebras of $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$, isomorphic to $\mathfrak{so}_n(K)$, and, provided $N(a)\neq 0$, to a form of $\mathfrak{gl}_n(K^q)$, respectively; the isomorphisms are defined by sending $x\otimes 1$ to x for $x\in M_n^-(K)$, and $m\otimes a$ to $\sqrt{-N(a)}\,m$ for $m\in M_n^+(K^q)$.

Moreover,
$$[M_n^-(K)\otimes 1, M_n^+(K)\otimes \mathbb{O}_\mu^-(K)] = SM_n(K)\otimes \mathbb{O}_\mu^-(K) \subset M_n^+(K)\otimes \mathbb{O}_\mu^-(K).$$

The subspace $M_n^+(K)\otimes\mathbb{O}_\mu^-(K)$ is not a subalgebra: for any $m,s\in M_n^+(K)$, $a,b\in\mathbb{O}_\mu^-(K)$, we have

(9) $$\begin{aligned}&[m\otimes a, s\otimes b]\\ &= \frac{1}{2}(ms-sm)\otimes(ab+ba) + \frac{1}{2}(ms+sm)\otimes(ab-ba) = -\frac{N(a,b)}{2}[m,s]\otimes 1 + (m\circ s)\otimes[a,b].\end{aligned}$$

Theorem 5. *The algebras* $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ *and* $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$ *are simple for any* $n\geq 1$.

Before we plunge into the proof, a few remarks are in order:

(i) The cases of $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ for $n=1,2,3$, and of $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$ for $n=1$ are well-known, due to the known structure of the algebras in question in these cases (see §1); however, our proofs, uniform for all n, appear to be new. The case of $\mathrm{S}^+(M_4(\mathbb{O}_\mu(K)),J)$ is stated without proof in [R, Satz 8.1].

(ii) In [St] it is proved that ideals of the tensor product $A\otimes B$ of two algebras A and B, where A is central (i.e., its centroid coincides with the ground field) and simple, and B satisfies some other conditions (like having a unit), are of the form $A\otimes I$, where I is an ideal of B. In particular, the tensor product of two central simple algebras, for example, $M_n(K)\otimes\mathbb{O}_\mu(K)$, is simple. Our method of proof of Theorem 5, based on application of the (version of) Jacobson density theorem, resembles those in [St].

(iii) Another related result about simplicity of nonassociative algebras is established in [R, Satz 5.1]: the matrix algebra over a composition algebra with respect to the Jordan product $\circ$, is simple; a particular case is the algebra $(M_n(\mathbb{O}_\mu(K)),\circ)$.

We will need the following version of the Jacobson density theorem.

Proposition 6. † *Let R be an associative algebra with unit, and $M_1,\dots,M_n$ pairwise non isomorphic right irreducible R-modules. Then for any linearly independent elements $x_1^{(i)},\dots,x_{k_i}^{(i)}\in M_i$, and any elements $y_1^{(i)},\dots,y_{k_i}^{(i)}\in M_i$, $i=1,\dots,n$, there is an element $a\in R$ such that $x_j^{(i)}\bullet a = y_j^{(i)}$ for any $i=1,\dots,n$, $j=1,\dots,k_i$.*

(Here $\bullet$ denotes the right action of A on its modules).

†Added March 27, 2022: As stated, the statement of the proposition is wrong. Like in the classical Jacobson density theorem, one needs to require that the module elements are independent over the ring of R-module endomorphisms of M, and not just over the ground field. Alternatively, one may require that the ground field is algebraically closed, as it is enough for our purposes. Thanks to are due Alberto Elduque for spotting this mistake.

Proof. This is, essentially, the Jacobson density theorem formulated for a completely reducible module $M = M_1 \oplus \dots \oplus M_n$. Perhaps, the easiest way to derive it in our formulation is the following. First, apply the classical Jacobson density theorem to each irreducible R-module M_i to get elements $a_i \in R$ such that $x_j^{(i)} \bullet a_i = y_j^{(i)}$ for any $i = 1, \dots, n$, $j = 1, \dots, k_i$. By [L, Chapter XVII, Theorem 3.7] (which is a consequence of the Jacobson density theorem for semisimple modules formulated in terms of bicommutants of modules, see [L, Chapter XVII, Theorem 3.2]), there are elements $e_i \in R$ such that e_i acts as the identity on M_i, and $M_j \bullet e_i = 0$ for $j \neq i$. Then $a = e_1 a_1 + \dots + e_n a_n$ is the required element. □

We now specialize this to our situation. Let A be an algebra, and M a right A-module. By the *multiplication algebra* $\mathfrak{M}(A,M)$ we mean the unital subalgebra in the associative algebra of all linear transformations of M, generated by actions of all elements of A on M. If A acts on itself via right multiplications, i.e., $M = A$, then $\mathfrak{M}(A,A)$ is called the *multiplication algebra of* A.

Lemma 7.

(i) *For any linearly independent elements $m_1, \dots, m_k \in M_n^+(K)$, $x_1, \dots, x_\ell \in M_n^-(K)$, and any elements $m'_1, \dots, m'_k \in M_n^+(K)$, $x'_1, \dots, x'_\ell \in M_n^-(K)$, there is a map $R \in \mathfrak{M}(M_n^+(K), M_n(K))$ such that $R(m_i) = m'_i$ for $i = 1, \dots, k$, and $R(x_i) = x'_i$ for $i = 1, \dots, \ell$.*

(ii) *Let $n \neq 4$. For any linearly independent elements $m_1, \dots, m_k \in SM_n(K)$, $x_1, \dots, x_\ell \in M_n^-(K)$, and any elements $m'_1, \dots, m'_k \in SM_n(K)$, $x'_1, \dots, x'_\ell \in M_n^-(K)$, there is a map $R \in \mathfrak{M}(\mathfrak{so}_n(K), M_n(K))$ such that $R(m_i) = m'_i$ for $i = 1, \dots, k$, and $R(x_i) = x'_i$ for $i = 1, \dots, \ell$.*

(Here the Jordan algebra $M_n^+(K)$, respectively the Lie algebra $\mathfrak{so}_n(K)$, acts via Jordan multiplications, respectively commutators, on its ambient algebra $M_n(K)$.)

