A COMPENDIUM OF LIE STRUCTURES ON TENSOR PRODUCTS

PASHA ZUSMANOVICH

Abstract. We demonstrate how a simple linear-algebraic technique used earlier to compute low-degree cohomology of current Lie algebras, can be utilized to compute other kinds of structures on such Lie algebras, and discuss further generalizations, applications, and related questions. While doing so, we touch upon such seemingly diverse topics as associative algebras of infinite representation type, Hom-Lie structures, Poisson brackets of hydrodynamic type, Novikov algebras, simple Lie algebras in small characteristics, and Koszul dual operads.

Introduction

Earlier, we have demonstrated how a simple linear-algebraic technique, somewhat resembling separation of variables in differential equations, allows to obtain general results about the low-degree cohomology of current Lie algebras, i.e. Lie algebras of the form $L \otimes A$, where L is a Lie algebra, and A is an associative commutative algebra, with an obvious Lie bracket

$$[x \otimes a, y \otimes b] = [x, y] \otimes ab$$

for $x, y \in L$, $a, b \in A$. On a pedestrian level, kept through the most of [Z1] and [Z3], this technique amounts to symmetrization of cocycle equations with respect to variables belonging to L and A separately. On a more sophisticated level, the method can be formulated in terms of a certain spectral sequence defined with the help of Young symmetrizers on the underlying spaces L and A (described briefly in [Z1, §4]).

In this, somewhat eclectic, paper we discuss further generalizations, applications and extensions of this method. We start in §1 by providing examples of finite-dimensional Lie algebras having an infinite number of cohomologically nontrivial non-isomorphic modules of a fixed finite dimension. This answers a question and corrects a statement from an (old) paper by Dzhumadil'daev [D1]. The Lie algebras in our example are current Lie algebras, and our technique allows to reduce the question about the number of non-isomorphic cohomologically nontrivial "current" modules over such algebras to a similar, but purely representation-theoretic (i.e. not appealing to any cohomological condition) question about the number of non-isomorphic modules over the corresponding associative commutative algebra A.

Further, we consider a question of description of Poisson structures on a given Lie algebra – i.e., description of all (not necessary associative) algebra structures which, together with a given Lie structure, form a Poisson algebra. The Poissonity condition resembles the cocycle equation, and this allows to apply the same methods used previously to compute low-degree cohomology of current Lie algebras. In §2 we describe Poisson structures on current and Kac–Moody Lie algebras, and in §3 – on Lie algebras of the form $\mathsf{sl}_n(A)$. Associative Poisson structures on Kac–Moody algebras were described earlier by Kubo in [Ku1] via case-by-case computations, and we generalize and provide a conceptual explanation of Kubo's result.

In §4 we compute, in a similar way, Hom-Lie structures on current Lie algebras.

In the last §5 we discuss a possibility of extension of all the previous results from current Lie algebras to Lie algebras obtained from the tensor product of algebras over Koszul dual binary quadratic operads, with a special emphasis on Novikov algebras. This is the most speculative

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part of the paper: it does not contain a single theorem, but numerous examples, speculations, and questions, including an observation that the classical construction of "Poisson brackets of hydrodynamic type" due to Gelfand–Dorfman and Balinskii–Novikov, arises in this context. (However, this simple observation is, perhaps, the most important result of the paper).

This is a modest contribution in honor of Nikolai Vavilov, an amazing man and amazing mathematician. There is an obvious similarity between current Lie algebras, the main heroes of this paper, and linear and close to them groups over rings, Vavilov's favorite objects of study. The latters, however, offer much deeper difficulties, and it is my hope to extend in the future the methods and ideas of this paper to be able to tackle some questions close to Vavilov's heart.

Notation and conventions

Our notation is mostly standard. The base field is denoted by the letter K. By $\mathrm{H}^n(L,M)$ is denoted the nth degree Chevalley-Eilenberg cohomology of a Lie algebra L with coefficients in an L-module M. The symbol Hom_A for a (Lie or associative) algebra A is understood in the category of A-modules (in particular, Hom_K is just the vector space of linear maps between its arguments). The notation $\mathrm{End}_A(V)$ serves as a shortcut for $\mathrm{Hom}_A(V,V)$. All unadorned tensor products are assumed over K. Direct sum $\oplus$ is always understood in the category of vector spaces over K. An operator of multiplication by an element u in an associative commutative algebra is denoted by R_u. For a (not necessary associative) algebra A, $A^{(-)}$ denotes its *skew-symmetrization*, i.e. an algebra structure on the same underlying vector space A subject to the new multiplication $[a,b] = ab - ba$ for $a,b \in A$. An algebra is called *Lie-admissible*, if its skew-symmetrization is a Lie algebra.

All other necessary notation and notions are defined as they are introduced in the text.

1. On the finiteness of the number of cohomologically nontrivial Lie modules

In 1980s, Askar Dzhumadil'daev has initiated a study of cohomology of finite-dimensional Lie algebras over fields of positive characteristic, which showed a (long time anticipated) drastic difference with the characteristic zero case. It turned out that cohomology in positive characteristic does not vanish much more often that in the characteristic zero case (for example, Whitehead lemmas do not hold), and a natural question arose about the number of cohomologically nontrivial modules.

One of the papers of that period is [D1], and we start our discussion by indicating one inaccuracy there. In [D1, Theorem 1] it is claimed that, for any finite-dimensional Lie algebra L over a field of positive characteristic, the number of cohomologically nontrivial non-isomorphic L-modules of a fixed finite dimension is finite. There is a following gap in the proof of this statement: when arguing inductively about indecomposable modules over a Lie algebra L, it is claimed that non-equivalent extensions of L-modules $0 \to V_1 \to V \to V_2 \to 0$ are described by the space $\mathrm{H}^1(L, \mathrm{Hom}_K(V_1,V_2))$ which is finite-dimensional, and hence the set of such extensions is finite provided the set of modules V_1 and V_2 is finite. This is obviously incorrect: the extensions are in bijective correspondence with the space $\mathrm{H}^1(L, \mathrm{Hom}_K(V_1,V_2))$ as a *set*, which is infinite (provided the base field K is infinite).

This does not, however, exclude the possibility that the set of *non-isomorphic* modules would be finite, as there can be ad hoc module isomorphisms which do not follow from the equivalence of modules.

In the category of associative algebras, where extensions of modules are described, up to equivalence, by the corresponding first Hochschild cohomology, it is known that algebras can have both finite and infinite number of indecomposable modules of a fixed finite dimension.

The situation in the category of finite-dimensional Lie p-algebras was studied in [FeS]. It turns out that Lie p-algebras having a finite number of indecomposable p-modules of a fixed finite dimension, form quite a narrow class.

Still, from all this it is not clear whether the statement of [D1, Theorem 1] is true. The aim of this section is to show that it is not: we provide a recipe to construct Lie algebras, of arbitrarily high finite dimension, having an infinite number of non-isomorphic cohomologically nontrivial modules of a fixed finite dimension. These Lie algebras come as current Lie algebras $L \otimes A$, and the modules appear as the corresponding "current modules", i.e. $L \otimes A$-modules of the form $M \otimes V$, where M is an L-module and V is an A-module, with an obvious action

$$(x \otimes a) \bullet (m \otimes v) = (x \bullet m) \otimes (a \bullet v)$$

for $x \in L$, $a \in A$, $m \in M$, $v \in V$; the bullets denote, by abuse of notation, the respective module actions. By choosing suitably L and M, we are able to reduce the question to those about the finiteness of the number of modules of a given dimension over A, i.e. to a similar (but simpler, as it does not involve any cohomological condition) question in the category of associative commutative algebras.

This construction works in any characteristic of the base field, so it also answers affirmatively Question 1 from [D1]: does there exist finite-dimensional Lie algebras over a field of characteristic zero having an infinite number of cohomologically nontrivial non-isomorphic modules of a given finite dimension?

Lemma 1.1. *Let L be a Lie algebra, A an associative commutative algebra with unit, M an L-module, and V an unital A-module. Then there is an embedding of* $\mathrm{H}^*(L, M) \otimes V$ *into* $\mathrm{H}^*(L \otimes A, M \otimes V)$.

Proof. Define a map of the Chevalley–Eilenberg cochain complexes

$$\xi : \mathrm{C}^n(L, M) \otimes V \to \mathrm{C}^n(L \otimes A, M \otimes V)$$

as follows:

$$\xi(\varphi \otimes v)(x_1 \otimes a_1, \dots, x_n \otimes a_n) = \varphi(x_1, \dots, x_n) \otimes (a_1 \cdots a_n) \bullet v$$

where $\varphi \in \mathrm{C}^n(L, M)$, $v \in V$, $x_i \in L$, $a_i \in A$.

Let us see how differentials in the complexes $\mathrm{C}^*(L \otimes A, M \otimes V)$ and $\mathrm{C}^*(L, M)$, denoted as $d_{L\otimes A}$ and d_L respectively, interact with this map:

$$\begin{aligned}
& d_{L\otimes A}\xi(\varphi \otimes v)(x_1 \otimes a_1, \dots, x_{n+1} \otimes a_{n+1}) \\
&= \sum_{i=1}^{n+1} (-1)^{i+1}(x_i \otimes a_i) \bullet \xi(\varphi \otimes v)(x_1 \otimes a_1, \dots, \widehat{x_i \otimes a_i}, \dots, x_{n+1} \otimes a_{n+1}) \\
&+ \sum_{1 \le i < j \le n+1} (-1)^{i+j}\xi(\varphi \otimes v)([x_i \otimes a_i, x_j \otimes a_j], x_1 \otimes a_1, \dots, \widehat{x_i \otimes a_i}, \dots, \widehat{x_j \otimes a_j}, \\
&\qquad \dots, x_{n+1} \otimes a_{n+1}) \\
&= \sum_{i=1}^{n+1} (-1)^{i+1} x_i \bullet \varphi(x_1, \dots, \widehat{x}_i, \dots, x_{n+1}) \otimes (a_i a_1 \cdots \widehat{a}_i \cdots a_{n+1}) \bullet v \\
&+ \sum_{1 \le i < j \le n+1} (-1)^{i+j} \varphi([x_i, x_j], x_1, \dots, \widehat{x}_i, \dots, \widehat{x}_j, \dots, x_{n+1}) \\
&\qquad \otimes (a_i a_j a_1 \cdots \widehat{a}_i \cdots \widehat{a}_j \cdots a_{n+1}) \bullet v \\
&= d_L\varphi(x_1, \dots, x_{n+1}) \otimes (a_1 \cdots a_{n+1}) \bullet v = \xi(d_L\varphi \otimes v)(x_1 \otimes a_1, \dots, x_{n+1} \otimes a_{n+1}).
\end{aligned}$$

Thus ξ is a homomorphism of cochain complexes, and hence it induces the map between their cohomology:

$$\overline{\xi} : \mathrm{H}^*(L, M) \otimes V \to \mathrm{H}^*(L \otimes A, M \otimes V).$$

The kernel of $\overline{\xi}$ consists of classes of cocycles $\sum_k \varphi_k \otimes v_k \in \mathrm{Z}^n(L,M) \otimes V$ such that

$$\sum_k \varphi_k(x_1,\dots,x_n) \otimes (a_1 \cdots a_n) \bullet v_k = d_{L\otimes A}\Omega(x_1 \otimes a_1, \dots, x_n \otimes a_n)$$

for some $\Omega \in \mathrm{C}^{n-1}(L \otimes A, M \otimes V)$ and all $x_1,\dots,x_n \in L$, $a_1,\dots,a_n \in A$. Setting in this equality all a_i's to 1, we get

$$\sum_k \varphi_k(x_1,\dots,x_n) \otimes v_k = \sum_\ell d_L\omega_\ell(x_1,\dots,x_n) \otimes u_\ell, \tag{1.1}$$

where $\sum_\ell \omega_\ell \otimes u_\ell$ is an image of the restriction of Ω to $\bigwedge^{n-1}(L \otimes 1)$, under the canonical isomorphism $\mathrm{C}^{n-1}(L, M \otimes V) \simeq \mathrm{C}^{n-1}(L,M) \otimes V$.

The equality (1.1) implies that some nontrivial linear combination of φ_k's is a coboundary, hence $\overline{\xi}$ is injective, and the statement of the lemma follows. $\square$

As simple as it is, it is curious to note that this is essentially the same reasoning as in [Z4, Lemma] about cohomology of generalized de Rham complex.

Generally, a full computation of cohomology $\mathrm{H}^*(L \otimes A, M \otimes V)$, except for some special cases in low degrees, is a hopeless task, and this embedding is very far from being surjective (for cohomology of degree 2, see [Z1, Proposition 3.1]).

Lemma 1.2. *Let L be a Lie algebra, A an associative commutative algebra with unit, M an L-module, and V_1, V_2 two unital A-modules. Suppose that either M, or both V_1, V_2 are finite-dimensional. Then*

$$\begin{aligned} &\mathrm{Hom}_{L\otimes A}(M \otimes V_1, M \otimes V_2) \\ &\simeq \{\varphi \in \mathrm{End}_K(M) \,|\, \varphi(LM) = 0; L\varphi(M) = 0\} \otimes \mathrm{Hom}_K(V_1,V_2) + \mathrm{End}_L(M) \otimes \mathrm{Hom}_A(V_1,V_2). \end{aligned} \tag{1.2}$$

At this point we start to use repeatedly linear-algebraic reasonings similar to those in [Z1] and [Z3]. This simple case concerns maps of one argument, and thus does not involve any symmetrization, unlike the other cases considered further, in subsequent sections.

