

MELIKYAN ALGEBRA IS A DEFORMATION OF A POISSON ALGEBRA

HAYK MELIKYAN AND PASHA ZUSMANOVICH

ABSTRACT. We prove, using computer, that the restricted Melikyan algebra of dimension 125 is a deformation of a Poisson algebra.

INTRODUCTION

The classification of finite-dimensional simple Lie algebras over an algebraically close field of characteristic $p > 3$ spanned the period of more than 50 years, and was achieved relatively recently (see [PS] and [S]). Roughly, each simple algebra is either of classical type or of Cartan type. As it is customary in many branches of mathematics, small characteristics provide additional difficulties: in characteristic $p = 5$, there is a series of simple Lie algebras, discovered by the first author ([M]), and dubbed thereafter as Melikyan algebras (see [S, Vol. I, §4.3]), which do not have analogs in higher characteristics. They are, however, intimately related to Lie algebras of Cartan type, and the present note provides an additional evidence of such a connection.

It was observed long time ago that simple Lie algebras of Cartan type can be deformed into each other (see, for example, [D] and references therein), and such kind of relationship, being interesting for its own sake, is also proved to be useful in structure theory. A similar connection exists between Melikyan algebras and Lie algebras of Cartan type: though they are different, Melikyan algebras are deformations of certain kind of Poisson algebras, the latter being extensions of Hamiltonian algebras. This was stated without proof in [KD, §5]. Unfortunately, the proof was never published, and more than decade later, the details seem to be lost.

In the present note we give a proof of this statement in the case of the smallest (and also the only restricted one) algebra in the series of dimension 125, $\mathcal{M}(1,1)$. The strategy of our proof is the following. We construct on computer the deformed algebra \mathcal{D} in question. Then we verify, using computer, that the constructed Lie algebra is simple. Over an algebraically closed field of characteristic 5, there are only three simple Lie algebras of dimension 125 – the algebra of general type $\mathcal{W}_1(3)$, the contact algebra $\mathcal{K}_3(1,1,1)$, and the Melikyan algebra $\mathcal{M}(1,1)$, the latter has a symmetric invariant bilinear form, and the formers do not. We check that the constructed algebra \mathcal{D} has a symmetric invariant bilinear form, and hence it is isomorphic to the Melikyan algebra. After giving all necessary definitions and constructions in §1, the computer calculations are explained in detail in §2. In the last §3 we outline some further questions and possible lines of development.

1. POISSON ALGEBRAS, COCYCLES AND DEFORMATIONS

We remind some standard definitions and facts concerning Poisson algebras, in a slightly more general, than usual, setting which will be useful for our purposes.

Let A be a commutative associative algebra with unit 1. Let us call a linear map $D : A \rightarrow A$ a *generalized derivation* of A , if it satisfies the identity

$$D(ab) = D(a)b + aD(b) - abD(1)$$

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for any $a, b \in A$. Note that this class of maps includes ordinary derivations (when $D(1) = 0$), and multiplications by an element of A . It is straightforward to check that the set of all generalized derivations of an algebra A forms a Lie algebra under the operation of commutator (which contains the usual derivations as a subalgebra).

Given a set $\{D_1, \dots, D_n, F_1, \dots, F_n\}$ of pairwise commuting generalized derivations of A , the bracket

$$(1) \quad [a, b] = \sum_{i=1}^n D_i(a)F_i(b) - F_i(a)D_i(b)$$

defines a Lie algebra structure on A which will be denoted as $A_{\{D_i, F_i\}}$. In the particular case when A is a polynomial algebra in even number of variables $x_1, \dots, x_n, y_1, \dots, y_n$ (or, more generally, an algebra of smooth functions on a symplectic manifold), and generalized derivations are the usual ones, the partial derivatives with respect to x_i, y_i , $i = 1, \dots, n$, the so obtained algebra is the ubiquitous Poisson algebra.