Proof. (i) As follows from §1.3, $M_n(K)$ is decomposed, as an $M_n^+(K)$-module, into the direct sum of two irreducible non isomorphic Jordan modules: $M_n(K) = M_n^+(K) \oplus M_n^-(K)$. Apply Proposition 6 to $R = \mathfrak{M}(M_n^+(K), M_n(K))$, and $M_1 = M_n^+(K)$, $M_2 = M_n^-(K)$.

(ii) The statement is vacuous for $n = 1$, and easily verified directly for $n = 2$, so assume $n \geq 3$. As follows from §1.3, $M_n(K)$ is decomposed, as an $\mathfrak{so}_n(K)$-module, into the direct sum of three non-isomorphic modules:

$$M_n(K) = KE \oplus SM_n(K) \oplus M_n^-(K).$$

Apply Proposition 6 to

$$R = \mathfrak{M}\big(\mathfrak{so}_n(K), M_n(K)\big) = \mathfrak{M}\big(\mathfrak{so}_n(K), SM_n(K) \oplus M_n^-(K)\big),$$

and $M_1 = SM_n(K)$, $M_2 = M_n^-(K)$. □

Note that the restriction $n \neq 4$ in Lemma 7(ii) is essential. As noted in §1.3, the adjoint module of $\mathfrak{so}_4(K)$ decomposes into the direct sum of two irreducible isomorphic modules, so Proposition 6 is not applicable as is. It is possible to devise more sophisticated versions of Proposition 6 and Lemma 7 which are trying to take account of this, but we found it easier to treat the case $n = 4$ below in a different way, avoiding more sophisticated versions of the Jacobson density theorem.

Proof of Theorem 5. As a form of a simple algebra is simple, it is enough to prove the theorem when the ground field K is algebraically closed. In this case, due to isomorphism (6), we may assume $\mathbb{O}_\mu(K) = \mathbb{O}(K)$.

Case of $\mathrm{S}^+(M_n(\mathbb{O}(K)), J)$. Let I be an ideal of $\mathrm{S}^+(M_n(\mathbb{O}(K)), J)$. We argue in terms of the decomposition (7). Assume first that $I \subseteq M_n^-(K) \otimes \mathbb{O}^-(K)$. Consider an element

$$\sum_{i=1}^{7} x_i \otimes e_i \in I,$$

where $x_i \in M_n^-(K)$, and $e_1,\dots,e_7$ are elements of the standard basis of $\mathbb{O}(K)$, as described in §1.4. For any $y \in M_n^-(K)$, and any $k = 1,\dots,7$, we have

$$(y \otimes e_k) \circ (\sum_{i=1}^{7} x_i \otimes e_i) = -(x_k \circ y) \otimes 1 + \text{ terms lying in } M_n^-(K) \otimes \mathbb{O}^-(K).$$

Hence, $x_k \circ y = 0$ for any $y \in M_n^-(K)$, and by Lemma 1, $x_k = 0$. This shows that $I = 0$, and we may assume $I \not\subseteq M_n^-(K) \otimes \mathbb{O}^-(K)$.

Now take an element

$$m \otimes 1 + \sum_{i \in \mathbb{I}} x_i \otimes a_i \in I,$$

where $m \in M_n^+(K)$, $m \neq 0$, $x_i \in M_n^-(K)$, $i \in \mathbb{I}$ are linearly independent, and $a_i \in \mathbb{O}^-(K)$. By Lemma 7(i), for any $m' \in M_n^+(K)$ there is a linear map $R : M_n(K) \to M_n(K)$, represented as the sum of products of the form $R_{s_1} \dots R_{s_\ell}$, where each s_i belongs to $M_n^+(K)$, and R_s is the Jordan multiplication by the element s, such that $R(m) = m'$ and $R(x_i) = 0$ for any $i = 1,\dots,7$. We form the corresponding map $\widetilde{R}$ from the multiplication algebra of $\mathrm{S}^+(M_n(\mathbb{O}(K)),J)$ by replacing each R_{s_i} by $R_{s_i \otimes 1}$. Then $\widetilde{R}(m \otimes 1) = m' \otimes 1$ and $\widetilde{R}(x_i \otimes a_i) = 0$. Consequently, $m' \otimes 1 \in I$, and I contains $M_n^+(K) \otimes 1$. This, in its turn, implies

$$M_n^-(K) \otimes \mathbb{O}^-(K) = (M_n^+(K) \otimes 1) \circ (M_n^-(K) \otimes \mathbb{O}^-(K)) \subseteq I,$$

and hence I coincides with the whole algebra $\mathrm{S}^+(M_n(\mathbb{O}(K)),J)$.

Case of $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$. The proof goes largely along the same route as in the previous case, but with some complications and modifications, notably in the case $n = 4$. If $n = 1$, the algebra in question is isomorphic to the 7-dimensional Malcev algebra $\mathbb{O}^-(K)$, whose simplicity is well known (and can be established by an easy modification of some of the reasonings below), so assume $n \geq 2$.