Proof. Let $\Phi : M \otimes V_1 \to M \otimes V_2$ be a homomorphism of $L \otimes A$-modules. Because of the finite-dimensionality condition,

$$\mathrm{Hom}_K(M \otimes V_1, M \otimes V_2) \simeq \mathrm{End}_K(M) \otimes \mathrm{Hom}_K(V_1,V_2),$$

so we may write

$$\Phi = \sum_{i\in I} \varphi_i \otimes \alpha_i \tag{1.3}$$

for some finite set I, and some $\varphi_i \in \mathrm{End}_K(M)$, $\alpha_i \in \mathrm{Hom}_K(V_1,V_2)$. The condition of Φ to be a homomorphism of $L \otimes A$-modules can be then written as

$$\sum_{i\in I} \varphi_i(x \bullet m) \otimes \alpha_i(a \bullet_1 v) - \sum_{i\in I} (x \bullet \varphi_i(m)) \otimes (a \bullet_2 \alpha_i(v)) = 0 \tag{1.4}$$

for any $x \in L$, $a \in A$, $m \in M$, $v \in V_1$, where $\bullet_1$ and $\bullet_2$ denote the A-action on V_1 and V_2, respectively. Substitute in this equality $a = 1$:

$$\sum_{i\in I} (\varphi_i(x \bullet m) - x \bullet \varphi_i(m)) \otimes \alpha_i(v) = 0.$$

Hence, we may assume that $\varphi_i(x \bullet m) = x \bullet \varphi_i(m)$ holds for all φ_i's and any $x \in L$, $m \in M$; in other words, each φ_i's is an endomorphism of M as an L-module. Substituting this back to (1.4), we have:

$$\sum_{i\in I} \varphi_i(x \bullet m) \otimes (\alpha_i(a \bullet_1 v) - a \bullet_2 \alpha_i(v)) = 0.$$

Hence there is a partition of the set of indices $I = I_1 \cup I_2$ such that $\varphi_i(LM) = 0$ for $i \in I_1$, and $\alpha_i(a \bullet_1 v) = a \bullet_2 \alpha_i(v)$ for $i \in I_2$, and the statement of the lemma follows. $\square$

It seems to be difficult to derive similarly general results about isomorphisms between current modules over a current Lie algebra, as it is not clear how the condition of bijectivity of the map (1.3) could be related to conditions imposed on the summands φ_i's and α_i's. However, in the particular case where the number of homomorphisms between the respective modules is rather "small", we can use the previous lemma to establish

Theorem 1. *Let L be a Lie algebra having a cohomologically nontrivial finite-dimensional module M such that either $LM = M$, or M does not contain a trivial submodule, and A an associative commutative algebra with unit having an infinite set $\mathcal{V}$ of non-isomorphic unital modules of a fixed finite dimension. Then in each of the following cases:*

(i) $\dim \mathrm{End}_L(M) = 1$,
(ii) *for any two non-isomorphic modules V_1, $V_2 \in \mathcal{V}$, $\dim \mathrm{Hom}_A(V_1, V_2) \le 1$,*

the Lie algebra $L \otimes A$ has an infinite number of cohomologically nontrivial non-isomorphic modules of a fixed finite dimension.

Proof. Consider the set of $L\otimes A$-modules of the form $M \otimes V$, where $V \in \mathcal{V}$. By Lemma 1.1, all these modules are cohomologically nontrivial. Due to the condition imposed on M, the first summand in the right-hand side of (1.2) vanishes, hence by Lemma 1.2, in both cases (i) and (ii), each $L \otimes A$-homomorphism between two modules $M \otimes V_1$ and $M \otimes V_2$ can be represented as a decomposable map $\varphi \otimes \alpha$, where $\varphi \in \mathrm{End}_L(M)$ and $\alpha \in \mathrm{Hom}_A(V_1, V_2)$. Such a map is bijective (i.e. is an isomorphism of $L \otimes A$-modules) if and only if both tensor components φ and α are bijective. Hence two $L \otimes A$-modules $M \otimes V_1$ and $M \otimes V_2$ are isomorphic if and only if the A-modules V_1 and V_2 are isomorphic, whence the conclusion of the theorem. $\square$

Thus, to exhibit an example of a finite-dimensional Lie algebra having an infinite number of cohomologically nontrivial non-isomorphic modules of a fixed finite dimension (and thus answer Question 1 from [D1] in the case of characteristic zero, and provide a counterexample to the claim of Theorem 1 from [D1] in the case of positive characteristic), it is enough to exhibit algebras and modules satisfying the conditions of Theorem 1.

Take, for example, a Lie algebra L having a nontrivial, irreducible, cohomologically nontrivial module M: in characteristic zero, according to a converse to the Whitehead theorem ([Z2, Theorem]), we may take any Lie algebra which is not a direct sum of a semisimple algebra and a nilpotent algebra; in positive characteristic, by combination of Dzhumadil'daev's results – that any finite-dimensional Lie algebra has a finite-dimensional cohomologically nontrivial module ([D2, §1, Corollary 2]; also established independently in [FaS, Corollary 2.2]), and that triviality of any irreducible cohomologically nontrivial module implies nilpotency ([D2, §4, Theorem]) – we may take any nonnilpotent Lie algebra.

Over an algebraically closed base field, the condition (i) of Theorem 1 is satisfied then due to Schur's lemma, so in this case it is enough to pick any finite-dimensional associative commutative algebra A with an infinite number of modules of a fixed finite dimension – for example, any algebra of infinite representation type. Over an arbitrary (infinite) field, however, we exhibit a concrete example of an algebra A having an infinite number of modules satisfying the property (ii).

Example. Let A be a 3-dimensional local algebra whose radical is square-zero: that is, A has a basis $\{1, x, y\}$, with multiplication $x^2 = xy = yx = y^2 = 0$. This is one of the simplest examples of algebras of infinite representation type. Finite-dimensional indecomposable representations of A are known: to our knowledge, they were described first in [HR, Proposition 5]. (A more general problem about canonical forms of a pair of mutually annihilating nilpotent – and not necessary square-zero – matrices was solved in the famous paper [GP]; see also

[NRSB]). From them, it is possible to pick an infinite family $\mathcal{V}$ satisfying the property (ii) of Theorem 1. This can be done in multiple ways, and one of the simplest is the following.

Consider a family V_t of unital 2-dimensional A-modules, parametrized by element of the base field t, defined as follows: V_t has a basis $\{u, v\}$ with the following A-action:

$$x \bullet u = v, \quad y \bullet u = tv, \quad x \bullet v = 0, \quad y \bullet v = 0.$$

Straightforward computation show that for different $t, s \in K$, the modules V_t and V_s are non-isomorphic, and that $\mathrm{Hom}_A(V_t, V_s)$ is 1-dimensional, linearly spanned by the map $u \mapsto v$, $v \mapsto 0$.

Note that for any finite-dimensional Lie algebra of positive characteristic, the number of *irreducible* cohomologically nontrivial finite-dimensional modules is finite, by the same arguments used in the proof of [D1, Theorem 1]. In view of this, perhaps the Question 1 from [D1] should be reformulated as follows:

Problem 1. *Do there exist finite-dimensional Lie algebras over a field of characteristic zero having an infinite number of cohomologically nontrivial irreducible non-isomorphic modules of a fixed finite dimension?*

We conjecture that the answer to this question is negative, and a possible way to prove this is to employ the standard ultraproduct considerations, allowing to deduce the zero characteristic case from the cases of positive characteristics.

Problem 2. *What would be a "noncommutative" version of Theorem 1, where current Lie algebras are replaced by Lie algebras of the form $\mathsf{sl}_n(A)$ or $\mathsf{gl}_n(A)$ for an associative (and not necessary commutative) algebra A?*

We expect that in this way one may to obtain further interesting connections between cohomology of Lie algebras and representation theory of associative algebras.

2. Poisson structures on current and Kac-Moody algebras

In this section, the characteristic of the base field K is assumed to be as big as needed, in particular, to allow all denominators appearing in the formulas. Thus, in the results about general current Lie algebras below (Theorem 2, Corollary 2.1, Lemmas 2.1–2.3), we assume the characteristic is $\neq 2, 3$, while when dealing with simple Lie algebras and, in particular, with formulas like (2.3), the characteristic is assumed to be zero.

The classical example of a vector space of functions on a manifold equipped, from one hand, with operation of point-wise function multiplication, and, from another hand, with Poisson bracket, leads to the abstract notion of a Poisson algebra: a vector space A with two algebra structures: one, Lie, denoted by $[\cdot, \cdot]$, and another, associative commutative, denoted by $\star$ with compatibility condition saying that Lie multiplication by each element is a derivation of a commutative algebra structure:

$$[z, x \star y] = [z, x] \star y + x \star [z, y] \tag{2.1}$$

for any $x, y, z \in A$.

It is only natural to consider a more general situation, when condition of commutativity of $\star$ is dropped, arriving at so-called *noncommutative Poisson algebras*. Such algebras were studied in many papers ([Ku1] is just one of them). As in [Ku1], we adopt a "Lie-centric" (as opposed to "associative-centric") viewpoint, according to which one fixes a Lie algebra L and considers all possible multiplications $\star$ on the vector space L such that $(L, [\cdot, \cdot], \star)$ forms a Poisson algebra.

We go, however, a bit further, and drop associativity condition of $\star$, thus retaining only the compatibility condition (2.1) (and, of course, Lieness of $[\cdot, \cdot]$). So, given a Lie algebra L, we call a *Poisson structure on* L any algebra structure $\star$ on the vector space L such that

(2.1) holds. If $\star$ satisfies additional conditions like commutativity, associativity, etc., we will speak about commutative, associative, etc., Poisson structures on L.

Note that a Poisson structure on a Lie algebra L is nothing but an L-module homomorphism from $L \otimes L$ to L. As such, the set of all Poisson structures on L forms a vector space under usual operations of additions and multiplication on scalar of linear maps from $L \otimes L$ to L.

There is a canonical decomposition of an L-module $L \otimes L$ into the direct sum of commutative and anti-commutative components: $L \otimes L = (L \wedge L) \oplus (L \vee L)$. Accordingly, the space of Poisson structures on L decomposes into the direct sum of two spaces, consisting of commutative and anti-commutative Poisson structures on L.

Obviously, for every Lie algebra L, a bilinear map $L \times L \to L$ proportional to the Lie multiplication, is a Poisson structure on L (what amounts to the Jacobi identity). Of course, such Poisson structures are not very interesting, and one wishes to consider all Poisson structures on L modulo them. A bit more generally, we may wish to eliminate Poisson structures which can be written in the form $x \star y = \omega([x,y])$ for some linear map $\omega : L \to L$. Then (2.1) together with the Jacobi identity implies that

$$[z, \omega([x,y])] = \omega([z,[x,y]]) \tag{2.2}$$

for any $x, y, z \in L$. Note that this is nothing but the *centroid* (i.e., the set of linear maps on an algebra commuting with all multiplications) of the commutant $[L, L]$. Moreover, when L is simple and the base field is algebraically closed, the centroid $\mathrm{Cent}(L)$ of L coincides with multiplications on the elements of the base field ([Ja, Chapter 10, §1, Theorem 1]), so we arrive in this case at Poisson structures which are proportional to Lie multiplications we started with.

Centroid also appears in a different way in our context: if $\star$ is a Poisson structure on a Lie algebra L with a nonzero center $\mathrm{Z}(L)$, then (2.1) implies that for any element $z \in \mathrm{Z}(L)$, the left and right Poisson multiplication on z, i.e. the maps $x \mapsto z \star x$ and $x \mapsto x \star z$, are elements of the centroid of L.

Poisson structures of type (2.2) will be called *trivial*, and the quotient of the space of all Poisson structures on a Lie algebra L by the trivial ones, will be denoted by $\mathscr{P}(L)$. Poisson structures, whose representatives form a basis of $\mathscr{P}(L)$, will be called *basic Poisson structures*. Note that trivial Poisson structures are always anti-commutative, so any nonzero commutative Poisson structure is nontrivial.

Somewhat similarly, on a skew-symmetrization $A^{(-)}$ of an associative algebra A, one always have a commutative Poisson structure defined by the "Jordan" multiplication: $x \circ y = \frac{1}{2}(xy + yx)$ for $x, y \in A$. Though such "Jordan" Poisson structures are defined only for Lie algebras which are skew-symmetrizations of associative ones, for the Lie algebra $\mathsf{sl}_n(K)$ we can define a related Poisson structure by adding a term "correcting" for tracelessness:

$$X \star Y = \frac{1}{2}(XY + YX) - \frac{1}{n}\,\mathrm{Tr}(XY)E, \tag{2.3}$$

where X, Y are traceless matrices of order n, and E is the identity matrix. It is easy to check that this is indeed a Poisson structure on $\mathsf{sl}_n(K)$ (what amounts to the fact that $(X,Y) \mapsto \mathrm{Tr}(XY)$ is a symmetric invariant form on $\mathsf{sl}_n(K)$), and it will be called a *standard commutative Poisson structure*. Note that this is no longer a Jordan structure. The case of $\mathsf{sl}_2(K)$ is degenerate, as multiplication defined by (2.3) vanishes.

Our goal is to describe Poisson structures on current Lie algebras $L \otimes A$ in terms of L and A. We do this under mild technical assumptions. The Poissonity condition (2.1) is similar to the cocycle equation, and the same technique used to compute low-degree cohomology of $L \otimes A$, applies.

