ON THE UTILITY OF ROBINSON–AMITSUR ULTRAFILTERS. II

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ABSTRACT. We extend the result of the previous paper under the same title about embedding of ideal-determined algebraic systems into ultraproducts, to arbitrary algebraic systems, and to ultraproducts over κ-complete ultrafilters. We also discuss the scope of applicability of this result, and correct a mistake from the previous paper concerning homomorphisms from ultraproducts.

0. **Introduction.** In [Z], an old and simple trick, used by A. Robinson and S. Amitsur (see [A1, Proof of Theorem 15] and [A2, Theorem 3]) to establish, in the context of ring theory, an embedding of certain rings in ultraproducts, was generalized to algebraic systems with ideal-determined congruences. Here we push it to the full generality, for arbitrary algebraic systems without any restrictions on their congruences.

1. **Recollection on congruences, ultrafilters, and ultraproducts.** We refer to [C.B] and [CK] for the rudiments of universal algebra and model theory we need.

Let us recall some basic notions and fix notation. We consider the most general algebraic systems, i.e. sets with a number of operations Ω defined on them, of, generally, various arity. The signature Ω is arbitrary, but fixed (so, in what follows all algebraic systems are supposed to be of signature Ω). The set of all congruences on a given algebraic system A forms a partially ordered set (actually, a lattice), and the minimal element in this set is the *trivial congruence*, coinciding with the diagonal $\{(a,a) \mid a \in A\}$ in $A \times A$. The intersection of all congruences containing a given relation ρ, i.e. a set of pairs of elements from $A \times A$, is called a *congruence generated by* ρ and is denoted by $\mathrm{Con}(\rho)$. In the case of one-element relation, i.e. a single pair $(a,b) \in A \times A$ with $a \neq b$, we shorten this notation to $\mathrm{Con}(a,b)$ and speak about *principal congruences*.

Given a cardinal $\kappa > 2$, let us call a set $\mathscr{S}$ of sets κ*-complete*, if the intersection of any nonempty set of fewer than κ elements of $\mathscr{S}$ belongs to $\mathscr{S}$. If a set $\mathscr{S}$ of subsets of a set $\mathbb{I}$ satisfies a weaker condition – that the intersection of any nonempty set of fewer than κ elements of $\mathscr{S}$ contains an element of $\mathscr{S}$ – then the set of subsets of $\mathbb{I}$ which are oversets of all such intersections is a κ-complete filter on $\mathbb{I}$ containing $\mathscr{S}$ (this is an obvious generalization of the standard and frequently employed fact that any set of subsets satisfying the finite intersection property can be extended to a filter).

An algebraic system A is called κ*-subdirectly irreducible*, if either of the following two equivalent conditions is satisfied:

- The set of nontrivial congruences of A is κ-complete.
- If A embeds in the direct product of $< \kappa$ algebraic systems $\prod_{i\in\mathbb{I}} B_i$, $|\mathbb{I}| < \kappa$, then A embeds in one of B_i's.

Obviously, the condition of ω-subdirect irreducibility, also called *finite subdirect irreducibility*, is equivalent to n-subdirect irreducibility for any finite $n > 2$. Note that ω-complete filters (ultrafilters) are just the usual filters (ultrafilters).

Given a filter (respectively, ultrafilter) $\mathscr{F}$ on a set $\mathbb{I}$, the quotient of the direct product of algebraic systems $\prod_{i\in\mathbb{I}} A_i$ by the congruence

$$\theta_{\mathscr{F}} = \{(a,b) \in (\prod_{i\in\mathbb{I}} A_i) \times (\prod_{i\in\mathbb{I}} A_i) \mid \{i \in \mathbb{I} \mid a(i) = b(i)\} \in \mathscr{F}\}$$

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is called *filtered product* (respectively, *ultraproduct*) of the corresponding algebraic systems, and is denoted by $\prod_{\mathscr{F}} A_i$.