Let I be an ideal of $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$. Assume first $I \subseteq M_n^+(K) \otimes \mathbb{O}^-(K)$. Consider an element

$$\sum_{i=1}^{7} m_i \otimes e_i \in I,$$

where $m_i \in M_n^+(K)$. For any $s \in M_n^+(K)$, and any $k = 1,\dots,7$, we have

$$[s \otimes e_k, \sum_{i=1}^{7} m_i \otimes e_i] = [m_k, s] \otimes 1 + \text{terms lying in } M_n^+(K) \otimes \mathbb{O}^-(K).$$

Hence, $[m_k, s] = 0$ for any $s \in M_n^+(K)$, and by Lemma 2, $m_k = \lambda_k E$ for some $\lambda_k \in K$. Therefore, any element of I is of the form $\sum_{i=1}^{7} \lambda_i E \otimes e_k \in E \otimes \mathbb{O}^-(K)$, and $I = E \otimes S$ for some subspace $S \subseteq \mathbb{O}^-(K)$. But then

$$[M_n^+(K) \otimes \mathbb{O}^-(K), E \otimes S] = M_n^+(K) \otimes [\mathbb{O}^-(K), S] \subseteq E \otimes S,$$

this can happen only if $[\mathbb{O}^-(K), S] = 0$, hence $S = 0$ and $I = 0$. Therefore, we may assume $I \not\subseteq M_n^+(K) \otimes \mathbb{O}^-(K)$.

Consider an element

$$x \otimes 1 + \sum_{i \in \mathbb{I}} m_i \otimes a_i \in I, \tag{10}$$

where $x \in M_n^-(K)$ is non-zero, $m_i \in M_n^+(K)$ for $i \in \mathbb{I}$ are linearly independent, and $a_i \in \mathbb{O}^-(K)$ are non-zero. Taking the commutator of this element with an element $y \otimes 1$, where $y \in M_n^-(K)$ is such that $[x,y] \neq 0$, we may assume that $m_i \in SM_n(K)$.

Assume $n \neq 4$. By Lemma 7(ii), for any $x' \in M_n^-(K)$ there is a linear map $R : M_n(K) \to M_n(K)$ of the form

$$R = \lambda \,\mathrm{id} + R', \tag{11}$$

where $\lambda \in K$, and R' is the sum of products of the form $\mathrm{ad}\, y_1 \dots \mathrm{ad}\, y_\ell$, where each y_i belongs to $M_n^-(K)$, and $\mathrm{ad}\, y$ denotes the commutator with y, such that $R(x) = x'$, and $R(m_i) = 0$ for each $i = 1,\dots,7$. (Note that the term $\lambda\,\mathrm{id}$ in (11) occurs from the necessity to adjoin the unit to the multiplication algebra generated by commutators with elements of $\mathfrak{so}_n(K)$; this term does not occur in the previous case, where the multiplication algebra was formed by Jordan multiplications by elements of $M_n^+(K)$, as the latter already contains the unit: the Jordan product with the unit matrix.)

We have $R'(x) = x' - \lambda x$, and $R'(m_i) = -\lambda m_i$. Replacing in R' each $\mathsf{ad}\, y_i$ by $\mathsf{ad}(y_i \otimes 1)$, we get the map $\widetilde{R}$ in the multiplication algebra of $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$ such that $\widetilde{R}(x\otimes 1) = (x' - \lambda x)\otimes 1$ and $\widetilde{R}(m_i \otimes a_i) = -\lambda m_i \otimes a_i$, and thus

$$\widetilde{R}\,(x\otimes 1 + \sum_{i=1}^{7} m_i \otimes a_i) = (x' - \lambda x)\otimes 1 - \lambda \sum_{i=1}^{7} m_i \otimes a_i \in I.$$

Adding to this element the element (10) multiplied by λ, we get $x'\otimes 1 \in I$ for any $x' \in M_n^-(K)$, i.e., I contains $M_n^-(K)\otimes 1$. Hence,

$$SM_n(K)\otimes \mathbb{O}^-(K) = [M_n^-(K)\otimes 1, M_n^+(K)\otimes \mathbb{O}^-(K)] \subseteq I.$$

The formula (9), in its turn, implies

$$[m\otimes e_i, s\otimes e_j] = \pm 2(m\circ s)\otimes e_{i*j},$$

for any $m,s \in SM_n(K)$, and $i,j = 1,\dots,7$. Since $SM_n(K)\circ SM_n(K) = M_n^+(K)$, and $i*j$ runs through all the range $1,\dots,7$, we conclude that I contains $M_n^+(K)\otimes \mathbb{O}^-(K)$, and hence coincides with the whole algebra $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$.

Now consider the case $n = 4$. Consider an element of I of the form (10), where $m_i \in SM_4(K)$ for any $i \in \mathbb{I}$. By the (classical) Jacobson density theorem for the case of an irreducible module (or, equivalently, by Lemma 7(ii) in the case $n = 4$ where the "$M_n^-(K)$ part" is ignored), for any $m \in SM_4(K)$, and any $k \in \mathbb{I}$, there is a map of the form (11), where R' is formed by the commutators with elements of $M_4^-(K)$, such that $R(m_k) = m$, and $R(m_i) = 0$, $i \ne k$. Deriving from this the map $\widetilde{R}$ in the multiplication algebra of $\mathrm{S}^-(M_4(\mathbb{O}(K)),J)$ as above, we get:

$$\widetilde{R}(M_4^-(K)\otimes 1) \subseteq M_4^-(K)\otimes 1$$
$$\widetilde{R}(m_k\otimes a_k) = (m - \lambda m_k)\otimes a_k$$
$$\widetilde{R}(m_i\otimes a_i) = -\lambda m_i \otimes a_i\,,\, i \ne k.$$

Consequently, $\widetilde{R}$, being applied to the element (10), produces the element

$$x'\otimes 1 + (m - \lambda m_k)\otimes a_k - \lambda \sum_{i\in \mathbb{I}\setminus\{k\}} m_i \otimes a_i \in I,$$

where $x' \in M_4^-(K)$. Adding to this element the element (10) multiplied by λ, we get the element

$$x''\otimes 1 + m\otimes a_k \in I,$$

where $x'' \in M_4^-(K)$.