As the centroid, essentially, is "almost" isomorphic to the space of trivial Poisson structures, we first determine the centroid of current Lie algebras. This is generally known from the literature, perhaps in a slightly less general or explicit form.

Lemma 2.1. *Let L be a Lie algebra, A an associative commutative algebra with unit, and one of L, A is finite-dimensional. Then*

$$\mathrm{Cent}(L \otimes A) \simeq \mathrm{Cent}(L) \otimes A + \mathrm{Hom}_K(L/[L,L], (\mathrm{Z}(L) + [L,L])/[L,L]) \otimes \mathrm{End}_K(A).$$

Any element of $\mathrm{Cent}(L \otimes A)$ *can be written as the sum of decomposable maps $\varphi \otimes \alpha$, $\varphi \in \mathrm{End}_K(L)$, $\alpha \in \mathrm{End}_K(A)$, of one of the following form:*

(i) $\varphi \in \mathrm{Cent}(L)$ *and* $\alpha = \mathrm{R}_u$ *for some* $u \in A$*;*

(ii) $\varphi(L) \subseteq \mathrm{Z}(L)$ *and* $\varphi([L,L]) = 0$.

This generalizes [Kr, Lemma 5.1], and overlaps with [BeNe, Corollary 2.23] and [Gü, Remark 2.19.1].

Proof. Let $\Phi \in \mathrm{Cent}(L \otimes A)$. Due to the finite-dimensionality assumption, we may write $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$ for suitable linear maps $\varphi_i \in \mathrm{End}_K(L)$ and $\alpha_i \in \mathrm{End}_K(A)$. The centroidity condition then reads

$$\sum_{i \in I} \varphi_i([x,y]) \otimes \alpha_i(ab) - [x, \varphi_i(y)] \otimes a\alpha_i(b) = 0 \tag{2.4}$$

for any $x, y \in L$, $a, b \in A$. Substituting here $a = 1$, we get

$$\sum_{i \in I} (\varphi_i([x,y]) - [x, \varphi_i(y)]) \otimes \alpha_i(b) = 0,$$

and hence we may assume that all φ_i's belong to $\mathrm{Cent}(L)$. Taking this into account, (2.4) can be rewritten as

$$\sum_{i \in I} \varphi_i([x,y]) \otimes (\alpha_i(ab) - a\alpha_i(b)) = 0.$$

Hence there is a partition of the set of indices $I = I_1 \cup I_2$ such that $\varphi_i([x,y]) = [x, \varphi_i(y)] = 0$, $x, y \in L$ for $i \in I_1$, and $\alpha_i(ab) = a\alpha_i(b)$, $a, b \in A$ for $i \in I_2$. The latter condition is equivalent to $\alpha_i(a) = au_i$ for some $u_i \in A$. $\square$

If L is perfect, i.e. $[L,L] = L$, then the second summand in the isomorphism of Lemma 2.1 disappears:

Corollary 2.1. *Let L be a perfect Lie algebra, A an associative commutative algebra with unit, and one of L, A is finite-dimensional. Then* $\mathrm{Cent}(L \otimes A) \simeq \mathrm{Cent}(L) \otimes A$.

After this warm-up, let us turn to computation of Poisson structures on current and close to them Lie algebras. Let us call a Poisson structure $\star$ on a Lie algebra L *left skew* if it satisfies the identity $[x,y] \star z = [y,z] \star x$, and *right skew* if it satisfies the identity $x \star [y,z] = y \star [z,x]$.

Theorem 2. *Let L be a perfect Lie algebra not having nonzero left skew and right skew Poisson structures, A an associative commutative algebra with unit, and one of L, A is finite-dimensional. Then $\mathscr{P}(L \otimes A) \simeq \mathscr{P}(L) \otimes A$, and basic Poisson structures on $L \otimes A$ can be chosen as*

$$(x \otimes a) \bigstar (y \otimes b) = (x \star y) \otimes abu, \tag{2.5}$$

where $x, y \in L$, $a, b \in A$, for some Poisson structure $\star$ on L, and $u \in A$.

Proof. Let $\Phi = \sum_{i \in I} \varphi_i \otimes \alpha_i$, where $\varphi_i : L \times L \to L$, $\alpha_i : A \times A \to A$ are bilinear maps, be a Poisson structure on $L \otimes A$. Writing the Poissonity condition (2.1) for triple $x \otimes a$, $y \otimes b$, $z \otimes c$, we get:

$$\sum_{i \in I} [z, \varphi_i(x,y)] \otimes c\alpha_i(a,b) - \varphi_i([z,x],y) \otimes \alpha_i(ca,b) - \varphi_i(x,[z,y]) \otimes \alpha_i(a,cb) = 0. \tag{2.6}$$

Substitute here $c = 1$:

$$\sum_{i \in I} ([z, \varphi_i(x,y)] - \varphi_i([z,x],y) - \varphi_i(x,[z,y])) \otimes \alpha_i(a,b) = 0.$$

Hence, replacing φ_i's by their appropriate linear combinations, we get that each φ_i satisfies $[z, \varphi_i(x, y)] = \varphi_i([z, x], y) + \varphi_i(x, [z, y])$ and hence is a Poisson structure on L. Substituting this back to (2.6), we get:

$$\sum_{i \in I} \varphi_i([z, x], y) \otimes (c\alpha_i(a, b) - \alpha_i(ca, b)) + \varphi_i(x, [z, y]) \otimes (c\alpha_i(a, b) - \alpha_i(a, cb)) = 0. \tag{2.7}$$

Symmetrizing this with respect to x, z and y, z, we get respectively:

$$\sum_{i \in I} (\varphi_i(x, [z, y]) + \varphi_i(z, [x, y]) \otimes (c\alpha_i(a, b) - \alpha_i(a, cb)) = 0$$

and

$$\sum_{i \in I} (\varphi_i([z, x], y) + \varphi_i([y, x], z)) \otimes (c\alpha_i(a, b) - \alpha_i(ca, b)) = 0.$$

The vanishing of the first tensor factors here is equivalent for φ_i to be right and left skew respectively, and hence they cannot vanish for a nonzero φ_i. Hence for each $i \in I$, $c\alpha_i(a, b) - \alpha_i(a, cb) = 0$ and $c\alpha_i(a, b) - \alpha_i(ca, b) = 0$, what implies $\alpha_i(ab) = ab\alpha_i(1)$. It is easy to check that each $\varphi_i \otimes \alpha_i$ for such α_i's is a Poisson structure on $L \otimes A$.

So, we see that each Poisson structure on $L \otimes A$ can be written as a sum of Poisson structures of the form (2.5), and hence the space of all Poisson structures on $L \otimes A$ is isomorphic to the tensor product of the space of all Poisson structures on L with A. To prove the asserted isomorphism, note that perfectness of L implies perfectness of $L \otimes A$, and hence the space of trivial Poisson structures on $L \otimes A$ is isomorphic to the centroid of $L \otimes A$, and by Corollary 2.1, to $\mathrm{Cent}(L) \otimes A$. But $\mathrm{Cent}(L)$ is precisely the space of trivial Poisson structures on L, so factoring out the isomorphism of the spaces of Poisson structures established above by the spaces of trivial Poisson structures, we get the desired isomorphism. ☐

Though it seems that it is impossible to get a definitive result describing Poisson structures on $L \otimes A$ in terms of L and A in the most general case, by using much more elaborate (and cumbersome) arguments along these lines, one may significantly relax restrictions on L. That way, nonperfectness of L becomes responsible for Poisson structures on $L \otimes A$ of the form $\psi \otimes \alpha$ with conditions like $\psi([L, L], L) = 0$, skew Poisson structures φ on L become responsible for Poisson structures on $L \otimes A$ of the form $\varphi \otimes \alpha$, where α is composed from various generalized derivations on A, etc.

Corollary 2.2. *Let $\mathfrak{g}$ be a finite-dimensional simple Lie algebra and A an associative commutative algebra with unit, defined over an algebraically closed field K of characteristic 0. Then $\mathscr{P}(\mathfrak{g} \otimes A) \simeq A$ in the case $\mathfrak{g} = \mathsf{sl}_n(K)$, $n \geq 3$, and vanishes in all other cases. In the former case the basic Poisson structures can be chosen as*

$$(X \otimes a) \star (Y \otimes b) = \Big(\frac{1}{2}(XY + YX) - \frac{1}{n}\operatorname{Tr}(XY)E\Big) \otimes abu,$$

where $X, Y \in \mathsf{sl}_n(K)$, $a, b \in A$, for some $u \in A$.

Note that in the case $A = K$, this corollary reduces to [BO, Lemma 3.1], which can be interpreted as a description of Poisson structures on simple finite-dimensional Lie algebras (also rediscovered in [LR, Lemma 8] and [Ku2]). In fact, we base on that result. It is also curious to note that exactly the same dichotomy between $\mathsf{sl}_n(K)$, $n \geq 3$, and the other simple types occurs in the second and third degree (co)homology $\mathrm{H}_2(\mathfrak{g} \otimes A, \mathfrak{g} \otimes A)$ and $\mathrm{H}_3(\mathfrak{g} \otimes A, K)$ (see [C, Propositions 1 and 2]; also follows from a careful substitution of $\mathfrak{g}$ into the isomorphism displayed at p. 94 in the published version, and p. 17 in the arXiv version of [Z1]).

Proof of Corollary 2.2. To be able to apply Theorem 2, we should demonstrate that $\mathfrak{g}$ does not have nonzero left and right skew Poisson structures. This is not difficult: consider, for example, a left skew Poisson structure $\star$ on $\mathfrak{g}$. By the above-mentioned [BO, Lemma 3.1], for the case $\mathfrak{g} = \mathsf{sl}_n(K)$, $n \geq 3$, we have

$$X \star Y = \frac{\lambda}{2}(XY + YX) - \frac{\lambda}{n}\,\mathrm{Tr}(XY)E + \mu(XY - YX)$$

for some $\lambda, \mu \in K$. Taking X and Z to be the block-diagonal matrices with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in the upper-left corner, and zero everywhere else, respectively (these are just a semisimple element and a corresponding root vector in the subalgebra isomorphic to $\mathsf{sl}_3(K)$), we get $[X, X] \star Z = 0$ and $[X, Z] \star X = Z \star X$ is equal to the block-diagonal matrix with $\begin{pmatrix} 0 & 0 & \frac{\lambda}{2} - \mu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in the upper-left corner, whence $\lambda - 2\mu = 0$.

Taking now X' to be the block-diagonal matrix with $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$ in the upper-left corner, and zero everywhere else, we get $[X', X'] \star Z = 0$ and $[X', Z] \star X' = 3Z \star X'$ is equal to the block-diagonal matrix with $\begin{pmatrix} 0 & 0 & -\frac{3}{2}\lambda - 9\mu \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ in the upper-left corner, whence $\lambda + 6\mu = 0$.

Thus $\lambda = \mu = 0$ and $\star$ vanishes.

For the all other types of $\mathfrak{g}$, $\star$ is proportional to the Lie multiplication on $\mathfrak{g}$, so the left skewness amounts to the identity $[[x, y], z] = [[y, z], x]$, which obviously does not hold in a simple Lie algebra.

The case of right skew Poisson structures is similar.

A consecutive application of Theorem 2 and [BO, Lemma 3.1] concludes the proof. $\square$

In order to apply this further to Kac–Moody algebras, we need to consider semidirect sums of a current Lie algebra with an algebra of derivations of A, and their central extensions. Consider a Lie algebra defined as the vector space $(L \otimes A) \oplus Kz \oplus \mathcal{D}$, where $\mathcal{D}$ is a nontrivial Lie algebra of derivations of A, with the following multiplication:

$$\begin{aligned} [x \otimes a, y \otimes b] &= [x, y] \otimes ab + \langle x, y\rangle \xi(a, b) z \\ [x \otimes a, d] &= x \otimes d(a) \end{aligned} \tag{2.8}$$

for $x, y \in L$, $a, b \in A$, $d \in \mathcal{D}$, and z is a central element. Here $\langle \cdot, \cdot \rangle$ is a nonzero symmetric bilinear invariant form on L, i.e., a symmetric bilinear map satisfying the equality $\langle [x, y], z\rangle + \langle y, [x, z]\rangle = 0$ for any $x, y, z \in L$; and ξ is a nonzero $\mathcal{D}$-invariant element of $\mathrm{HC}^1(A)$, the first-degree cyclic cohomology of A, i.e., a skew-symmetric bilinear map $\xi : A \times A \to K$ satisfying the following conditions: $\xi(ab, c) + \xi(ca, b) + \xi(bc, a) = 0$, and $\xi(d(a), b) + \xi(a, d(b)) = 0$ for any $a, b, c \in A$, $d \in \mathcal{D}$. Note that the former condition implies $\xi(1, A) = 0$.

Such algebra is a nonsplit central extension of the semidirect sum $(L \otimes A) \Subset \mathcal{D}$. For brevity, we will call algebras with multiplication (2.8) *extended current Lie algebras*. Specializing this construction to the case where K is an algebraically closed field of characteristic zero, $L = \mathfrak{g}$, a simple finite-dimensional Lie algebra, $\langle \cdot, \cdot \rangle$ is the Killing form on $\mathfrak{g}$, $A = K[t, t^{-1}]$, the algebra of Laurent polynomials, $\mathcal{D} = Kt\frac{d}{dt}$, and $\xi(f, g) = Res(g\frac{df}{dt})$ for $f, g \in K[t, t^{-1}]$, we get non-twisted affine Kac-Moody algebras (see [Ka, Chapter 7]).