As any κ-complete ultrafilter on a set of cardinality $< \kappa$ is principal, the condition of κ-subdirect irreducibility of an algebraic system A may be trivially reformulated as follows:

- If A embeds in the direct product of $< \kappa$ algebraic systems $\prod_{i \in \mathbb{I}} B_i$, $|\mathbb{I}| < \kappa$, then A embeds in the ultraproduct $\prod_{\mathscr{U}} B_i$ for some ultrafilter $\mathscr{U}$ on $\mathbb{I}$.

What is, perhaps, surprising, is that under a suitable set-theoretic assumption on the cardinal κ, the latter condition is equivalent to the same condition with an arbitrary index set $\mathbb{I}$, without a restriction on its cardinality. This equivalence is what constitutes the "Robinson–Amitsur theorem" (Theorem 1 below).

2. **Embeddings in direct products.** The next proof follows the proof of Theorem 1.1 from [Z], with ideals being replaced by congruences, and ω being replaced by an arbitrary cardinal κ. On this level of generality, the proof is even simpler than an already simple proof from [Z].

Theorem 1 (ROBINSON–AMITSUR)**.** *Let κ be a cardinal > 2 and such that any κ-complete filter can be extended to a κ-complete ultrafilter. Then for any algebraic system A the following are equivalent:*

(i) *A is κ-subdirectly irreducible.*
(ii) *For any embedding f of A in the direct product $\prod_{i \in \mathbb{I}} B_i$ of a set of algebraic systems $\{B_i\}_{i \in \mathbb{I}}$, there is a κ-complete ultrafilter $\mathscr{U}$ on the set $\mathbb{I}$ such that the composition of f with the canonical homomorphism $\prod_{i \in \mathbb{I}} B_i \to \prod_{\mathscr{U}} B_i$, is an embedding.*

Proof. (ii) $\Rightarrow$ (i): take $|\mathbb{I}| < \kappa$ and observe, as above, that then the ultrafilter $\mathscr{U}$ is principal.

(i) $\Rightarrow$ (ii). For any two elements $a, b \in A$, define $\mathbb{S}_{a,b} = \{i \in \mathbb{I} \,|\, a(i) \neq b(i)\}$, and let $\mathscr{S} = \{\mathbb{S}_{a,b} \,|\, a, b \in A, a \neq b\}$. Let us verify that the intersection of $< \kappa$ elements of $\mathscr{S}$ contains an element of $\mathscr{S}$.

Let a, b be two elements of A such that $a \neq b$. If for some $i_0 \in \mathbb{I}$ we have $a(i_0) = b(i_0)$, then (a, b) belongs to the congruence of $\prod_{i \in \mathbb{I}} B_i$ which is the kernel of the canonical projection Pr_{i_0} to B_{i_0}, and by the Third Isomorphism Theorem, belongs to the congruence $\mathrm{Ker}(\mathrm{Pr}_{i_0}) \cap (A \times A)$ of A, whence $\mathrm{Con}(a, b) \subseteq \mathrm{Ker}(\mathrm{Pr}_{i_0})$. The latter means that for any $(c, d) \in \mathrm{Con}(a, b)$, we have $\mathbb{S}_{c,d} \subseteq \mathbb{S}_{a,b}$.

Due to κ-subdirect irreducibility of A, for any set of pairs of different elements $\{a_i, b_i\}$ of A of cardinality $< \kappa$, the intersection $\bigcap_{i < \kappa} \mathrm{Con}(a_i, b_i)$ is a nontrivial congruence. Pick an element (c, d), $c \neq d$, from this intersection. Then by just proved we have $\mathbb{S}_{c,d} \subseteq \bigcap_{i < \kappa} \mathbb{S}_{a_i, b_i}$, as required.