To summarize: for any $a \in \mathbb{O}^-(K)$ which appears as one of a_i's in the decomposition (10) of some nonzero element of I, and any $m \in SM_4(K)$, there is an element $x \in M_4^-(K)$ such that $x\otimes 1 + m\otimes a \in I$. Fixing here a and varying m, we also vary x, but since

$$\dim SM_4(K) = 9 > \dim M_4^-(K) = 6,$$

we will get nonzero elements with vanishing x, i.e., of the form $m\otimes a$. Now taking commutators of such an element with elements from $M_4^-(K)\otimes 1$, we get the whole $SM_4(K)\otimes a \subseteq I$.

This means that the ideal I is homogeneous with respect to the decomposition (8), i.e., is of the form

$$I = T\otimes 1 \dotplus S_1\otimes e_1 \dotplus \dots \dotplus S_7 \otimes e_7,$$

where T is a nonzero linear subspace of $M_4^-(K)$, and each of the linear subspaces $S_i \subseteq M_4^+(K)$ is either zero, or contains $SM_4(K)$. Taking commutators of elements from $T\otimes 1$ with elements from $M_4^+(K)\otimes e_i$, we see that each S_i is nonzero. Now,

$$[SM_4^+(K)\otimes e_i, SM_4^+(K)\otimes e_i] = [SM_4^+(K), SM_4^+(K)]\otimes 1 = M_4^-(K)\otimes 1;$$

thus, $T = M_4^-(K)$. Finally, according to (9), for any $i \neq j$, we have

$$[SM_4(K) \otimes e_i, SM_4(K) \otimes e_j] = (SM_4(K) \circ SM_4(K)) \otimes e_{i*j} = M_4^+(K) \otimes e_{i*j};$$

thus, $S_i = M_4^+(K)$ for each i, and I coincides with the whole algebra $\mathrm{S}^-(M_4(\mathbb{O}(K)), J)$. □

3. δ-DERIVATIONS

In [P], derivations of the algebras $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)), J)$ and $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)), J)$ were computed. Here we extend this result by computing δ-derivations of these algebras. Recall that a *δ-derivation* of an algebra A is a linear map $D : A \to A$ such that

$$D(xy) = \delta D(x)y + \delta x D(y) \tag{12}$$

for any $x, y \in A$ and some fixed $\delta \in K$. This notion generalizes simultaneously the notions of derivation and of centroid (any element of the centroid is, obviously, a $\frac{1}{2}$-derivation).

The set of δ-derivations of an algebra A, denoted by $\mathrm{Der}_\delta(A)$, is a vector space. Moreover, as noted, for example, in [F2, §1],

$$[\mathrm{Der}_\delta(A), \mathrm{Der}_{\delta'}(A)] \subseteq \mathrm{Der}_{\delta\delta'}(A),$$

so the vector space $\Delta(A)$ linearly spanned by all δ-derivations, for all possible values of δ, is a Lie algebra, an extension of the Lie algebra $\mathrm{Der}(A)$ of (the ordinary) derivations of A.

Theorem 8. *Let D be a nonzero δ-derivation of the algebra $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)), J)$ or $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)), J)$. Then either $\delta = 1$ (i.e., D is a derivation), or $\delta = \frac{1}{2}$ and D is a multiple of the identity map.*

Note that δ-derivations do not change under field extensions. Namely, an obvious argument, the same as in the case of ordinary derivations, cocycles, or any other "linear" structures, shows that

$$\mathrm{Der}_\delta(A) \otimes_K \overline{K} \simeq \mathrm{Der}_\delta(A \otimes_K \overline{K})$$

for any K-algebra A, and $\delta \in K$. In view of this, it is enough to prove the theorem in the case when K is algebraically closed, and $\mathbb{O}_\mu(K) = \mathbb{O}(K)$.

The case of $\mathrm{S}^+(M_n(\mathbb{O}(K)), J)$ is easier, as the algebra contains a unit, and δ-derivations of algebras with unit are tackled by the simple

Lemma 9. *Let D be a δ-derivation of a commutative algebra A with unit. Then either $\delta = 1$ (i.e., D is a derivation), or $\delta = \frac{1}{2}$ and $D = R_a$ for some $a \in A$ such that*

$$2(xy)a - (xa)y - (ya)x = 0 \tag{13}$$

for any pair of elements $x, y \in A$.

Proof. This is, essentially, [Kay, Theorem 2.1] with a bit more (trivial) details. Repeatedly substituting the unit 1 in the equality (12) gives that either $\delta = 1$ and $D(1) = 0$, or $\delta = \frac{1}{2}$ and $D(x) = xD(1)$ for any $x \in A$. In the latter case, denoting $D(1) = a$, the condition (12) is equivalent to (13). □

Proof of Theorem 8 in the case of $\mathrm{S}^+(M_n(\mathbb{O}(K)), J)$. Due to Lemma 9, it amounts to description of the algebra elements satisfying the condition (13). Let

$$a = m \otimes 1 + \sum_{i=1}^{7} x_i \otimes e_i$$

be such an element, where $m \in M_n^+(K)$, $x_i \in M_n^-(K)$. Writing the condition (13) for the pair of elements $s \otimes 1$, $t \otimes 1$ where $s, t \in M_n^+(K)$, and collecting terms lying in $M_n^+(K) \otimes 1$, we get

$$2(s \circ t) \circ m - (s \circ m) \circ t - (t \circ m) \circ s = 0$$

for any $s, t \in M_n^+(K)$. The latter condition means that R_m is a $\frac{1}{2}$-derivation of the Jordan algebra $M_n^+(K)$, and by [Kay, Theorem 2.5], $m = \lambda E$ for some $\lambda \in K$. Since the set of elements satisfying the condition (13) forms a vector space (as, generally, the set of $\frac{1}{2}$-derivations does), by subtracting from a the element $\lambda E \otimes 1$, we get an element still satisfying the condition (13), so we may assume $\lambda = 0$.