In the next series of lemmas and corollaries to Theorem 2, we gradually approach to computation of Poisson structures on some class of extended current Lie algebras.

Lemma 2.2. *Let $\mathcal{L}$ be an extended current Lie algebra such that L is a perfect centerless Lie algebra, and one of L, A is finite-dimensional. Then any element in* $\mathrm{Cent}(\mathcal{L})$ *is of the form*

$$
\begin{aligned}
x \otimes a &\mapsto x \otimes au \\
d &\mapsto ud + \varphi(d)z \\
z &\mapsto \lambda z
\end{aligned}
\tag{2.9}
$$

where $x \in L$, $a \in A$, $d \in \mathcal{D}$, for some $u \in A$ such that $\mathcal{D}(u) = 0$, a linear map $\varphi : \mathcal{D} \to K$ such that $\varphi([\mathcal{D}, \mathcal{D}]) = 0$, and $\lambda \in K$ such that $\xi(au, b) = \lambda\xi(a, b)$ for any $a, b \in A$.

Proof. Let $\Phi \in \mathrm{Cent}(\mathcal{L})$. Note the the center of $\mathcal{L}$ coincides with Kz. It is straightforward that Φ preserves the center of $\mathcal{L}$, and hence induces the map on the quotient $\mathcal{L}/Kz$, which belongs to the centroid of $\mathcal{L}/Kz \simeq (L \otimes A) \inplus \mathcal{D}$.

Let us determine the structure of the latter first. Let $\Psi \in \mathrm{Cent}((L \otimes A) \inplus \mathcal{D})$. Then

$$\Psi(L \otimes A) = \Psi([L, L] \otimes A) = [L \otimes A, \Psi(L \otimes A)] \subseteq L \otimes A.$$

Consequently Ψ, being restricted to $L \otimes A$, belongs to $\mathrm{Cent}(L \otimes A)$, and by Corollary 2.1, $\Psi(x \otimes a) = \psi(x) \otimes au$ for some $\psi \in \mathrm{Cent}(L)$ and $u \in A$. The centroidity condition $[\Psi(x \otimes a), d] = \Psi([x \otimes a, d])$, for any $x \in L$, $a \in A$, $d \in \mathcal{D}$, implies that $\mathcal{D}(u) = 0$. Next, the centroidity condition $[x \otimes a, \Psi(d)] = \Psi([x \otimes a, d])$ is equivalent to

$$[x \otimes a, \Psi(d)] = \psi(x) \otimes d(a)u. \tag{2.10}$$

Writing the $L \otimes A$-valued component of the restriction of Ψ to $\mathcal{D}$ in the form $\sum_{i \in I} x_i \otimes \alpha_i$ for some $x_i \in L$ and linear maps $\alpha_i : \mathcal{D} \to A$, and substituting in (2.10) $a = 1$, we get

$$\sum_{i \in I} [x, x_i] \otimes \alpha_i(d) = 0.$$

Thus all x_i's belong to the center of L and hence vanish. This shows that $\Psi(\mathcal{D}) \subseteq \mathcal{D}$, and the condition (2.10) can be rewritten as

$$x \otimes \Psi(d)(a) = \psi(x) \otimes d(a)u$$

for any $x \in L$, $a \in A$, $d \in \mathcal{D}$. This implies $\psi(x) = \alpha x$ and $\Psi(d) = \alpha ud$ for some $\alpha \in K$.

Consequently, $\mathrm{Cent}((L \otimes A) \inplus \mathcal{D}) \simeq A^{\mathcal{D}}$, and any element in that centroid is of the form

$$x \otimes a + d \mapsto x \otimes au + ud,$$

where $x \in L$, $a \in A$, $d \in \mathcal{D}$, for some $u \in A$ such that $\mathcal{D}(u) = 0$.

Returning to the algebra $\mathcal{L}$, we may now write Φ in the form

$$
\begin{aligned}
\Phi(x \otimes a) &= x \otimes au + \tau(x \otimes a)z \\
\Phi(d) &= ud + \varphi(d)z \\
\Phi(z) &= \lambda z
\end{aligned}
$$

for some $\tau \in \mathrm{Hom}_K(L \otimes A, K)$, $\varphi \in \mathrm{Hom}_K(\mathcal{D}, K)$, $\lambda \in K$, and $u \in A$ as above. Then the centroidity condition $\Phi([x \otimes a, y \otimes b]) = [\Phi(x \otimes a), y \otimes b]$ is equivalent to

$$\tau([x, y] \otimes ab) = \langle x, y \rangle (\xi(au, b) - \lambda\xi(a, b))$$

for any $x, y \in L$, $a, b \in A$. The left-hand side here is skew-symmetric in x, y, while the right-hand side is symmetric, therefore they vanish separately, what implies the vanishing of τ, and the identity $\xi(au, b) = \lambda\xi(a, b)$.

The centroidity condition $\Phi([d, f]) = [\Phi(d), f]$ for any $d, f \in \mathcal{D}$ is equivalent to $\varphi([d, f]) = 0$.

It is easy to check that the map defined by (2.9) subject the all specified conditions, indeed belongs to the centroid of $\mathcal{L}$. □

We will need also the following statement about bilinear invariant forms on current Lie algebras.

Lemma 2.3. *Let L be a Lie algebra, A an associative commutative algebra with unit, and one of L, A is finite-dimensional. Then each bilinear invariant form on $L \otimes A$ can be represented as the sum of decomposable forms $\varphi \otimes \alpha$, $\varphi : L \times L \to K$, $\alpha : A \times A \to K$ satisfying one of the two following conditions:*

(i) *$\varphi([x,y],z) = \varphi([z,x],y)$ for any $x,y,z \in L$, and $\alpha(a,b) = \beta(ab)$ for some linear map $\beta : A \to K$;*

(ii) *$\varphi([L,L],L) = 0$.*

Proof. The proof is absolutely similar to those of Theorem 2, and, in fact, is given in [Z3, Theorem 2]. The condition of symmetricity of bilinear forms used there at the very end and does not affect the main body of the proof. Specializing to the case where A contains a unit, we get the statement of the lemma. ☐

Corollary 2.3. *Let A be an associative commutative algebra with unit, and $\mathcal{D}$ an abelian Lie algebra of derivations of A. Then*

$$\mathscr{P}((\mathfrak{g} \otimes A) \Subset \mathcal{D}) \simeq \begin{cases} U(\mathcal{D},A) \oplus U(\mathcal{D},A) \oplus \{u \in A \mid \mathcal{D}(u) = \mathcal{D}(A)u = 0\}, & \\ \hfill \mathfrak{g} = \mathsf{sl}_n(K), n \geq 3 & \\ U(\mathcal{D},A) \oplus U(\mathcal{D},A), \text{ otherwise,} & \end{cases}$$

where $U(\mathcal{D},A) = \{\beta \in \mathrm{Hom}_K(\mathcal{D},A) \mid \mathcal{D}(\beta(\mathcal{D})) = 0\}$. The basic Poisson structures can be chosen as follows (assuming $x,y \in \mathfrak{g}$, $a,b \in A$, $d,d' \in \mathcal{D}$):

(i) *$(x \otimes a) \star (y \otimes b) = (\frac{1}{2}(xy + yx) - \frac{1}{n}\mathrm{Tr}(xy)E) \otimes abu$, where $u \in A$ such that $\mathcal{D}(u) = 0$ and $\mathcal{D}(A)u = 0$;*
$\mathcal{D} \star (\mathfrak{g} \otimes A) = (\mathfrak{g} \otimes A) \star \mathcal{D} = \mathcal{D} \star \mathcal{D} = 0$;
these Poisson structures exist in the case $\mathfrak{g} = \mathsf{sl}_n(K)$, $n \geq 3$ only;

(ii) *$d \star (x \otimes a) = x \otimes a\beta(d)$, $d \star d' = \beta(d)d'$, where $\beta \in U(\mathcal{D},A)$;*
$(\mathfrak{g} \otimes A) \star (\mathfrak{g} \otimes A) = (\mathfrak{g} \otimes A) \star \mathcal{D} = 0$;

(iii) *$(x \otimes a) \star d = x \otimes a\gamma(d)$, $d \star d' = \gamma(d')d$, where $\gamma \in U(\mathcal{D},A)$;*
$(\mathfrak{g} \otimes A) \star (\mathfrak{g} \otimes A) = \mathcal{D} \star (\mathfrak{g} \otimes A) = 0$.

Proof. Each Poisson structure on $(\mathfrak{g} \otimes A) \Subset \mathcal{D}$, being restricted to $(\mathfrak{g} \otimes A) \times (\mathfrak{g} \otimes A)$, can be decomposed into the sum of two maps Φ and Ψ with values in $\mathfrak{g} \otimes A$ and $\mathcal{D}$, respectively. Writing the Poissonity condition for elements of $\mathfrak{g} \otimes A$, we get

$$\begin{aligned} (2.11)\quad [z \otimes c, \Phi(x \otimes a, y \otimes b)] + [z \otimes c, \Psi(x \otimes a, y \otimes b)] \\ = \Phi([z,x] \otimes ca, y \otimes b) + \Phi(x, [z,y] \otimes ca) \end{aligned}$$

and

$$\Psi([z,x] \otimes ca, y \otimes b) + \Psi(x, [z,y] \otimes ca) = 0$$

for any $x,y,z \in \mathfrak{g}$, $a,b,c, \in A$. Hence, Ψ is a $\mathcal{D}$-valued invariant bilinear form on $\mathfrak{g} \otimes A$. Now apply Lemma 2.3. As $\mathfrak{g}$ is perfect, forms of type (ii) vanish. Bilinear maps on $\mathfrak{g}$ satisfying the condition in (i) can be decomposed into the sum of symmetric and skew-symmetric ones. Symmetric maps coincide with symmetric invariant forms on $\mathfrak{g}$ and hence are proportional to the Killing form $\langle \cdot, \cdot \rangle$, and skew-symmetric maps vanish by [Z3, Lemma 2]. Hence Ψ can be written in the form $\Psi(x \otimes a, y \otimes b) = \langle x,y \rangle \beta(ab)$ for some linear map $\beta : A \to \mathcal{D}$.

Now we may proceed as in the proof of Theorem 2. Writing Φ in the form $\sum_{i \in I} \varphi_i \otimes \alpha_i$ and substituting this into (2.11), we get the equality (2.6) with

$$(2.12)\qquad \langle x,y \rangle z \otimes \beta(ab)(c)$$

instead of 0 at the right-hand side. This expression vanishes under the substitution $c = 1$, and we may proceed exactly as in the proof of Theorem 2 to arrive at the equality (2.7) with the same (2.12) instead of 0 at the right-hand side. Setting in the latter $x = y = z$, we get the vanishing left-hand side, and $\langle x,x \rangle x \otimes \beta(ab)(c)$ at the right-hand side. Now picking

$x \in \mathfrak{g}$ such that $\langle x, x\rangle \neq 0$, we get $\beta = 0$. Hence $\Psi = 0$, Φ is a Poisson structure on $\mathfrak{g} \otimes A$, and by Corollary 2.2,

$$\Phi(x \otimes a, y \otimes b) = (x \star y) \otimes abu + [x, y] \otimes abv$$

in the case of $\mathfrak{g} = \mathsf{sl}_n(K)$, $n \geq 3$, where $\star$ is the standard commutative Poisson structure on $\mathsf{sl}_n(K)$, and $u, v \in A$, and $\Phi(x \otimes a, y \otimes b) = [x, y] \otimes abv$ for the all other types of $\mathfrak{g}$.

Now writing the Poissonity condition for triple $y \otimes b, d, x \otimes a$, where $x, y \in \mathfrak{g}$, $a, b \in A$, $d \in \mathcal{D}$, we get that $d(u) = d(v) = 0$ for any $d \in \mathcal{D}$.