Thus $\mathscr{S}$ can be extended to a κ-complete filter, and then to a κ-complete ultrafilter $\mathscr{U}$ on $\mathbb{I}$. Factoring the embedding of A in $\prod_{i \in \mathbb{I}} B_i$ by the congruence $\theta_{\mathscr{U}}$, we get, again by the Third Isomorphism Theorem, an embedding of the quotient $A/(\theta_{\mathscr{U}} \cap (A \times A))$ in the ultraproduct $\prod_{\mathscr{U}} B_i$. But if $(a, b) \in \theta_{\mathscr{U}} \cap (A \times A)$, then, by definition, $\{i \in \mathbb{I} \,|\, a(i) = b(i)\} \in \mathscr{U}$, and, since $\mathscr{U}$ is an ultrafilter, $\{i \in \mathbb{I} \,|\, a(i) \neq b(i)\} \notin \mathscr{U}$, and hence $\{i \in \mathbb{I} \,|\, a(i) \neq b(i)\} \notin \mathscr{S}$, what, in its turn, implies $a = b$. This shows that the congruence $\theta_{\mathscr{U}} \cap (A \times A)$ is trivial, and hence A embeds in $\prod_{\mathscr{U}} B_i$. □

Coupling the case $\kappa = \omega$ of this theorem with the Birkhoff theorem about varieties of algebraic systems, and some rudimentary model theory, we get:

Corollary (CRITERION FOR ABSENCE OF NONTRIVIAL IDENTITIES)**.** *Let $\mathfrak{A}$ be a variety of algebraic systems such that any free system in $\mathfrak{A}$ is finitely subdirectly irreducible. Then for an algebraic system $A \in \mathfrak{A}$ the following three conditions are equivalent:*

(i) *A does not satisfy a nontrivial identity within $\mathfrak{A}$.*
(ii) *Any free system in $\mathfrak{A}$ embeds in an ultrapower of A.*
(iii) *Any free system in $\mathfrak{A}$ embeds in a filtered power of A.*

Proof. (i) $\Rightarrow$ (ii) is established exactly as in [Z, Corollaries 1.2 and 1.3]: by the Birkhoff theorem, a free system in $\mathfrak{A}$ embeds in the direct power of A. Apply Theorem 1, with $\kappa = \omega$.

(ii) $\Rightarrow$ (iii): obvious.

(iii) $\Rightarrow$ (i): follows from the fact that any nontrivial identity is a first-order Horn sentence, and Horn sentences are preserved under filtered products (see, e.g., [CK, Proposition 6.2.2]). □

3. **No way to move beyond ω. Discussion.** Let us make a few remarks about the scope of applicability of Theorem 1 and Corollary. By the Łoś theorem, an algebraic system is elementarily equivalent to its ultrapower, so if the condition of Corollary is met, it allows to reduce the question about the absence of nontrivial identities in an algebraic system A to the question whether a system elementary equivalent to A contains a free subsystem. This was demonstrated to be useful in some situations in [Z], but the usefulness is severely restricted by the fact that any property of an algebraic system A we can use in the process should be a first-order property. Most of the interesting properties of algebraic systems are second- or higher-order.

Note that by the analog of Łoś theorem for ultraproducts over κ-complete ultrafilters, the ultrapower $A^{\mathscr{U}}$ of an algebraic system A is elementarily equivalent to A in the sense of certain higher-order logic (denoted by $\mathbf{L}_\kappa$ in [CK, §4.2], and by $\mathbf{L}_{\kappa\kappa}$ in [D, §3.3]). But unfortunately, we cannot derive an analog of Corollary to Theorem 1 for $\kappa > \omega$, at least in, arguably, the most interesting for applications cases – groups and algebras over fields (the main protagonists of [Z]). The reason is that free systems in these varieties are residually nilpotent (the intersection of all terms of the lower central series is trivial) and hence are already not ω_1-subdirectly irreducible.