Now writing the condition (13) for $a = \sum_{i=1}^{7} x_i \otimes e_i$, and the pair $x \otimes e_k$, $y \otimes e_\ell$, where $x, y \in M_n^-(K)$ and $k, \ell = 1, \dots, 7$, $k \neq \ell$, and again collecting terms lying in $M_n^+(K) \otimes 1$, we get $[x,y] \circ x_{k*\ell} = 0$. Since $[M_n^-(K), M_n^-(K)] = M_n^-(K)$, and the values of $k*\ell$ run over all $1, \dots, 7$, we see that $M_n^-(K) \circ x_i = 0$ for any $i = 1, \dots, 7$. By Lemma 1, $x_i = 0$, which shows that any element $a \in \mathrm{S}^+(M_n(\mathbb{O}(K), J))$ satisfying (13), is a multiple of the unit. □

Before turning to the proof of the $\mathrm{S}^-(M_n(\mathbb{O}(K)), J)$ case, we need a couple of auxiliary lemmas.

Lemma 10. *Let $n > 2$.*

(i) *If $\delta \neq 1, \frac{1}{2}$, then the vector space $\mathrm{Der}_\delta(\mathfrak{gl}_n(K))$ is 1-dimensional, and each δ-derivation is a multiple of the map ξ vanishing on $\mathfrak{sl}_n(K)$, and sending E to itself.*

(ii) *The vector space $\mathrm{Der}_{\frac{1}{2}}(\mathfrak{gl}_n(K))$ is 2-dimensional, with a basis consisting of the two maps: the map ξ from part* (i)*, and the map coinciding with the identity map on $\mathfrak{sl}_n(K)$, and vanishing on E.*

Proof. This follows immediately from the fact that $\mathfrak{gl}_n(K)$ is the split central extension of $\mathfrak{sl}_n(K)$: $\mathfrak{gl}_n(K) = \mathfrak{sl}_n(K) \oplus KE$, and the fact, established in numerous places, that each nonzero δ-derivation of $\mathfrak{sl}_n(K)$, $n > 2$, is either an ordinary derivation ($\delta = 1$), or an element of the centroid ($\delta = \frac{1}{2}$) (see, for example, [LL, Corollary 4.16] or [F2]). □

Lemma 11. *Let $D : M_n^+(K) \to M_n^+(K)$ be a linear map such that*

$$D([x,m]) = \delta[x, D(m)] \tag{14}$$

for any $x \in M_n^-(K)$, $m \in M_n^+(K)$, and some fixed $\delta \in K$, $\delta \neq 0, 1$. Then the image of D lies in the one-dimensional linear space spanned by E.

Proof. Replacing in the equality (14) x by $[x,y]$, where $x, y \in M_n^-(K)$, and using the Jacobi identity, we get:

$$D([x,[y,m]]) - D([y,[x,m]]) = \delta[[x,y], D(m)].$$

Using the fact that $[x,m], [y,m] \in M_n^+(K)$, applying again (14) to each term at the left-hand side twice, and using the Jacobi identity, we get $[[x,y], D(m)] = 0$. Since $[M_n^-(K), M_n^-(K)] = M_n^-(K)$, the latter equality is equivalent to $[M_n^-(K), D(m)] = 0$. By Lemma 2, $D(m)$ is a multiple of E for any $m \in M_n^+(K)$. □

When considering restrictions of δ-derivations to subalgebras, we arrive naturally at the necessity to consider a more general notion of δ-derivations with values in not necessary the algebra itself, but in a module over the algebra. Generally, this require to consider bimodules, but as we will need this generalization only in the case of anticommutative (in fact, Lie) algebras, we confine ourselves here with the following definition. Let A be an anticommutative algebra, and M a left A-module, with the action of A on M denoted by $\bullet$. A δ-derivation of A with values in M is a linear map $D : A \to M$ such that

$$D(xy) = -\delta y \bullet D(x) + \delta x \bullet D(y)$$

for any $x, y \in A$.

Proof of Theorem 8 in the case of $\mathrm{S}^-(M_n(\mathbb{O}(K)), J)$. If $n = 1$, the algebra in question is the 7-dimensional simple Malcev algebra $\mathbb{O}^-(K)$, and the result is covered by [F3, Lemma 3].

Let $n > 2$ and $\delta \neq 1$. We may write

$$D(x \otimes 1) = d(x) \otimes 1 + \sum_{i=1}^{7} d_i(x) \otimes e_i$$

$$D(m \otimes e_k) = f_k(m) \otimes 1 + \sum_{i=1}^{7} f_{ki}(m) \otimes e_i$$

for any $x \in M_n^-(K)$, $m \in M_n^+(K)$, $k = 1, \dots, 7$, and some linear maps $d : M_n^-(K) \to M_n^-(K)$, $d_i : M_n^-(K) \to M_n^+(K)$, $f_k : M_n^+(K) \to M_n^-(K)$, and $f_{ki} : M_n^+(K) \to M_n^+(K)$.

For a fixed $k = 1,\dots,7$, consider the Lie subalgebra

$$\mathscr{L}^-(e_k) = M_n^-(K) \otimes 1 \dotplus M_n^+(K) \otimes e_k$$

of $\mathrm{S}^-(M_n(\mathbb{O}),J)$, isomorphic, as noted in §2, to $\mathfrak{gl}_n(K)$ (remember that K is algebraically, and, in particular, quadratically, closed). According to decomposition (8), $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$ is decomposed, as an $\mathscr{L}^-(e_k)$-module, into the direct sum of the adjoint module $\mathscr{L}^-(e_k)$, and the module $M_n^+(K) \otimes B_k$ (note, however, that the latter is not a Lie module). This implies that the restriction of D to $\mathscr{L}^-(e_k)$, being composed with the canonical projection $\mathrm{S}^-(M_n(\mathbb{O}(K)),J) \to \mathscr{L}^-(e_k)$, i.e., the map

$$\begin{aligned} x \otimes 1 &\mapsto d(x) \otimes 1 \quad + d_k(x) \otimes e_k \\ m \otimes e_k &\mapsto f_k(m) \otimes 1 + f_{kk}(m) \otimes e_k, \end{aligned}$$

is a δ-derivation of $\mathscr{L}^-(e_k)$ (with values in the adjoint module).