Now denoting, by abuse of notation, the whole Poisson structure on $(\mathfrak{g} \otimes A) \Subset \mathcal{D}$ by the same letter Φ, let us see what happens when one of the arguments of Φ lies in $\mathcal{D}$. Writing the Poissonity condition for triple $y \otimes b, d, x \otimes a$, we get

$$[y \otimes b, \Phi(d, x \otimes a)] = \Phi(y \otimes d(b), x \otimes a) + \Phi(d, [y, x] \otimes ba), \tag{2.13}$$

what implies $\Phi(\mathcal{D}, \mathfrak{g} \otimes A) = \Phi(\mathcal{D}, [\mathfrak{g}, \mathfrak{g}] \otimes A) \subseteq \mathfrak{g} \otimes A$. We may write

$$\Phi(d, x \otimes a) = \sum_{i \in I} \varphi_i(x) \otimes \alpha_i(d, a)$$

for suitable (bi)linear maps $\varphi_i : \mathfrak{g} \to \mathfrak{g}$ and $\alpha_i : \mathcal{D} \times A \to A$. Substituting this into (2.13) and setting there $b = 1$, we get

$$\sum_{i \in I} (\varphi_i([y, x]) - [y, \varphi_i(x)]) \otimes \alpha_i(d, a) = 0.$$

Thus we may assume that $\varphi_i([y, x]) - [y, \varphi_i(x)] = 0$ for each $i \in I$, i.e. each φ_i belongs to the centroid of $\mathfrak{g}$ and hence is a multiplication by an element of the base field. But then we have $\Phi(d, x \otimes a) = x \otimes \alpha(d, a)$ for a suitable bilinear map $\mathcal{D} \times A \to A$ and all $x \in \mathfrak{g}$, $a \in A$, $d \in \mathcal{D}$, and the Poissonity condition (2.13) reduces to

$$[y, x] \otimes (b\alpha(d, a) - \alpha(d, ba)) = \Phi(y \otimes d(b), x \otimes a)$$

for any $x, y \in \mathfrak{g}$, $a, b \in A$, $d \in \mathcal{D}$. If $\mathfrak{g} = \mathsf{sl}_n(K)$, $n \geq 3$, then setting in this equality $x = y$, we get $(x \star x) \otimes d(b)au = 0$, and picking x such that $x \star x \neq 0$ (for example, $x = \mathrm{diag}(1, 1, -2, 0, \dots, 0)$), we get that $\mathcal{D}(A)u = 0$, and the whole equality reduces to

$$b\alpha(d, a) - \alpha(d, ba) = ad(b)v. \tag{2.14}$$

If $\mathfrak{g}$ is different from $\mathsf{sl}_n(K)$, $n \geq 3$, then the term with the standard commutative Poisson structure is absent in Φ, and we also get the equality (2.14). Setting in the latter equality $a = 1$, we see that it is equivalent to the condition $\alpha(d, a) = a\beta(d) - d(a)v$ for some linear map $\beta : \mathcal{D} \to A$. The Poissonity condition for triple $d, d', x \otimes a$ for $x \in \mathfrak{g}$, $a \in A$, $d, d' \in \mathcal{D}$, together with the condition $\mathcal{D}(v) = 0$ and abelianity of $\mathcal{D}$, implies then that $\mathcal{D}(\beta(\mathcal{D})) = 0$.

Quite similarly, we have that

$$\Phi(x \otimes a, d) = a\gamma(d) + d(a)v$$

for any $x \in \mathfrak{g}$, $a \in A$, $d \in \mathcal{D}$, and a suitable linear map $\gamma : \mathcal{D} \to A$ such that $\mathcal{D}(\gamma(\mathcal{D})) = 0$.

Finally, the Poissonity condition written for three elements of $\mathcal{D}$ implies that $[\mathcal{D}, \Phi(\mathcal{D}, \mathcal{D})] = 0$ and hence $\Phi(\mathcal{D}, \mathcal{D}) \subseteq \mathcal{D}$, and the Poissonity condition for triple $x \otimes a, d, d'$ implies

$$\Phi(d, d')(a) = d(a)\gamma(d') + d'(a)\beta(d)$$

for any $a \in A, d, d' \in \mathcal{D}$.

Summarizing all this together, it is easy to single out the trivial Poisson structures: those are precisely the maps involving the element v. Writing the generic form of the Poisson structure modulo these maps, we get the desired statement. ☐

Corollary 2.4. *Let $\mathfrak{L}$ be an extended current Lie algebra over an algebraically closed base field K of characteristic zero, L is simple finite-dimensional Lie algebra, $\mathcal{D} = Kd$ is one-dimensional,* $\dim A > 2$*, and $A^{\mathcal{D}} = K$. Then the basic Poisson structures on $\mathcal{L}$ can be chosen from the following list:*

(i) $(x \otimes a) \star (y \otimes b) = \langle x, y \rangle f(ab) z$,
$z \star d = -d \star z = z$,
$(\mathfrak{g} \otimes A) \star d = d \star (\mathfrak{g} \otimes A) = d \star d = 0$,
where $f : A \to K$ *is a linear map such that* $f(ad(b)) = \xi(a, b)$ *for any* $a, b \in A$*;*

(ii) $(x \otimes a) \star (y \otimes b) = \langle x, y \rangle f(ab) z$,
$d \star \mathcal{L} = \mathcal{L} \star d = 0$,
where $f : A \to K$ *is a linear map such that* $f(ad(b)) = 0$ *for any* $a, b \in A$*;*

(iii) $d \star (x \otimes a) = x \otimes a$,
$d \star d = d$,
$(\mathfrak{g} \otimes A) \star (\mathfrak{g} \otimes A) = (\mathfrak{g} \otimes A) \star d = z \star d = d \star z = 0$*;*

(iv) $(x \otimes a) \star d = x \otimes a$,
$d \star d = d$,
$(\mathfrak{g} \otimes A) \star (\mathfrak{g} \otimes A) = d \star (\mathfrak{g} \otimes A) = z \star d = d \star z = 0$*;*

(v) $d \star d = z$,
$(\mathfrak{g} \otimes A) \star \mathcal{L} = \mathcal{L} \star (\mathfrak{g} \otimes A) = z \star d = d \star z = 0$.

Additionally, for all of them $(\mathfrak{g} \otimes A) \star z = z \star (\mathfrak{g} \otimes A) = z \star z = 0$.

Proof. If Φ is a Poisson structure on $\mathcal{L}$, then $\Phi(z, \cdot)$ and $\Phi(\cdot, z)$ are elements of the centroid of $\mathcal{L}$. According to Lemma 2.2, $\mathrm{Cent}(\mathcal{L})$ is 2-dimensional, linearly spanned by the identity map, and the map sending d to z, and $(\mathfrak{g} \otimes A) \oplus Kz$ to zero. Hence we may write

$$
\begin{aligned}
\Phi(x \otimes a, z) = \Phi(z, x \otimes a) &= \alpha x \otimes a \\
\Phi(d, z) &= \alpha d + \beta z \\
\Phi(z, d) &= \alpha d + \gamma z \\
\Phi(z, z) &= \alpha z
\end{aligned}
\tag{2.15}
$$

where $x \in \mathfrak{g}$, $a \in A$, for some $\alpha, \beta, \gamma \in K$.

Restrict Φ to $((\mathfrak{g} \otimes A) \inplus Kd) \times ((\mathfrak{g} \otimes A) \inplus Kd)$, and decompose the restriction as the sum of a $((\mathfrak{g} \otimes A) \inplus Kd)$-valued bilinear map Ψ, and a Kz-valued bilinear map. The Poissonity condition, taking into account equalities (2.15), then implies

$$
\begin{aligned}
&[t \otimes c, \Psi(x \otimes a, y \otimes b)] \\
&\quad = \Psi([t, x] \otimes ca, y \otimes b) + \Psi(x \otimes a, [t, y] \otimes cb) + \alpha(\langle t, x \rangle \xi(c, a) y \otimes b + \langle t, y \rangle \xi(c, b) x \otimes a)
\end{aligned}
\tag{2.16}
$$

and

$$
\begin{aligned}
[d, \Psi(x \otimes a, y \otimes b)] &= \Psi([d, x \otimes a], y \otimes b) + \Psi(x \otimes a, [d, y \otimes b]) \\
[x \otimes a, \Psi(d, y \otimes b)] &= \Psi([x \otimes a, d], y \otimes b) + \Psi(d, [x, y] \otimes ab) + \alpha \langle x, y \rangle \xi(a, b) d \\
[x \otimes a, \Psi(y \otimes b, d)] &= \Psi([x, y] \otimes ab, d) + \Psi(y \otimes b, [x \otimes a, d]) + \alpha \langle x, y \rangle \xi(a, b) d \\
[d, \Psi(d, x \otimes a)] &= \Psi(d, [d, x \otimes a]) \\
[d, \Psi(x \otimes a, d)] &= \Psi([d, x \otimes a], d) \\
[x \otimes a, \Psi(d, d)] &= \Psi([x \otimes a, d], d) + \Psi(d, [x \otimes a, d]) \\
[d, \Psi(d, d)] &= 0
\end{aligned}
\tag{2.17}
$$

for any $x, y, t \in \mathfrak{g}$, $a, b, c \in A$.

Decompose further Ψ into the $(\mathfrak{g} \otimes A)$-valued and Kd-valued components, and proceed as in the proof of Corollary 2.3. By the same reasoning, the Kd-valued component, being a Kd-valued invariant form on $\mathfrak{g} \otimes A$, can be written as $\langle x, y \rangle \otimes \beta(ab) d$ for some linear map $\beta : A \to K$. Substituting this into equality (2.16), we get the Poissonity condition for the $\mathfrak{g} \otimes A$-valued component, up to the term

$$
\langle x, y \rangle t \otimes \beta(ab) d(c) + \alpha(\langle t, x \rangle \xi(c, a) y \otimes b + \langle t, y \rangle \xi(c, b) x \otimes a)
$$

instead of (2.12) at the right-hand side. Proceeding exactly as in the proof of Corollary 2.3, we get that this term vanishes whenever $x = y = t$, i.e.

$$\alpha(\xi(c,a)b + \xi(c,b)a) = \beta(ab)d(c)$$

for any $a, b, c \in A$. An easy calculation yields $\beta(a)d(b) + \beta(b)d(a) = 0$ for any $a, b \in A$. If β is nonzero, this implies that d vanishes on $\operatorname{Ker}\beta$, whence $\operatorname{Ker}\beta = K$ and $\dim A = 2$, a contradiction. Hence β vanishes, $\alpha = 0$, and equalities (2.16)–(2.17) imply that Ψ is a Poisson structure on $(\mathfrak{g} \otimes A) \Subset \mathcal{D}$.

Now look at Corollary 2.3. Since $\mathcal{D}$ is one-dimensional, and $A^{\mathcal{D}} = K$, the space $U(\mathcal{D}, A)$ is one-dimensional, linearly spanned by the map sending d to 1. The conditions on u in the structures of type (i) imply the vanishing of structures of that type. Hence the space of Poisson structures on $(\mathfrak{g} \otimes A) \Subset Kd$ is linearly spanned by the Lie bracket on that algebra, and the following two bilinear maps:

$$\begin{aligned} (d, x \otimes a) &\mapsto x \otimes a \\ (x \otimes a, d) &\mapsto 0 \end{aligned}$$

and

$$\begin{aligned} (d, x \otimes a) &\mapsto 0 \\ (x \otimes a, d) &\mapsto x \otimes a, \end{aligned}$$

both of them sending (d, d) to d, and $(\mathfrak{g} \otimes A) \times (\mathfrak{g} \otimes A)$ to zero.

Going back to Φ, we may now write a generic Poisson structure on $\mathcal{L}$, modulo the maps proportional to the Lie bracket on $\mathcal{L}$, as:

$$\begin{aligned} \Phi(x \otimes a, y \otimes b) &= \Theta(x \otimes a, y \otimes b)z \\ \Phi(d, x \otimes a) &= \lambda x \otimes a + \Gamma(x \otimes a)z \\ \Phi(x \otimes a, d) &= \mu x \otimes a + \Gamma'(x \otimes a)z \\ \Phi(d, d) &= (\lambda + \mu)d + \eta z \end{aligned}$$

for some (bi)linear maps $\Theta : (\mathfrak{g} \otimes A) \times (\mathfrak{g} \otimes A) \to K$, $\Gamma, \Gamma' : \mathfrak{g} \otimes A \to K$, and $\lambda, \mu, \eta \in K$. Taking into account (2.15) (with $\alpha = 0$), the Poissonity condition for Φ now amounts to the following equalities:

$$\Theta([t,x] \otimes ac, y \otimes b) + \Theta(x \otimes a, [t,y] \otimes bc) = 0 \tag{2.18}$$

$$\Theta(x \otimes d(a), y \otimes b) + \Theta(x \otimes a, y \otimes d(b)) = 0 \tag{2.19}$$

$$\Theta(x \otimes d(a), y \otimes b) + \Gamma([x,y] \otimes ab) + \beta\langle x, y\rangle \xi(a,b) = 0 \tag{2.20}$$

$$\Theta(y \otimes b, x \otimes d(a)) + \Gamma'([x,y] \otimes ab) + \gamma\langle x, y\rangle \xi(a,b) = 0 \tag{2.21}$$

for any $x, y, t \in \mathfrak{g}$, $a, b, c \in A$. Substituting into (2.20) $a = 1$, we get that Γ vanishes, similarly for Γ'. The equality (2.18) is the condition of invariance of the bilinear form Θ. Referring to Lemma 2.3 and [Z3, Lemma 2] in the same way as in the proof of Corollary 2.3, we have $\Theta(x \otimes a, y \otimes b) = \langle x, y\rangle f(ab)$ for some linear map $f : A \to K$.

The rest is straightforward: (2.19) now amounts to $f(d(A)) = 0$, and (2.20) and (2.21), to $f(d(a)b) + \beta\xi(a,b) = 0$ and $f(bd(a)) - \gamma\xi(a,b) = 0$, respectively. Hence $\gamma = -\beta$, and the statement of the corollary follows. □

Corollary 2.5. *The space of nontrivial Poisson structures on an affine non-twisted Kac-Moody algebra is* 3*-dimensional, spanned by structures defined in* (iii), (iv) *and* (v) *of Corollary* 2.4.

Proof. It is easy to see that the map f satisfying the conditions in cases (i) and (ii) of Corollary 2.4, vanishes in the Kac-Moody case. □

This generalizes the result from [Ku1], where it is proved that any *associative* Poisson structure on an affine Kac-Moody algebra is a linear combination of the specified 3 structures. The proof in [Ku1] goes by lengthy case-by-case computations with the corresponding root systems. As any of the structures of types (iii), (iv) and (v) is associative, it turns out that any Poisson structure on an affine Kac–Moody algebra is decomposed into the sum of an associative structure, and a trivial structure proportional to the Lie bracket.