Moreover, it seems that no statement similar to Corollary, where the (usual) ultrapower construction is replaced by whatever other construction providing us with elementary equivalence in the sense of second- or higher-order logic of some sort, is possible. Indeed, any such logic $\mathbf{L}$ should be a priori weak enough not to allow encode the fact that an algebraic system contains a free subsystem: otherwise, if in the condition (ii) of Corollary we would be able to replace the ultrapower of A by another construction elementary equivalent to A in the sense of $\mathbf{L}$, this would imply that for any algebraic system the absence of a nontrivial identity is equivalent to having a free subsystem – an obviously false statement. While, in general, free systems in arbitrary varieties are not necessary countable, they are, essentially, so in all cases of interest (groups and algebras over fields): all identities in a given variety are captured by free systems of no more than countable rank; such free groups are, obviously, countable, and such free algebras are countable-dimensional over the base field. In the latter case, as identities of algebras do not change under field extensions (except, possibly, the cases of finite base fields of "small" cardinality), we may always assume the base field to be countable, and, hence, the whole free algebra to be countable. To summarize: the hypothetical logic $\mathbf{L}$ should not allow to encode the fact that an algebraic system contains a countable subsystem – a very weak logic indeed, on the verge not to be qualified as the "second-order" one!

What about the condition on κ in Theorem 1 (sometimes called *strong compactness*)? In the case $\kappa = \omega$ this amounts to the well-known fact that any filter can be extended to an ultrafilter (true by the Zorn lemma). In the general case we are in deep waters of set theory. It seems that this is one of the statements about which one does not make sense to speak if they are "true" or "false", but it is rather a matter of which model of set theory is adopted. See [D, §3.3, Parts C,D] for this and many other conditions of such sort, and relationships between them. Note also that the very existence of non-principal κ-complete ultrafilters for $\kappa > \omega$, being equivalent to the existence of measurable cardinals, entails rather strange properties of the corresponding model(s) of set theory (see, again, [CK, §4.2] and [D, §0.4]).

Note finally that it would be very interesting to extend arguments used in the proof of Theorem 1 to the case of metric ultraproducts. This could provide an approach to various questions related to sofic and hyperlinear groups.

4. **Semigroups.** Arguably, the most interesting class of algebraic systems whose congruences are not ideal-determined, i.e. not covered by [Z], but by Theorem 1 and Corollary above, is the class of semigroups. Unfortunately, the Corollary is not applicable to the variety of all semigroups, as free semigroups of rank > 1 are not finitely subdirectly irreducible. For example, if x and y are among free generators of a free semigroup G, then the congruences of G generated by pairs (x,x^2) and (y,y^2) intersect trivially. It seems to be interesting to investigate for which classes of semigroups (inverse semigroups?

Burnside varieties?) the Corollary would be applicable, and what good it could do. The same question for the class of loops.

5. **Homomorphisms from direct products.** Finally, we take the opportunity to note a mistake in [Z]. Theorem 7.1 there contains a statement dual, in a sense, to Theorem 1: if there is a surjective homomorphism $\alpha : \prod_{i \in \mathbb{I}} B_i \to A$, where A is finitely subdirectly irreducible, then A is a surjective homomorphic image of an ultraproduct of the same family $\{B_i\}$. The proof of this theorem is in error. At the top of p. 283 in the published version (at the bottom of p. 13–top of p. 14 in the arXiv version), it is, essentially, claimed (keeping the notation and terminology of the original paper) that any equality in A of the form $\alpha(u) = t(b_1, \dots, b_n, \alpha(f), \dots, \alpha(f))$, where $u, f \in \prod_{i \in \mathbb{I}} B_i$, $b_1, \dots, b_n \in A$, and t is an ideal term in arguments occupied by $\alpha(f)$'s, can be lifted to $\prod_{i \in \mathbb{I}} B_i$: $u = t(h_1, \dots, h_n, f, ..., f)$ for some $h_i \in \prod_{i \in \mathbb{I}} B_i$ such that $\alpha(h_i) = b_i$. There is no reason whatsoever for this to be true in general: for example, in the simplest possible case $t(x) = x$ (the ideal term just in one variable, without arguments occupied by b_i's), this claim amounts to saying that α is an isomorphism.