By Lemma 10, either $\delta \neq \frac{1}{2}$, and each such map is of the form

$$\begin{aligned} x \otimes 1 &\mapsto 0 \\ m \otimes e_k &\mapsto 0, \quad m \in SM_n(K) \\ E \otimes e_k &\mapsto \mu_k E \otimes e_k \end{aligned}$$

for some $\mu_k \in K$; or $\delta = \frac{1}{2}$, and each such map is of the form

$$\begin{aligned} x \otimes 1 &\mapsto \lambda_k x \otimes 1 \\ m \otimes e_k &\mapsto \lambda_k m \otimes e_k, \quad m \in SM_n(K) \\ E \otimes e_k &\mapsto \mu_k E \otimes e_k \end{aligned}$$

for some $\lambda_k, \mu_k \in K$. (Recall from §1.3, that $SM_n(K)$ denotes the space of matrices from $M_n^+(K)$ with trace zero.) Taking into account that one of these alternatives holds uniformly for all values of k, we arrive at the following two cases:

Case 1. $\delta \neq 1, \frac{1}{2}$, and $D(M_n^-(K) \otimes 1) = 0$.

Case 2. $\delta = \frac{1}{2}$, and $D(x \otimes 1) = \lambda x \otimes 1$ for any $x \in M_n^-(K)$ and some fixed $\lambda \in K$.

Moreover, in both cases

$$D\big(M_n^+(K) \otimes \mathbb{O}^-(K)\big) \subseteq M_n^+(K) \otimes \mathbb{O}^-(K).$$

We will handle these two cases together, keeping in mind that $\lambda = 0$ if $\delta \neq \frac{1}{2}$.

Consider now the restriction of D to $M_n^+(K) \otimes \mathbb{O}^-(K)$. Since

$$\mathrm{Hom}\,\big(M_n^+(K) \otimes \mathbb{O}^-(K), M_n^+(K) \otimes \mathbb{O}^-(K)\big) \simeq \mathrm{Hom}\,\big(M_n^+(K), M_n^+(K)\big) \otimes \mathrm{Hom}\,\big(\mathbb{O}^-(K), \mathbb{O}^-(K)\big),$$

we may write

$$D(m \otimes a) = \sum_{i \in \mathbb{I}} d_i(m) \otimes \alpha_i(a)$$

for any $m \in M_n^+(K)$, $a \in \mathbb{O}^-(K)$, some index set $\mathbb{I}$, and linear maps $d_i : M_n^+(K) \to M_n^+(K)$, $\alpha_i : \mathbb{O}^-(K) \to \mathbb{O}^-(K)$, $i \in \mathbb{I}$. Writing the condition of δ-derivation (12) for the pair $x \otimes 1$, $m \otimes a$, where $x \in M_n^-(K)$, $m \in M_n^+(K)$, $a \in \mathbb{O}^-(K)$, we get

$$\sum_{i \in \mathbb{I}} \Big(d_i([x,m]) - \delta[x, d_i(m)]\Big) \otimes \alpha_i(a) = \delta\lambda\,[x,m] \otimes a. \tag{15}$$

In Case 1 the right-hand side of (15) vanishes, and hence we may assume $d_i([x,m]) = \delta[x, d_i(m)]$ for any $x \in M_n^-(K)$, $m \in M_n^+(K)$, and any $i \in \mathbb{I}$. By Lemma 11, each $d_i(m)$ is a multiple of E, and hence $D(M_n^+(K) \otimes \mathbb{O}^-(K)) \subseteq E \otimes \mathbb{O}^-(K)$. But then writing (12) for the pair $m \otimes a$, $s \otimes b$, where $m, s \in M_n^+(K)$, $a, b \in \mathbb{O}^-(K)$, and taking into account (9), we get $D((m \circ s) \otimes [a,b]) = 0$. Since $(M_n(K), \circ)$ and $(\mathbb{O}^-(K), [\cdot,\cdot])$ are perfect (in fact, simple) algebras, the latter equality implies vanishing of D on the whole $M_n^+(K) \otimes \mathbb{O}^-(K)$, and thus on the whole $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$, a contradiction.

Hence, we are in Case 2, and $\delta = \frac{1}{2}$. Setting in this case $d_\star = -\lambda\,\mathrm{id}_{M_n^+(K)}$, and $\alpha_\star = \mathrm{id}_{\mathbb{O}^-(K)}$, the equality (15) can be rewritten as

$$\sum_{i\in\mathbb{I}\cup\{\star\}} \Big(d_i([x,m]) - \frac{1}{2}[x,d_i(m)]\Big)\otimes\alpha_i(a) = 0.$$

As in the previous case, this means that there are new linear maps $\widetilde{d_i}$, $\widetilde{\alpha}_i$ which are linear combinations of d_i and α_i, respectively, and such that

$$\sum_{i\in\mathbb{I}\cup\{\star\}} \widetilde{d_i}\otimes\widetilde{\alpha}_i = \sum_{i\in\mathbb{I}\cup\{\star\}} d_i\otimes\alpha_i, \tag{16}$$

and $\widetilde{d_i}([x,m]) = \frac{1}{2}[x,\widetilde{d_i}(m)]$. Lemma 11 tells us, as previously, that each $\widetilde{d_i}(m)$ is a multiple of E, and hence the image of the map in the left-hand side of (16) lies in $E\otimes\mathbb{O}^-(K)$. Since the right-hand side of (16) is equal to $D + d_\star\otimes\alpha_\star$, we have

$$D(m\otimes a) = \lambda m\otimes a + E\otimes\beta(m,a)$$

for any $m\in M_n^+(K)$, $a\in\mathbb{O}^-(K)$, and some bilinear map $\beta : M_n^+(K)\times\mathbb{O}^-(K)\to\mathbb{O}^-(K)$. Replacing D by the $\frac{1}{2}$-derivation $D-\lambda\,\mathrm{id}$, we arrive at the situation as in the previous case: a δ-derivation (with $\delta=\frac{1}{2}$) vanishing on $M_n^-(K)\otimes 1$, and taking values in $E\otimes\mathbb{O}^-(K)$ on $M_n^+(K)\otimes\mathbb{O}^-(K)$. Hence, $D-\lambda\,\mathrm{id}$ vanishes on the whole $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$, and $D=\lambda\,\mathrm{id}$, as claimed.