In fact, every summand in (1) defines a Lie algebra structure on A . In other words, they form *compatible Poisson structures*. This means that first, for any two subsets $\{D_i, F_i \mid i \in I\}$, $\{D_i, F_i \mid i \in J\}$ of the initial set of commuting generalized derivations, the bracket defining the Lie algebra structure on $A_{\{D_i, F_i \mid i \in J\}}$, is a 2-cocycle on the Lie algebra $A_{\{D_i, F_i \mid i \in I\}}$, and second, the Massey square of this cocycle is zero (or, in other words, the infinitesimal deformation defined by this cocycle is prolonged trivially).

Now, following [KD], consider a certain Poisson algebra structure on a 2-variable divided powers algebra. Recall that the divided powers algebra $\mathcal{O}_n(m_1, \dots, m_n)$, corresponding to n -tuple of positive integers m_1, \dots, m_n , is a commutative associative algebra defined over a field of characteristic $p > 0$, with the basis $\{x_1^{(i_1)} \dots x_n^{(i_n)} \mid 0 \leq i_k < p^{m_k}, k = 1, \dots, n\}$, and multiplication

$$x_1^{(i_1)} \dots x_n^{(i_n)} \cdot x_1^{(j_1)} \dots x_n^{(j_n)} = \binom{i_1 + j_1}{i_1} \dots \binom{i_n + j_n}{i_n} x_1^{(i_1 + j_1)} \dots x_n^{(i_n + j_n)},$$

where the standing assumption is that $x_k^{(0)} = 1$, the unit of the algebra, and $x_k^{(i)} = 0$ if i is outside the allowed range, i.e. $i < 0$ or $i \geq p^{m_k}$, $k = 1, \dots, n$.

Obviously,

$$(2) \quad \mathcal{O}_n(m_1, \dots, m_n) \simeq \mathcal{O}_1(m_1) \otimes \dots \otimes \mathcal{O}_1(m_n).$$

The algebra $\mathcal{O}_n(m_1, \dots, m_n)$ possess the following n derivations, lowering the power of each variable:

$$\partial_{x_k} : x_1^{(i_1)} \dots x_k^{(i_k)} \dots x_n^{(i_n)} \mapsto x_1^{(i_1)} \dots x_k^{(i_k - 1)} \dots x_n^{(i_n)},$$

$k = 1, \dots, n$. Obviously, these derivations pairwise commute. Of course, any map of $\mathcal{O}_n(m_1, \dots, m_n)$ of the form $a\partial_{x_k}$, where $a \in \mathcal{O}_n(m_1, \dots, m_n)$, is a derivation too (in fact, the derivation algebra of $\mathcal{O}_n(m_1, \dots, m_n)$ is linearly spanned by derivations of such form, and constitute the general Lie algebra of Cartan type $\mathcal{W}_n(m_1, \dots, m_n)$).

Let specialize the bracket (1) to the following situation: $A = \mathcal{O}_2(2, 1)$ (it will be more convenient to denote the generating variables in this case as x and y instead of x_1 and x_2), $p = 5$, $n = 1$, and $D_1 = \partial_x$, $F_1 = \partial_y$. The resulting Poisson Lie algebra \mathcal{P} of dimension 125 has one-dimensional center $K1$, and the commutator of the quotient $[\mathcal{P}/K1, \mathcal{P}/K1]$ is isomorphic to the simple 123-dimensional Lie algebra $\mathcal{H}_2(2, 1)$ of Hamiltonian type.

Now define the following 2-cochains $\varphi, \psi : \mathcal{P} \times \mathcal{P} \rightarrow \mathcal{P}$:

$$\begin{aligned} \varphi(a, b) &= \partial_x^2(a)\partial_x^3(b) - \partial_x^3(a)\partial_x^2(b) \\ \psi(a, b) &= (\text{id} - x\partial_x)(a)\partial_x^5(b) - \partial_x^5(a)(\text{id} - x\partial_x)(b) \end{aligned}$$

for $a, b \in \mathcal{O}_2(2, 1)$. Here id denotes the identity map. Note that both these cochains are of the form (1) for $n = 1$, but while ψ is formed by commuting generalized derivation $\text{id} - x\partial_x$ and derivation ∂_x^5 , and hence is automatically a cocycle on \mathcal{P} by the discussion above, the map φ is formed by the maps ∂_x^2 and ∂_x^3 which are not derivations. However, straightforward computations (which may be performed on a computer, see below) show that φ is a cocycle too.