The same approach can be used to describe Poisson structures in many other cases which involve tensor products in that or another way: twisted Kac–Moody algebras, toroidal Lie algebras, semisimple Lie algebras over a field of positive characteristic, variations of Kantor-Koecher-Tits construction, Lie algebras graded by root systems, etc. Some of these computations will be technically (much) more cumbersome, but all of them should follow the same scheme. Yet another similar computation will be presented in the next section.

3. Poisson structures on $\mathsf{sl}_n(A)$

We may try to consider a noncommutative analog of the problem in the previous section. Namely, for unital associative algebras A and B, whether one can describe Poisson structures on the Lie algebra $(A \otimes B)^{(-)}$ in terms of A and B, similar to those of Theorem 2? It seems that in general the answer is negative. Let, for example, both A and B be a semidirect sum of $K1$ and a nilpotent associative algebra N of degree 3 (i.e., $N^3 = 0$). It is not difficult to see that then $(A \otimes B)^{(-)}$ is an abelian Lie algebra. Poisson structures on an abelian Lie algebra L coincide with the whole space of bilinear maps $L \times L \to L$, so, in that case Poisson structures on $(A \otimes B)^{(-)}$ and on $A^{(-)}$ and $B^{(-)}$ appear to be not related at all.

This example suggests that to be able to get such a description, algebras A and B, or at least one of them, should be "far from nilpotent". We are indeed able to do so in the particular (and arguably one of the most interesting) case when B is, in a sense, as much not nilpotent as possible – that is, is isomorphic to the full matrix algebra. In this case the Lie algebra $(A \otimes B)^{(-)}$ is isomorphic to the Lie algebra $\mathsf{gl}_n(A)$. Actually, we will even consider not that algebra itself, but its close (and more interesting) cousin $\mathsf{sl}_n(A)$. One of the reasons to do so is that in the case of $\mathsf{gl}_n(A)$ the picture is obscured by a lot of "degenerate", not very interesting, and having a cumbersome formulation Poisson structures related to the fact that the center of $\mathsf{gl}_n(K)$ is not zero.

Thus, we will get, in a sense, a "noncommutative" version of Corollary 2.2.

Recall that for an associative algebra A, the Lie algebra $\mathsf{sl}_n(A)$ is defined as subalgebra of $\mathsf{gl}_n(A)$ consisting of matrices over A with trace lying in $[A, A]$. There is a split extension of Lie algebras

$$0 \to \mathsf{sl}_n(A) \to \mathsf{gl}_n(A) \to A/[A, A] \to 0,$$

and, as a vector space, $\mathsf{sl}_n(A)$ can be identified with

$$(\mathsf{sl}_n(K) \otimes A) \oplus (E \otimes [A, A]) \subseteq \mathsf{gl}_n(K) \otimes A \simeq \mathsf{gl}_n(A)$$

subject to multiplication

$$\begin{aligned}[] [X \otimes a, Y \otimes b] &= XY \otimes ab - YX \otimes ba \\ &= \Big(XY - \frac{1}{n}\operatorname{Tr}(XY)E\Big) \otimes ab - \Big(YX - \frac{1}{n}\operatorname{Tr}(XY)E\Big) \otimes ba + \frac{1}{n}\operatorname{Tr}(XY)E \otimes [a, b]. \end{aligned}$$

Note that the first tensor factors in the first two terms here are Poisson structures on $\mathsf{sl}_n(K)$.

Throughout this section, the characteristic of the base field K is assumed to be zero. This is to allow denominators in the formula above providing realization of $\mathsf{sl}_n(K)$ we prefer to work with. A more careful analysis would allow to prove essentially the same result assuming the characteristic $\neq 2, 3$, but we will not take this pain.

Theorem 3. *Let A be an associative algebra with unit. The basic Poisson structures on $\mathsf{sl}_n(A)$, $n \geq 3$, can be chosen as follows:*

(i) $(X\otimes a)\star(Y\otimes b) = \frac{1}{2}(XY\otimes abu + YX\otimes bau) + \mathrm{Tr}(XY)E\otimes\gamma(ab)$ *for either* $X \in \mathsf{sl}_n(K)$, $a \in A$, *or* $X = E, a \in [A,A]$, *the same for* Y,b, *where* $u \in Z(A)$ *and* $\gamma : A \to \mathrm{Z}(A)$ *is a linear map such that* $\gamma([A,A]) = \gamma([[A,A],A]A) = 0$, *and* $\frac{1}{n}au + \gamma(a) \in [A,A]$ *for any* $a \in A$;

(ii) $(E\otimes a)\star(X\otimes b) = X\otimes\alpha(a)b$ *for* $a \in [A,A]$ *and either* $X \in \mathsf{sl}_n(K)$, $b \in A$, *or* $X = E$, $b \in [A,A]$, *where* $\alpha : [A,A] \to \mathrm{Z}(A)$ *is a linear map such that* $\alpha([[A,A],[A,A]]) = 0$; $(\mathsf{sl}_n(K)\otimes A)\star\mathsf{sl}_n(A) = 0$.

(iii) $(X\otimes a)\star(E\otimes b) = X\otimes\beta(b)a$ *for* $b \in [A,A]$ *and either* $X \in \mathsf{sl}_n(K)$, $a \in A$, *or* $X = E$, $a \in [A,A]$, *where* $\beta : [A,A] \to \mathrm{Z}(A)$ *is a linear map such that* $\beta([[A,A],[A,A]]) = 0$; $\mathsf{sl}_n(A)\star(\mathsf{sl}_n(K)\otimes A) = 0$.

(iv) $(E\otimes a)\star(E\otimes b) = E\otimes\delta(a,b)$ *for* $a,b \in [A,A]$, *where* $\delta : [A,A]\times[A,A] \to \mathrm{Z}(A)\cap[A,A]$ *is a bilinear map such that* $\delta([c,a],b) + \delta(a,[c,b]) = 0$ *for any* $a,b,c \in [A,A]$; $(\mathsf{sl}_n(K)\otimes A)\star\mathsf{sl}_n(A) = \mathsf{sl}_n(A)\star(\mathsf{sl}_n(K)\otimes A) = 0$.

Here $\mathrm{Z}(A)$ denotes, as usual, the center of A.

Proof. Let Φ be a Poisson structure on $\mathsf{sl}_n(A)$. Identifying $\mathsf{sl}_n(A)$ with a subspace of $\mathsf{gl}_n(K)\otimes A$ as described above, extend Φ in an arbitrary way to the whole $(\mathsf{gl}_n(K)\otimes A)\times(\mathsf{gl}_n(K)\otimes A)$ (say, by picking a complementary subspace and letting Φ vanish on it).

Let

$$\Phi = \sum_{i\in I}\varphi_i\otimes\alpha_i, \tag{3.1}$$

where φ_i's are bilinear maps on $\mathsf{gl}_n(K)$, and α_i's are bilinear maps on A. Writing the Poissonity condition for triple $X\otimes a, Y\otimes b, Z\otimes c \in \mathsf{sl}_n(A)$ (that is, either $X \in \mathsf{sl}_n(K)$ and $a \in A$, or $X = E$ and $a \in [A,A]$, similarly for $Y\otimes b$ and $Z\otimes c$), we get:

$$\begin{aligned}\sum_{i\in I} & Z\varphi_i(X,Y)\otimes c\alpha_i(a,b) - \varphi_i(X,Y)Z\otimes\alpha_i(a,b)c \\ & -\varphi_i(ZX,Y)\otimes\alpha_i(ca,b) + \varphi_i(XZ,Y)\otimes\alpha_i(ac,b) \\ & -\varphi_i(X,ZY)\otimes\alpha_i(a,cb) + \varphi_i(X,YZ)\otimes\alpha_i(a,bc) = 0.\end{aligned} \tag{3.2}$$

Like in the proof of Theorem 2, substituting here $c = 1$, we can assume that each φ_i satisfies the Poissonity condition

$$[Z,\varphi_i(X,Y)] = \varphi_i([Z,X],Y) + \varphi_i(X,[Z,Y]) \tag{3.3}$$

for any $X,Y \in \mathsf{gl}_n(K), Z \in \mathsf{sl}_n(K)$.

Each φ_i, being restricted to $\mathsf{sl}_n(K)\times\mathsf{sl}_n(K)$, can be decomposed into the sum of two linear maps with values in $\mathsf{sl}_n(K)$ and KE, respectively. The $\mathsf{sl}_n(K)$-valued summand in this decomposition is a Poisson structure on $\mathsf{sl}_n(K)$, and the KE-valued one is an invariant bilinear form on $\mathsf{sl}_n(K)$. The latter is known to be proportional to the Killing form (see, for example, [Bo, Chap. 1, §6, Exercises 7(b) and 18(a,b)]), and hence is proportional to $\mathrm{Tr}(XY)$. Then by [BO, Lemma 3.1], each φ_i, being restricted to $\mathsf{sl}_n(K)\times\mathsf{sl}_n(K)$, belongs to the vector spaces of bilinear maps generated by $(X,Y)\mapsto XY$, $(X,Y)\mapsto YX$, and $(X,Y)\mapsto\mathrm{Tr}(XY)E$. Hence Φ, being restricted to $(\mathsf{sl}_n(K)\otimes A)\times(\mathsf{sl}_n(k)\otimes A)$, can be written in the form

$$\Phi(X\otimes a, Y\otimes b) = XY\otimes\alpha(a,b) + YX\otimes\beta(a,b) + \mathrm{Tr}(XY)E\otimes\gamma(a,b) \tag{3.4}$$

for some bilinear maps $\alpha, \beta, \gamma : A \times A \to A$. Then the Poissonity condition for triple $X \otimes a$, $Y \otimes b$, $Z \otimes c$, $X, Y, Z \in \mathsf{sl}_n(K)$, $a, b, c \in A$, reads

$$
\begin{aligned}
& XYZ \otimes (\alpha(a,bc) - \alpha(a,b)c) \\
& + XZY \otimes (\alpha(ac,b) - \alpha(a,cb)) \\
& + YXZ \otimes (\beta(ac,b) - \beta(a,b)c) \\
& + YZX \otimes (\beta(a,bc) - \beta(ca,b)) \\
& + ZXY \otimes (c\alpha(a,b) - \alpha(ca,b)) \\
& + ZYX \otimes (c\beta(a,b) - \beta(a,cb)) \\
& + \mathrm{Tr}(XY)Z \otimes [c, \gamma(a,b)] \\
& + E \otimes \Big(\mathrm{Tr}(XYZ)(\gamma(a,bc) - \gamma(ca,b)) + \mathrm{Tr}(XZY)(\gamma(ac,b) - \gamma(a,cb)) \Big) = 0.
\end{aligned}
\tag{3.5}
$$

Setting in (3.5) $X = Y = Z$, we get:

$$
\begin{aligned}
X^3 \otimes \Big(& [c, \alpha(a,b)] - \alpha([c,a],b) - \alpha(a,[c,b]) + [c, \beta(a,b)] - \beta([c,a],b) - \beta(a,[c,b]) \Big) \\
& + \mathrm{Tr}(X^2)X \otimes [c, \gamma(a,b)] - \mathrm{Tr}(X^3)E \otimes \Big(\gamma([c,a],b) + \gamma(a,[c,b]) \Big) = 0,
\end{aligned}
$$

for any $X \in \mathsf{sl}_n(K)$ and $a, b, c \in A$. Taking here, for example, $X = \mathrm{diag}(1, 1, -2, 0, \dots, 0)$, we see that each of the summands vanishes separately. In particular, the second tensor factor vanishes, and, so is the corresponding term in (3.5) (those containing $\mathrm{Tr}(XY)Z$). Moreover, taking into account the vanishing of the third tensor factor, the last term in (3.5) can be rewritten as

$$(\mathrm{Tr}(XYZ) - \mathrm{Tr}(XZY))E \otimes (\gamma(a,bc) - \gamma(ca,b)).$$

Now take X, Y, Z as follows: all their elements are zero besides the 3×3 upper left corner, and the latter is $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ for X, $\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ for Y, and $\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$ for Z (so, actually we are dealing with 3×3 matrices)†. It is straightforward to check that the 6 matrices formed by all triple products of X, Y, Z are linearly independent, and $\mathrm{Tr}(XYZ) - \mathrm{Tr}(XZY) = 0$. This implies that the second tensor factors in each of the first 6 terms of (3.5) vanish. But then (3.5) reduces to just the last term, and picking any 3 matrices such that $\mathrm{Tr}(XYZ) - \mathrm{Tr}(XZY) \neq 0$ shows that the second tensor factor in the last term vanishes too. By elementary manipulations with these vanishing conditions, we have $\alpha(a,b) = abu$ for some $u \in \mathrm{Z}(A)$, and $\beta(a,b) = bav$ for some $v \in \mathrm{Z}(A)$.

Now consider the equality

$$\mathrm{Tr}(XYZ)(\gamma(a,bc) - \gamma(ca,b)) + \mathrm{Tr}(XZY)(\gamma(ac,b) - \gamma(a,cb)) = 0.$$

Taking here, for example, $X = Z = \mathrm{diag}(1, -1, 0, \dots, 0)$ and $Y = \mathrm{diag}(\begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, 0, \dots, 0)$, we see that each of the two summands vanish separately, and, hence, $\gamma(a,b) = \gamma'(ab)$ for a linear map $\gamma' : A \to \mathrm{Z}(A)$ such that $\gamma'([A,A]) = 0$.