Finally, consider the case $n=2$. In this case Lemma 10 is not applicable: in addition to the cases described in Lemma, there is the 5-dimensional space of (-1)-derivations of $\mathfrak{sl}_2(K)$, and thus the corresponding 6-dimensional space of (-1)-derivations of $\mathfrak{gl}_2(K)$ (see [H, Example 1.5] or [F1, Example in §3]). In view of this, to proceed like in the proof of the case $n>2$, considering δ-derivations of the Lie subalgebras $\mathscr{L}^-(e_k)$, would be too cumbersome, and we are taking a somewhat alternative route.

Denote by $H = \begin{pmatrix} 0 & 1\\ -1 & 0\end{pmatrix}$ the basis element of the 1-dimensional space $M_2^-(K)$. Consider the subalgebra $E\otimes\mathbb{O}^-(K)$ of $\mathrm{S}^+(M_2(\mathbb{O}(K)),J)$, isomorphic to the 7-dimensional simple Malcev algebra $\mathbb{O}^-(K)$. As an $E\otimes\mathbb{O}^-(K)$-module, $\mathrm{S}^+(M_2(\mathbb{O}(K)),J)$ decomposes into the direct sum of the trivial 1-dimensional module $KH\otimes 1$, and the module $M_2^+(K)\otimes\mathbb{O}^-(K)$ which is isomorphic to the direct sum of 3 copies of the adjoint module ($\mathbb{O}^-(K)$ acting on itself). Thus D, being restricted to $E\otimes\mathbb{O}^-(K)$, is equal to the sum of a δ-derivation with values in the trivial module, which is obviously zero, and 3 δ-derivations of $\mathbb{O}^-(K)$. By the result mentioned at the beginning of this proof, the latter δ-derivations are zero if $\delta\neq 1,\frac{1}{2}$, and are multiples of the identity map if $\delta=\frac{1}{2}$. Consequently, $D(E\otimes a) = m_0\otimes a$ for any $a\in\mathbb{O}^-(K)$, and some fixed $m_0\in M_2^+(K)$.

Now write

$$D(H\otimes 1) = \lambda H\otimes 1 + \sum_{i=1}^{7} m_i\otimes e_i$$

for some $\lambda\in K$, and $m_i\in M_2^+(K)$. Writing the condition of δ-derivation (12) for the pair $H\otimes 1$, $E\otimes e_k$, where $k=1,\dots,7$, we get

$$2\sum_{1\le i\le 7, i\neq k} (\pm m_i\otimes e_{i*k}) + [H,m_0]\otimes e_k = 0.$$

It follows that $m_i=0$ for each $i=1,\dots,7$, and $D(H\otimes 1)=\lambda H\otimes 1$.

Now let

$$D(m\otimes a) = \beta(m,a)H\otimes 1 + \text{terms lying in } M_2^+(K)\otimes\mathbb{O}^-(K)$$

for any $m\in M_2^+(K)$, $a\in\mathbb{O}^-(K)$, and some bilinear map $\beta : M_2^+(K)\otimes\mathbb{O}^-(K)\to K$. Writing the condition of δ-derivation for the pair $H\otimes 1$, $m\otimes a$, and collecting terms which are multiples of $H\otimes 1$, we see that $\beta(m,a)H\otimes 1 = 0$. Thus,

$$D\big(M_2^+(K)\otimes\mathbb{O}^-(K)\big)\subseteq M_2^+(K)\otimes\mathbb{O}^-(K),$$

and we may proceed as in the generic case $n>2$ above. □

Note that it is also possible to pursue the case $\delta = 1$ along the same lines; this would give us an alternative proof of the results of [P], as well as of the classical result that derivation algebra of the 27-dimensional exceptional simple Jordan algebra is isomorphic to the simple Lie algebra of type F_4.

There is a vast literature devoted to δ-derivations of algebras and related notions (for a small, but representative sample, see [H], [F1]–[F3], [Kay], [LL]). Our strategy to prove Theorem 8 was to identify certain Lie subalgebras of the algebra $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$, and consider δ-derivations of those subalgebras with values in the whole $\mathrm{S}^-(M_n(\mathbb{O}(K)),J)$. Developing further the methods of the above-cited papers, it is possible to prove that δ-derivations of semisimple Lie algebras of classical type with coefficients in finite-dimensional modules are either (inner) derivations, or multiples of the identity map on irreducible constituents of the module isomorphic to the adjoint module of the algebra, or, in the case of the direct summands in the algebra isomorphic to $\mathfrak{sl}_2(K)$, (-1)-derivations with values in the irreducible constituents isomorphic to the adjoint $\mathfrak{sl}_2(K)$-modules. This general fact would allow us to further simplify the proof of Theorem 8, but establishing it would require considerable (though pretty much straightforward) efforts, and would lead us far away from the topic of this paper. We hope to return to this elsewhere.

Since by [P], both $\mathrm{Der}(\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J))$ for $n \geq 4$, and $\mathrm{Der}(\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J))$ for any n are isomorphic to the Lie algebra $G_2 \oplus \mathfrak{so}_n(K)$, then by Theorem 8, both $\Delta(\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J))$ and $\Delta(\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J))$ are isomorphic to the one-dimensional trivial central extension of $G_2 \oplus \mathfrak{so}_n(K)$.