Consider the cocycle $\varphi + 2\psi$. Direct computations (which, again, can be performed with the aid of computer) show that this cocycle can be prolonged trivially, and hence define a deformation of the algebra \mathcal{P} . Let us denote this deformation by \mathcal{D} .

Theorem (Kostrikin–Dzhumadil’daev). $\mathcal{D} \simeq \mathcal{M}(1, 1)$.

This theorem is proved on computer, as explained in the next section.

2. COMPUTER CALCULATIONS

We use GAP [G] which already contains a good deal of Lie-algebraic structures and functions for computation of various invariants of Lie algebras, and augment them by the necessary stuff. All GAP routines described in this section can be found at <http://justpasha.org/math/poisson-melikyan/>[†].

The base field in Theorem is algebraically closed, of characteristic 5. However, on computer we naturally work over the finite field $GF(5)$. While this difference is almost immaterial (all algebras in question are definable over $GF(5)$), in order to be rigorous, we distinguish between algebra \mathcal{X} , where $\mathcal{X} \in \{\mathcal{P}, \mathcal{D}\}$, defined over an algebraically closed field K , and its form \mathcal{X}' defined over the prime subfield $GF(5)$ (so $\mathcal{X}' \otimes_{GF(5)} K \simeq \mathcal{X}$ as K -algebras).

Step 1. Construction of the algebra \mathcal{P}' .

First, we provide two routines `PoissonAlgebra` and `SumOfAlgebraStructures` to construct Poisson algebras of kind (1). The routine `PoissonAlgebra` takes as arguments an algebra A , and two linear maps $D, F : A \rightarrow A$, and returns the bracket (1) in the particular case $n = 1$:

$$[a, b] = D(a)F(b) - F(a)D(b)$$

for $a, b \in A$. The routine `SumOfAlgebraStructures` takes as arguments several algebra structures defined on the same vector space V , and returns the algebra structure on V which is the sum of the supplied algebra structures.

Second, we provide two routines `DividedPowersAlgebra` and `TensorProductOfAlgebras` to construct divided powers algebras. The routine `DividedPowersAlgebra` takes as arguments a field K of positive characteristic, and a positive integer n , and returns the algebra $\mathcal{O}_1(n)$ over the field K . The routine `TensorProductOfAlgebras` takes as arguments several algebras defined over the same field, and returns their tensor product. Due to isomorphism (2), this enables us to construct an arbitrary divided powers algebra.

All this, together with a few other auxiliary routines implementing the derivations ∂_x and ∂_y as linear operators acting on $\mathcal{O}_2(2, 1)$, enables us to construct the algebra \mathcal{P}' .

Step 2. Construction of the algebra \mathcal{D}' .

Using the already employed routines `PoissonAlgebra` and `SumOfAlgebraStructures`, we construct cocycles φ and ψ , and then the bracket

$$\{a, b\} = [a, b] + \varphi(a, b) + 2\psi(a, b),$$

where $a, b \in \mathcal{O}_2(2, 1)$, what gives us the algebra \mathcal{D}' .

Step 3. Identifying algebras \mathcal{D} and $\mathcal{M}(1, 1)$.

[†]Added February 11, 2021: currently available at <https://web.osu.cz/~Zusmanovich/papers/poisson-melikyan/>

Generally, to establish on computer whether two given Lie algebras are isomorphic or not, amounts to solving a system of quadratic equations, and as such, is a difficult task. There are several algorithms for doing that, some of them quite sophisticated (see, for example, [E]), but neither of them, as of time of this writing, would work for algebras of dimension 125 on a reasonable computer within a reasonable amount of time.