Clearly, $\mathsf{sl}_n(A)$ is closed under Φ if and only if

$$\frac{1}{n}(abu + bav) + \gamma'(ab) \in [A,A]$$

for any $a, b \in A$, what can be rewritten as $\frac{1}{n}a(u+v) + \gamma'(a) \in [A,A]$ for any $a \in A$.

Now let us see what happens with values of Φ when one of the arguments belongs to $E \otimes [A,A]$. Substituting in the Poissonity condition for φ_i's (3.3) $X = E$ or $Y = E$, we get that the map $\varphi_i(E, \cdot)$ or $\varphi_i(\cdot, E)$, respectively, commutes with $\mathrm{ad}\, Z$ for any $Z \in \mathsf{sl}_n(K)$. But

†It seems that almost any other 3 matrices will do. See Problem 3 at the end of this section.

then it commutes with $\operatorname{ad} Z$ for any $Z \in \mathsf{gl}_n(K)$, i.e., belongs to the centroid of $\mathsf{gl}_n(K)$. Hence each of the $\varphi_i(E, X)$ and $\varphi_i(X, E)$ coincides with a scalar multiplication by X, and $\varphi_i(E, E)$ is proportional to E. This implies that the values of Φ for the arguments in question can be written in the form

$$\Phi(E \otimes a, X \otimes b) = X \otimes \beta'(a, b)$$
$$\Phi(X \otimes b, E \otimes a) = X \otimes \gamma'(a, b)$$
$$\Phi(E \otimes a, E \otimes b) = E \otimes \delta(a, b),$$

for $X \in \mathsf{sl}_n(K)$ and $a \in [A, A], b \in A$ in the first two cases, and $a, b \in [A, A]$ in the third one, and where $\beta', \gamma', \delta : A \to A$ are some bilinear maps.

Now writing the Poissonity condition for triples $E \otimes a$, $X \otimes b$, $Y \otimes c$, $X, Y \in \mathsf{sl}_n(K)$, $a \in [A, A]$, $b, c \in A$, and $E \otimes a$, $X \otimes b$, $E \otimes c$, $X \in \mathsf{sl}_n(K)$, $a, c \in [A, A]$, $b \in A$, we get respectively:

$$XY \otimes (\beta'(a, b)c - \beta'(a, bc) + b[c, a]v) + YX \otimes (\beta'(a, cb) - c\beta'(a, b) + [c, a]bu) \\ + \operatorname{Tr}(XY)E \otimes \alpha'([c, a]b) = 0$$

and

$$X \otimes ([c, \beta'(a, b)] - \beta'([c, a], b) - \beta'(a, [c, b])) = 0.$$

Then, obviously, each of the summands in the first of these two equalities vanishes separately (take, for example, $X = Y = \operatorname{diag}(1, -1, 0, \dots, 0)$, and $X = \operatorname{diag}(1, -1, 0, \dots, 0)$, $Y = \operatorname{diag}(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, 0, \dots, 0))$. Consequently, $\alpha'([[A, A], A]A) = 0$, and elementary transformations with conditions on β' entail that

$$\beta'(a, b) = abu + bav + \beta''(a)b$$

for any $a \in [A, A]$, $b \in B$, where $\beta'' : [A, A] \to \mathrm{Z}(A)$ is a linear map satisfying the condition $\beta''([[A, A], [A, A]]) = 0$.

Similarly, the Poissonity condition for triples $X \otimes a$, $E \otimes b$, $Y \otimes c$ and $X \otimes a$, $E \otimes b$, $E \otimes c$ entails that

$$\gamma'(a, b) = abu + bav + \gamma''(b)a$$

for any $a \in A$, $b \in [A, A]$, where $\gamma'' : [A, A] \to \mathrm{Z}(A)$ is a linear map satisfying the condition $\gamma''([[A, A], [A, A]]) = 0$.

At least, the Poissonity condition for triples $E \otimes a$, $E \otimes b$, $X \otimes c$, $a, b \in [A, A]$, $c \in A$, and $E \otimes a$, $E \otimes b$, $E \otimes c$, $a, b, c \in [A, A]$ yields

$$X \otimes ([c, \delta(a, b)] - \gamma'([c, a], b) - \beta'(a, [c, b])) = 0$$

and

$$E \otimes ([c, \delta(a, b)] - \delta([c, a], b) - \delta(a, [c, b])) = 0,$$

what implies

$$\delta(a, b) = abu + bav + \beta''(a)b + \gamma''(b)a + \delta'(a, b)$$

for any $a, b \in [A, A]$, where $\delta' : [A, A] \times [A, A] \to \mathrm{Z}(A)$ is a bilinear map such that $\delta'([c, a], b) + \delta'(a, [c, b]) = 0$ for any $a, b, c \in [A, A]$. For $\mathsf{sl}_n(A)$ to be closed under Φ, $abu + bav + \delta'(a, b)$ should lie in $[A, A]$, what is equivalent to $ab(u + v) + \delta'(a, b) \in [A, A]$ for any $a, b \in [A, A]$.

It remains to collect all the obtained maps, to check in a straightforward way that they are indeed Poisson structures on $\mathsf{sl}_n(A)$ (in fact, we already did that in almost all the cases by considering various Poissonity conditions on basic elements of $\mathsf{sl}_n(A)$), and to take the quotient by trivial Poisson structures. As $\mathsf{sl}_n(A)$ is perfect, the trivial Poisson structures are of the form $(x, y) \mapsto \omega([x, y])$ where ω belongs to the centroid of $\mathsf{sl}_n(A)$. The latter can be computed in exactly the same way as the space of Poisson structures on $\mathsf{sl}_n(A)$ (though the computations are much simpler): the centroid of $\mathsf{sl}_n(A)$, $n \ge 3$, is isomorphic to $\mathrm{Z}(A)$ and consists of the maps

$$(X \otimes a, Y \otimes b) \mapsto XY \otimes abu - YX \otimes bau$$

for some $u \in \mathrm{Z}(A)$, where either $X \in \mathsf{sl}_n(K)$ and $a \in A$, or $X = E$ and $a \in [A,A]$, the same for Y and b (this is also noted without proof in [Kr, Remark 5.6]). □

Remark 1. Corollary 2.2 and Theorem 3 overlap in the case $\mathfrak{g} = \mathsf{sl}_n(K)$, $n \ge 3$, and A commutative.

Remark 2. One may treat the case of $\mathsf{sl}_2(A)$ along the same lines, but this case, as it is often happens in such situations, is much more cumbersome. Let us briefly outline the relevant reasonings, without going into details. As every Poisson structure on $\mathsf{sl}_2(K)$ is trivial, we may assume in (3.5) $\beta = -\alpha$. Using that, and the identity $\mathrm{Tr}(XYZ) + \mathrm{Tr}(XZY) = 0$ which holds for any three 2×2 traceless matrices, one can easily single out in (3.5), like in the general case, the term containing $\mathrm{Tr}(XY)Z$, but the remaining reasonings are cumbersome: there are not "enough" elements in $\mathsf{sl}_2(K)$ to separate all the terms in (3.5). Say, substituting into (3.5) all possible values of the standard $\mathsf{sl}_2(K)$-basis $\left\{ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \right\}$, we get a (highly redundant) homogeneous linear system with rational coefficients of 27 equations in 7 unknowns, which reduces to 4 independent relations. Using these relations, one can derive that

$$\alpha(a,b) = u(ab + ba) + (ab + ba)u$$

for some $u \in A$ satisfying some additional conditions, but the relation between γ and α is more involved.

When dealing with the identity (3.5), we were able to separate the terms by picking concrete matrices X, Y, Z. Computer experiments suggest that almost any 3 traceless matrices will do.

Problem 3. *Provide an (elegant) algebro-geometric, or linear-algebraic proof of the following statement: for any three 3×3 traceless matrices X, Y, Z in general position, the 7 matrices $XYZ, XZY, YXZ, YZX, ZXY, ZYX$, and $(\mathrm{Tr}(XYZ) - \mathrm{Tr}(XZY))E$, are linearly independent.*

At the end of this section, let us mention another widely studied problem amenable to our methods. Namely, people studied a lot Lie-admissible third power-associative structures compatible with a given Lie algebra structure. (Lie-admissibility alone is too general, and third-power associativity is one of the most natural additional conditions leading to a meaningful theory). This is known to be equivalent to description of all symmetric bilinear maps $\Phi : L \times L \to L$ on a given Lie algebra L, such that

$$[\Phi(x,y), z] + [\Phi(z,x), y] + [\Phi(y,z), x] = 0$$

for any $x, y, z \in L$ (see, for example, [JKL, §1]). The resemblance with the 2-cocycle equation, as well as with Poissonity condition, is evident.

Problem 4. *Describe Lie-admissible third power-associative structures on current, extended current, Kac-Moody Lie algebras, and on Lie algebras of the form $\mathsf{sl}_n(A)$, similarly to how it is done for Poisson structures in this and preceding sections.*

This should provide, among other, a conceptual approach to results of [JKL], similarly to those to Kubo's results about Poisson structures discussed in §2. These computations should be pretty much straightforward, though, perhaps, technically somewhat challenging at places.

4. Hom-Lie structures

Hom-Lie algebras (under different names, notably "q-deformed Witt or Virasoro") started to appear long time ago in the physical literature, in the constant quest for deformed, in that or another sense, Lie algebra structures bearing a physical significance. Recently there was a surge of interest in them, starting with the paper [HLS].

Recall that a *Hom-Lie algebra* L is an algebra with a skew-symmetric multiplication $[\cdot\,,\cdot]$ and a linear map $\varphi : L \to L$ such that the following generalization of the Jacobi identity, called the *Hom-Jacobi identity*, holds:

$$[[x,y],\varphi(z)] + [[z,x],\varphi(y)] + [[y,z],\varphi(x)] = 0 \tag{4.1}$$

for any $x,y,z \in L$. Further variations of this notion arise by requiring φ to be a homomorphism, automorphism, etc., of an algebra L with respect to the multiplication $[\cdot\,,\cdot]$.

If $(L,[\cdot\,,\cdot])$ is a Lie algebra, then a linear map $\varphi : L \to L$ turning it into a Hom-Lie algebra – i.e. such that (4.1) holds – is called a *Hom-Lie structure on* L. The set of all Hom-Lie structures on L will be denoted by $\mathrm{HomLie}(L)$. Obviously, it is a subspace of $\mathrm{End}_K(L)$ containing the identity map.

In this section, the characteristic of the base field K is assumed $\neq 2,3$.

Theorem 4. *Let L be a Lie algebra, A is an associative commutative algebra with unit, and one of L, A is finite-dimensional. Then*

$$\mathrm{HomLie}(L\otimes A) \simeq \mathrm{HomLie}(L)\otimes A + \{\varphi \in \mathrm{End}_K(L) \mid [[L,L],\varphi(L)] = 0\} \otimes \mathrm{End}_K(A).$$

Each Hom-Lie structure on $L\otimes A$ can be represented as a sum of decomposable Hom-Lie structures of the form $\varphi\otimes\alpha$, where $\varphi \in \mathrm{End}_K(L)$, $\alpha \in \mathrm{End}_K(A)$, of the following two types:

(i) *$\varphi \in \mathrm{HomLie}(L)$, $\alpha = \mathrm{R}_u$ for some $u \in A$;*
(ii) *$[[L,L],\varphi(L)] = 0$.*

Proof. As usual in our approach, decompose a Hom-Lie structure Φ on $L\otimes A$ as $\Phi = \sum_{i\in I}\varphi_i\otimes\alpha_i$ for suitable linear maps $\varphi_i : L \to L$ and $\alpha_i : A \to A$. The Hom-Jacobi identity (4.1) then reads

$$\sum_{i\in I}[[x,y],\varphi_i(z)]\otimes ab\alpha_i(c) + [[z,x],\varphi_i(y)]\otimes ca\alpha_i(b) + [[y,z],\varphi_i(x)]\otimes bc\alpha_i(a) = 0 \tag{4.2}$$

for any $x,y,z \in L$, $a,b,c \in A$.

Cyclically permuting in this equality x,y,z, and summing up the obtained 3 equalities, we get:

$$\sum_{i\in I}\Big([[x,y],\varphi_i(z)] + [[z,x],\varphi_i(y)] + [[y,z],\varphi_i(x)]\Big)\otimes\Big(ab\alpha_i(c) + ca\alpha_i(b) + bc\alpha_i(a)\Big) = 0.$$

Easy transformations show (at this place the assumption that the characteristic of the ground field $\neq 2,3$ is essential) that the vanishing of the second tensor factor here, $ab\alpha_i(c)+ca\alpha_i(b)+bc\alpha_i(a)$, implies the vanishing of α_i, hence the first tensor factor vanishes for each φ_i, i.e. each φ_i is a Hom-Lie structure on L. Writing the latter condition as

$$[[z,x],\varphi_i(y)] = -[[x,y],\varphi_i(z)] - [[y,z],\varphi_i(x)]$$

and substituting this back to (4.2), we get:

$$\sum_{i\in I}[[x,y],\varphi_i(z)]\otimes\Big(ab\alpha_i(c) - ca\alpha_i(b)\Big) + [[y,z],\varphi_i(x)]\otimes\Big(bc\alpha_i(a) - ca\alpha_i(b)\Big) = 0$$

for any $x,y,z \in L$, $a,b,c \in A$.