Finally, note an important

Corollary. *The algebras* $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ *and* $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$ *are central simple.*

Proof. By Theorem 5, these algebras are simple, and by Theorem 8 their centroid coincides with the ground field. □

4. Symmetric associative forms

Let A be an algebra. A bilinear symmetric form $\varphi : A \times A \to K$ is called *associative*, if

$$\varphi(xy,z) = \varphi(x,yz) \tag{17}$$

for any $x,y,z \in A$. (In the context of Lie algebras, associative forms are usually called *invariant*, because in that case the condition (17) is equivalent to invariance of the form φ with respect to the standard action of the underlying Lie algebra on the space of symmetric bilinear forms.)

For a matrix $X = (a_{ij})$ from $M_n(\mathbb{O}_\mu(K))$, by $\overline{X}$ we will understand the matrix $(\overline{a_{ij}})$, obtained by element-wise application of conjugation in $\mathbb{O}_\mu(K)$.

Theorem 12. *Any bilinear symmetric associative form on* $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$*, or on* $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$*, is a scalar multiple of the form*

$$(X,Y) \mapsto \mathrm{Tr}(XY + \overline{X}\,\overline{Y}). \tag{18}$$

The form (18) is reminiscent of the Killing form on simple Lie algebras of classical type, and of the generic trace form on simple Jordan algebras (and *is* such a form when restricted from the algebra $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ to its Jordan subalgebra $M_n(K)$, and from the algebra $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$ to its Lie subalgebra $\mathfrak{so}_n(K)$, see below).

Proof. According to Corollary in §3, both algebras are central simple. The standard linear algebra arguments show that any bilinear symmetric associative form on a simple algebra is nondegenerate, and that any two nondegenerate symmetric associative forms on a finite-dimensional central algebra differ from each other by a scalar (see, e.g., [Kap, pp. 30–31, Exercise 15(b)]). Thus, the vector space of bilinear symmetric associative forms on a finite-dimensional central simple algebra is either 0- or 1-dimensional.

Now it remains to observe that in both cases this space is 1-dimensional by verifying that the form (18) is indeed associative. The most convenient way to do this is, perhaps, to rewrite the form in terms of decompositions (7) or (8). On the algebra $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ we obtain

$$\begin{aligned}(m\otimes 1, s\otimes 1) &\mapsto 2\,\mathrm{Tr}(ms)\\ (m\otimes 1, x\otimes a) &\mapsto 0\\ (x\otimes a, y\otimes b) &\mapsto (ab+ba)\,\mathrm{Tr}(xy),\end{aligned}$$

and on $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$,

$$\begin{aligned}(x\otimes 1, y\otimes 1) &\mapsto 2\,\mathrm{Tr}(xy)\\ (x\otimes 1, m\otimes a) &\mapsto 0\\ (m\otimes a, s\otimes b) &\mapsto (ab+ba)\,\mathrm{Tr}(ms).\end{aligned}$$

Here, as before in this paper, $x,y \in M_n^-(K)$, $m,s \in M_n^+(K)$, and $a,b \in \mathbb{O}_\mu^-(K)$. (For the algebra $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$, the associativity follows also from [R, Satz 5.2], where it is proved that the form (18) is a symmetric associative form on a larger algebra $(M_n(\mathbb{O}_\mu(K)),\circ)$.) □

Note that it is possible to get an alternative, direct proof of Theorem 12 without appealing to results of §3, in the linear algebra spirit of the proofs of Proposition 4 and Theorem 8.

5. FURTHER QUESTIONS

1) Compute automorphism groups of the algebras $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ and $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$. Are they isomorphic to $G_2 \times SO(n)$?

2) For $n > 3$, the algebras $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ are no longer Jordan. How "far" they are from Jordan algebras? Which identities these algebras do satisfy? (The last question was also asked in [BH], where it is proved that $\mathrm{S}^+(M_4(\mathbb{O}(\mathbb{Q})),J)$ does not satisfy nontrivial identities of degree ≤ 6.) A starting point could be investigation of (non-Jordan) representations of the Jordan subalgebras which are forms of the full matrix Jordan algebra $M_n(K)$, mentioned in § 2, in the whole $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$.

3) What can one say about subalgebras of the algebras in question? Say, what are the maximal subalgebras? Maximal Jordan subalgebras of $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$? Some low-dimensional subalgebras of $\mathrm{S}^+(M_4(\mathbb{O}(\mathbb{R})),J)$ were exhibited in [Jo, pp. 34–37] (see also [LRH, p. 37]). These subalgebras belong to the class of so-called elementary algebras, defined by a certain identity of degree 5. In that old and seemingly forgotten paper, Jordan suggested to investigate which other elementary subalgebras the octonionic matrix algebras may contain.

4) Idempotents play an important role in Jordan algebras. Find idempotents in $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$. This amounts to solving a system of quadratic equations in the Lie algebra $\mathfrak{so}_n(K)$.

5) In [Sa] it is proved that any anticommutative algebra with a bilinear symmetric associative form is isomorphic to a "minus" algebra $A^{(-)}$ of a noncommutative Jordan algebra A. In view of Theorem 12, which noncommutative Jordan algebras arise in this way in connection with the algebras $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$?

6) Investigate the case of characteristic 3. Though this case is, perhaps, of little interest for physics, in characteristic 3 the 7-dimensional algebra $\mathbb{O}_\mu^-(K)$ is not merely a Malcev algebra, but isomorphic to a form of the Lie algebra $\mathfrak{psl}_3(K)$ (see, for example, [EK, Theorem 4.26]). This suggests that the algebras $\mathrm{S}^+(M_n(\mathbb{O}_\mu(K)),J)$ and $\mathrm{S}^-(M_n(\mathbb{O}_\mu(K)),J)$ in this characteristic may satisfy a different set of identities than in the generic case, perhaps, more tractable and more closer to the classical identities (Jacobi, Jordan, etc.).

Note that, unlike the questions treated in this paper, some of these questions are sensitive to the ground field, and are related to the subtle behavior of quadratic forms, etc.

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