However, to establish whether two modules over the *same* algebra are isomorphic or not, is a linear problem, and as such, is much more tractable. Some little theory enables us to take advantage of the latter fact.

In dealing with modules over algebras, we use a highly efficient Meataxe suit of algorithms (see [HEO, §7.4]). Initially, Meataxe was developed for dealing with finite groups representations, but it works on the level of associative matrix algebras (via group algebras), so it can be utilized for study of representations of Lie algebras (via associative envelopes of Lie algebras) equally well. Using Meataxe routines available in GAP, we can compute whether a given Lie algebra is simple (i.e., the adjoint representation is irreducible), central simple (i.e., the adjoint representation is absolutely irreducible), and possess a symmetric invariant bilinear form (i.e., the adjoint representation is equivalent to its dual). We borrowed the idea to use Meataxe in the Lie-algebraic context from [E].

First, we establish that the algebra \mathcal{D}' is central simple (so \mathcal{D} is simple). According to the classification of simple modular Lie algebras, over an algebraically closed field of characteristic 5 there are only three simple Lie algebras of dimension 125 – the algebra of general type $\mathcal{W}_1(3)$, the algebra of contact type $\mathcal{K}_3(1, 1, 1)$, and the Melikyan algebra $\mathcal{M}(1, 1)$ (see, for example, [S, Vol. I, §4.2] for the dimensions of Lie algebras of Cartan type). In order to distinguish between them, we have to choose some invariant, and the suitable invariant in our case is the presence of a nonzero symmetric invariant bilinear form.

Recall that a symmetric bilinear form $\omega : L \times L \rightarrow K$ on a Lie algebra L is called *invariant*, if $\omega([x, z], y) + \omega(x, [y, z]) = 0$ for any $x, y, z \in L$. On simple Lie algebras, this form, if exists, is unique up to a scalar due to Schur lemma, and the existence of nonzero form is equivalent to isomorphism of L -modules $L \simeq L^*$. The algebras $\mathcal{W}_1(3)$ and $\mathcal{K}_3(1, 1, 1)$ do not have such a form (see [SF, §4.6, Theorems 6.3 and 6.6], and also [Z2] for a short alternative proof and further references), while the Melikyan algebra does ([PS, Proposition 6.1]).

As the last step, we establish that \mathcal{D}' has a symmetric invariant bilinear form, and hence so does \mathcal{D} . To summarize: \mathcal{D} is a simple Lie algebra of dimension 125 having nonzero symmetric invariant bilinear form, and hence have to be isomorphic to the Melikyan algebra, thus proving the Theorem.

The whole computation takes about 1.5 minutes on a machine with 2.40 GHz CPU.

3. WHAT NEXT?

It goes without saying that a more satisfactory proof of the Theorem would cover the whole series of Melikyan algebras, will not involve computer, and will provide an explicit isomorphism. We are planning to systematically compute deformations of Poisson Lie algebras of the kind appearing in this note, and identify among them those which are isomorphic to Melikyan algebras and other interesting Lie algebras in small characteristics. As a first step, we envisage a linear-algebraic approach to computation of (low-degree) cohomology of a more general class of Poisson algebras, in which the underlying commutative associative algebra A is decomposed as the tensor product of two algebras. This should be somewhat similar to computation of cohomology of current Lie algebras in [Z1].

Another interesting question is how the p -map comes into play. Note that the Poisson algebra \mathcal{P} is not restricted, while its deformation $\mathcal{M}(1, 1)$ is. This phenomenon, where a restricted Lie

algebra is a deformation of a not restricted one, and, generally, how the p -map behaves under deformations, deserves further investigation.

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DEPARTMENT OF MATHEMATICS AND PHYSICS, NORTH CAROLINA CENTRAL UNIVERSITY, DURHAM, NC 27707, USA

Email address: melikyan@ncsu.edu

INSTITUTE OF MATHEMATICS AND STATISTICS, UNIVERSITY OF SÃO PAULO, SÃO PAULO, BRAZIL

Email address, as of February 12, 2021: pasha.zusmanovich@gmail.com