Luckily, we can slightly modify the statement, and supply it with a new proof. While the new statement differs from the original one, it is more general for the most classes of algebraic systems of interest. The proof uses an ingenious (and barely utilizing anything beyond the definition of ultrafilter) argument from an array of recent papers by G. Bergman, some of them jointly with N. Nahlus (e.g., [BN] and [G.B]), devoted to factoring homomorphisms from direct product of groups, algebras over fields, and modules. The argument is valid for arbitrary algebraic systems, and we reproduce it here, throwing in κ-complete ultrafilters for good measure.

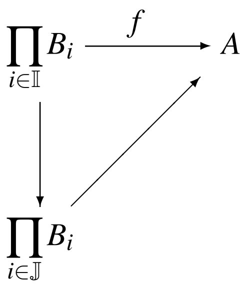
Theorem 2 (BERGMAN–NAHLUS). *For any cardinal $\kappa > 2$, and any algebraic system A consisting of more than one element, the following are equivalent:*

(i) *For any surjective homomorphism f of the direct product $\prod_{i \in \mathbb{I}} B_i$, $|\mathbb{I}| < \kappa$, to A, there is $i_0 \in \mathbb{I}$ such that f factors through the canonical projection $\prod_{i \in \mathbb{I}} B_i \to B_{i_0}$.*

(ii) *For any surjective homomorphism f of the direct product $\prod_{i \in \mathbb{I}} B_i$ to A, there is a κ-complete ultrafilter $\mathscr{U}$ on the indexing set $\mathbb{I}$ such that f factors through the canonical homomorphism $\prod_{i \in \mathbb{I}} B_i \to \prod_{\mathscr{U}} B_i$.*

Proof. (ii) $\Rightarrow$ (i) is obvious (like in a similar situation in Theorem 1, take $|\mathbb{I}| < \kappa$ and observe that $\mathscr{U}$ is necessary principal).

(i) $\Rightarrow$ (ii). Let $\mathscr{U}$ consist of subsets $\mathbb{J} \subseteq \mathbb{I}$ such that f factors through a surjective homomorphism from the (smaller) direct product of B_i's indexed over $\mathbb{J}$, i.e. the diagram



where the vertical arrow is the canonical projection, commutes. Utilizing the condition (i) with $|\mathbb{I}| = 2$, and reasoning as in [BN, Lemma 7 and Proposition 8] or [G.B, Lemma 1.2], we get that $\mathscr{U}$ is an ultrafilter, and f factors through the canonical homomorphism $\prod_{i \in \mathbb{I}} B_i \to \prod_{\mathscr{U}} B_i$.

To prove that $\mathscr{U}$ is κ-complete, it is sufficient to show that for any decomposition $\mathbb{I} = \bigcup_{j < \kappa} \mathbb{I}_j$ into the union of pairwise disjoint sets $\mathbb{I}_j$, $j < \kappa$, at least one of them belongs to $\mathscr{U}$. Since $\prod_{i \in \mathbb{I}} B_i = \prod_{j < \kappa} \left(\prod_{i \in \mathbb{I}_j} B_i \right)$, f factors through the canonical projection $\prod_{i \in \mathbb{I}} B_i \to \prod_{i \in \mathbb{I}_{j_0}} B_i$ for some $j_0 < \kappa$, i.e. $\mathbb{I}_{j_0} \in \mathscr{U}$, as required. □

Note that in the case of groups and algebras over fields, the condition (i) of Theorem 2 with $\kappa = \omega$ is weaker than the finite subdirect irreducibility. This follows from [BN, Lemma 5] in the case of algebras over fields, and in the case of groups the argument is repeated almost verbatim.

Note also that though Theorems 1 and 2 are, in a sense, dual to each other, this duality does not stretch to the proofs: the proofs are different, and each proof seemingly cannot be adapted to the dual situation.

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