Symmetrizing the last equality with respect to x,z, we get:

$$\sum_{i\in I}\Big([[x,y],\varphi_i(z)] - [[y,z],\varphi_i(x)]\Big)\otimes\Big(ab\alpha_i(c) - bc\alpha_i(a)\Big) = 0.$$

The vanishing of the first tensor factor here, $[[x,y],\varphi_i(z)] - [[y,z],\varphi_i(x)]$, together with the Hom-Jacobi identity (4.1), implies that

$$[[L,L],\varphi_i(L)] = 0. \tag{4.3}$$

The vanishing of the second tensor factor, $ab\alpha_i(c) - bc\alpha_i(a)$, implies that $\alpha_i(a) = a\alpha_i(1)$. Hence we may split the indexing set into two subsets: $I = I_1 \cup I_2$ such that φ_i satisfies (4.3) for $i \in I_1$, and $\varphi_i \in \mathrm{HomLie}(L)$ and $\alpha_i(a) = au_i$ for some $u_i \in A$, for $i \in I_2$. It is obvious that for each $i \in I_1$, $i \in I_2$, the decomposable map $\varphi_i \otimes \alpha_i$ satisfies the Hom-Jacobi identity (4.2), and we are done. ☐

Hom-Lie structures on simple classical Lie algebras were described in [JL], and it is natural to try to extend this result to Kac-Moody algebras. However, a direct attempt to generalize Theorem 4 to extended current Lie algebras, like it is done for Poisson structures in §2, meets certain technical difficulties.

Problem 5. *Describe Hom-Lie structures on extended current and Kac-Moody Lie algebras, and on Lie algebras of the form* $\mathsf{sl}_n(A)$.

5. Dual operads and affinizations of Novikov algebras

The following is a well-known phenomenon from the operadic theory: if A and B are algebras over binary quadratic operads (Koszul) dual to each other, then their tensor product $A \otimes B$ equipped with the bracket

$$[a \otimes b, a' \otimes b'] = aa' \otimes bb' - a'a \otimes b'b \tag{5.1}$$

for $a, a' \in A, b, b' \in B$, becomes a Lie algebra ([GK, Theorem 2.2.6(b)]). The most famous pairs of dual operads are (Lie, associative commutative) and (associative, associative). In the first case, this constructions leads to current Lie algebras, and in the second one – to Lie algebras of the form $(A \otimes B)^{(-)}$ for two associative algebras A, B, of which $\mathsf{gl}_n(A)$ and $\mathsf{sl}_n(A)$ are important special cases.

We are going now to describe another broad and interesting class of Lie algebras, which fits into this scheme for yet another pair of dual operads.

The ground field K is arbitrary, unless specified otherwise.

Recall that an algebra is called *left Novikov* if it satisfies two identities:

$$(xy)z - x(yz) = (yx)z - y(xz) \tag{5.2}$$

and

$$(xy)z = (xz)y.$$

If an algebra satisfies the opposite identities (the order of multiplication is reversed):

$$(xy)z - x(yz) = (xz)y - x(zy) \tag{5.3}$$

and

$$x(yz) = y(xz),$$

then it is called *right Novikov*. By just *Novikov algebras* we will mean algebras which are either left or right Novikov.

Every associative commutative algebra is (both left and right) Novikov. An important feature of Novikov algebras is that they are Lie-admissible.

Let N be a left Novikov algebra, G a commutative semigroup (written multiplicatively), and $\chi : G \to K$ a map. Define a bracket on the tensor product $N \otimes K[G]$, where $K[G]$ is the semigroup algebra, as follows:

$$[x \otimes a, y \otimes b] = \Big(\chi(a)xy - \chi(b)yx\Big) \otimes ab, \tag{5.4}$$

where $x, y \in N$ and $a, b \in G$. This bracket is obviously anticommutative.

Lemma 5.1. *For a fixed G and χ as above, the bracket* (5.4) *satisfies the Jacobi identity for an arbitrary left Novikov algebra N if an only if*

$$\chi(ab) - \chi(ac) = \chi(b) - \chi(c) \tag{5.5}$$

for any $a, b, c \in G$.

Proof. A tedious, but straightforward check. □

Remark. If G contains a unit e, then the property (5.5) is equivalent to

$$\chi(ab) = \chi(a) + \chi(b) - \chi(e)$$

for any $a, b \in G$.

A map satisfying the property (5.5) will be called a *quasi-character of G*. If χ is a quasi-character of a commutative semigroup G, then the Lie algebra defined by the bracket (5.4) will be denoted as $N_\chi[G]$. Let us list a few interesting special cases of this construction.

(i) *Witt algebras.* $G = (\mathbb{Z}, +)$ (so $K[G] \simeq K[t, t^{-1}]$, the algebra of Laurent polynomials), $\chi =$ degree of the monomial, and $N = K$. This is the famous infinite-dimensional (two-sided) Witt algebra W with the basis $\{e_m | m \in \mathbb{Z}\}$ and multiplication $[e_m, e_n] = (m-n)e_{m+n}$, $m, n \in \mathbb{Z}$. A variation of this constructions assumes K has characteristic $p > 0$, and $G = \mathbb{Z}/p\mathbb{Z}$ (so $K[G] \simeq K[t]/(t^p)$), leading to the p-dimensional Witt algebra.

(ii) *Current algebra over a Witt algebra.* More generally, taking in the previous example N as an arbitrary associative commutative algebra instead of K, we get the tensor product of the corresponding Witt algebra with N.

(iii) *Skew-symmetrization of a Novikov algebra.* $G = \{1\}$ and $\chi(1) = 1$. This is the Lie algebra $N^{(-)}$, a skew-symmetrization of N.

(iv) *Current algebra over a skew-symmetrization of a Novikov algebra.* More generally, letting G be arbitrary and χ is identically 1, we get the current Lie algebra

$$N^{(-)} \otimes K[G] \simeq (N \otimes K[G])^{(-)}.$$

(v) *Affinization of a Novikov algebra.* $G = (\mathbb{Z}, *)$, where $n * m = n + m - 1$ (so $K[G] \simeq K[t, t^{-1}]$ with multiplication $t^m t^n = t^{n+m-1}$), $\chi =$ degree of the monomial. The resulting Lie bracket is

$$[x \otimes t^m, y \otimes t^n] = (mxy - nyx) \otimes t^{m+n-1},$$

where $x, y \in N$, $n, m \in \mathbb{Z}$. This construction, which appeared (in a somewhat implicit form) in the pioneering papers [GD] and [BaNo] (in the latter, under the name "Poisson brackets of hydrodynamic type"), appears also in various physical questions, and in the theory of vertex operator algebras (see references in [PB1], [PB2]). It resembles the construction of untwisted affine Kac-Moody algebras, and we will call it *affinization of the Novikov algebra N*.

(vi) *The Heisenberg–Virasoro algebra* (see [PB1]). Specializing example (v) to the case where N is the algebra with the basis $\{x, y\}$ and multiplication table

	x	y
x	x	0
y	y	0

we get a nilpotent extension of the two-sided Witt algebra $\langle e_m, h_m \,|\, m \in \mathbb{Z}\rangle$ with multiplication table

$$[e_m, e_n] = (m-n)e_{m+n}, \qquad [e_m, h_n] = -nh_{m+n}, \qquad [h_m, h_n] = 0 \tag{5.6}$$

for $m, n \in \mathbb{Z}$.

(vii) *The Schrödinger–Virasoro algebra* (see [PB2]). Specializing example (v) to the case where N is the algebra with the basis $\{x, y, z\}$ and multiplication table

	x	y	z
x	x	0	$\frac{1}{2}z$
y	y	0	0
z	z	y	0

we get a nilpotent extension of the two-sided Witt algebra $\langle e_m, h_m, f_r \,|\, m \in \mathbb{Z}, r \in \frac{1}{2} + \mathbb{Z}\rangle$ with multiplication rules between the basic elements, in addition to relations (5.6), as follows:

$$[e_m, f_r] = \big(\frac{m}{2} - r\big) f_{m+r}, \qquad [f_r, f_s] = (r - s) h_{r+s}, \qquad [h_m, f_r] = 0$$

for $m \in \mathbb{Z}$, $r, s \in \frac{1}{2} + \mathbb{Z}$.

As we see, most of the interesting instances of $N_\chi[G]$ are infinite-dimensional, physically-motivated Lie algebras of characteristic zero. We suggest that this construction may serve as an organizing principle in an entirely different, at the first glance, context – in the unruly world of simple finite-dimensional Lie algebras over fields of small characteristics.

Problem 6. *Is it possible to realize the so-called bi-Zassenhaus algebras over a field of characteristic* 2 *introduced in* [Ju]*, as Lie algebras of the form $N_\chi[G]$ for suitable N, G and χ?*

Bi-Zassenhaus algebras form a family parametrized by two integers $g \geq 2, h \geq 1$ and are constructed as follows. Consider a Lie algebra with the basis $\{e_{(i,\alpha)} \,|\, i \in \mathbb{Z}/2\mathbb{Z}, \alpha \in \mathbb{Z}/2^{g+h}\mathbb{Z}\}$ and multiplication table:

$$\begin{aligned}
[e_{(0,\alpha)}, e_{(0,\beta)}] &= (\alpha + \beta) e_{(0,\alpha+\beta)} \\
[e_{(0,\alpha)}, e_{(1,\beta)}] &= (\alpha + \beta) e_{(1,\alpha+\beta)} \\
[e_{(1,\alpha)}, e_{(1,\beta)}] &= (\alpha^{2^g-1} + \beta^{2^g-1}) e_{(0,\alpha+\beta)}.
\end{aligned}$$

This algebra possess an ideal which is a simple Lie algebra looking, at the first glance, very different from the simple modular Lie algebras of Cartan type†. Note that it is not a skew-symmetrization of a Novikov algebra, so G in such realization should contain more than one element. On the other hand, dimension of N should be at least 3, as a seemingly direct approach – take a 2-dimensional Novikov algebra to parametrize the first coordinate in the set of indices – does not work, as a quick glance at the list of such algebras (for example, in [BM, §2]) reveals.

Problem 7. *The same question for Melikyan algebras – a series of simple Lie algebras over a field of characteristic* 5 *which are neither of classical, nor of Cartan type, and is peculiar to that characteristic (see* [S, §4.3]*).*

This question is, apparently, more difficult.

It is remarkable (though pretty much straightforward) that, actually, the bracket (5.4) is a particular case of (5.1). Namely, as noted in [D3, §4], the left Novikov and right Novikov operads are dual to each other, hence if A is a left Novikov algebra and B is a right Novikov one, their tensor product $A \otimes B$ equipped with the bracket (5.1), is a Lie algebra. If G is a commutative semigroup and χ its quasi-character, then the semigroup algebra $K[G]$ equipped with multiplication $a \cdot b = \chi(a)ab$ for $a, b \in G$, becomes a right Novikov algebra, and the bracket (5.4) becomes a particular case of the bracket (5.1) with $A = N$ and $B = (K[G], \cdot)$.

In [PB1], [PB2] it was demonstrated how central extensions of algebras from the examples (vi) and (vii) above, can be realized in terms of some bilinear forms on the underlying Novikov

†However, recently is was shown that it is isomorphic to a suitable Hamiltonian algebra under a highly non-trivial isomorphism, see [Gr] and [BLLS, §3].

algebra N. We suggest that these results are instances of a more general principle, which is a generalization of results of [Z1], [Z3] and §§2-4 of this paper, from current Lie algebras and Lie algebras of the form $\mathsf{sl}_n(A)$, to Lie algebras with the bracket (5.1):

Problem 8. *Describe, as much as possible, low-degree (co)homology, invariant bilinear forms, Poisson structures, Hom-Lie structures on Lie algebras of the form* $(A \otimes B)^{(-)}$*, where* A*,* B *are algebras over dual binary quadratic operads, in terms of the underlying algebras* A *and* B*.*

As we have seen at the beginning of §3, even in the "classical" case, when both A and B are associative, this can be unfeasible in some cases, so it is interesting to understand which concrete properties of operads turn that or another problem of that type to a tractable one.

At least, we expect that all this is entirely tractable for algebras of the form $N_\chi[G]$:

Problem 9. *Describe low-degree (co)homology, invariant bilinear forms, Poisson structures, Hom-Lie structures on Lie algebras of the form* $N_\chi[G]$ *in terms of* N *and* G*.*

Specializing these conjectural description further to the case (v) above, we should get the corresponding results for affinizations of Novikov algebras, similarly to how the results for Kac-Moody algebras are derived from those for current Lie algebras.

We conclude with yet another couple of problems.

Arguing from the physical perspective, Bai, Meng and He in [BMH] called the left Novikov algebras *bosonic*, and introduced a new class of so-called fermionic Novikov algebras as algebras satisfying two identities: the left-symmetric identity (5.2), and

$$(xy)z = -(xz)y.$$

We will call such algebras *left fermionic Novikov*. The dual operad to the left fermionic Novikov operad is *right fermionic Novikov*, governed by two identities: the right-symmetric identity (5.3), and

$$x(yz) = -y(xz).$$

Problem 10. *Are there "interesting" Lie algebras realized as* (5.1) *in this context, i.e. as skew-symmetrization of the tensor product of left and right fermionic Novikov algebras?*

Problem 11. *To superize constructions of this section. Consider Problems 8 and 9 for superalgebras.*

We expect that this conjectural superization will provide an alternative view (and probably much more) on the so-called stringy Lie superalgebras, see [KL], [GLS] and [Be, Appendix D3, §8].

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Email address, as of October 22, 2022: `pasha.zusmanovich@gmail.com`