QUESTIONS ABOUT SIMPLE LIE ALGEBRAS IN CHARACTERISTIC 2 BY ALEXANDER GRISHKOV

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The ground field is assumed algebraically closed of characteristic 2, unless stated otherwise, and all algebras and modules assumed to be finite-dimensional. $W_1$ and $H_2$ are 7-dimensional simple Lie algebras, the Zassenhaus algebra and the Hamiltonian algebra, respectively (see [GG2] and [GGA]), the Skryabin algebra is a certain simple 15-dimensional algebra (see [GGRZ]).

1. LIE ALGEBRAS OF LOW DIMENSION

1.1. (asked in [GGA]). Describe irreducible representations of $W_1$ and $H_2$.

1.2. Elements in the 15-dimensional Skryabin algebra. For the 2-envelope of the 15-dimensional Skryabin algebra, describe conjugacy classes of the following elements:

1. (toral elements);
2. elements satisfying $x^{[2]} \in \langle x \rangle$;
3. elements satisfying $x^{[2^n]} = 0$, for all $n$;

1.3. Gradings. Describe gradings of the 7-dimensional Hamiltonian algebra and the 15-dimensional Skryabin algebra (the latter question is asked in [GGRZ]). The $\mathbb{Z}/2\mathbb{Z}$-gradings probably could be handled with the approach from [KL].

2. CLASSIFICATION OF SIMPLE LIE ALGEBRAS

2.1. Describe simple Lie algebras of dimension 7. Conjecture: they are isomorphic either to $W_1$, or to $H_2$. For algebras of absolute toral rank 3, this is proved in [GGA].

2.2. For each known simple Lie algebra, compute its absolute toral rank and the automorphism group.

2.3. (posed in [GG1]). Let $L$ be a Lie algebra over a field of characteristic $p > 0$ such that $\text{Der}(L)$ contains a subalgebra $S$ isomorphic to $\mathfrak{sl}_2$ (note that if $p = 2$, then $\mathfrak{sl}_2$ is nilpotent). Let us call the pair $(S, L)$ semisimple, if $L$ decomposed as an $S$-module as

$$L = \bigoplus_i S_i \oplus \bigoplus_j V_j,$$

where each $S_i$ is a submodule of $M_2(K)$, and each $V_j$ is an irreducible $S$-module.

Conjecture: $L$ is of classical type if and only if there exists a semisimple pair $(L, S)$.

2.4. Simple Lie algebras of toral rank 1. Describe simple Lie algebras $L$ having a Cartan subalgebra of toral rank 1 in $L$. According to [S, Theorem 6.3], they are either Zassenhaus, or Hamiltonian, or filtered deformations of semisimple Lie algebras $G$ of the form

$$S \otimes \mathcal{O}_1(n) \subset G \subseteq \text{Der}(S) \otimes \mathcal{O}_1(n) + K \partial,$$

where either $n = 2$ and $S \simeq W'_1(m)$, or $n = 1$ and $S \simeq H''_2(m_1, m_2)$. In [GZ] those deformations are computed in the simplest case when $S = W'_1(2)$, the 3-dimensional simple algebra. So the question is reduced to computation of those deformations in general case.

Which of those simple algebras admit a thin decomposition?

2.5. Classify finite-dimensional simple Lie 2-algebras.

2.6. (asked in [GGRZ]). Classify finite-dimensional simple Lie algebras having a $\mathbb{Z}$-grading with all homogeneous components of dimension < 3.

3. ROOT SPACE DECOMPOSITIONS

3.1. Conjecture. Let $L$ be a simple Lie algebra, $T$ a torus of the maximal dimension in the 2-envelope of $L$, and $T \cap L = 0$. Then in the root space decomposition $L = \bigoplus \alpha L_\alpha$, $\dim L_\alpha$ is a constant (i.e., does not depend on $\alpha$).

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3.2. Thin decompositions. Derivations. (asked in [GGRZ]). Conjecture. If \( L \) is a simple Lie algebra admitting a thin decomposition, then \( \text{Der}(L) \cong T \oplus L \).

3.3. Thin decompositions. Classification. (asked in [GGRZ]). To classify simple Lie algebras over an algebraically closed field admitting a thin decomposition, i.e., when \( \dim L = 2^n - 1 \), \( \dim T = n \), the roots are exactly \( GF(2)^n \setminus (0, \ldots, 0) \), and \( \dim L_\alpha = 1 \) for any root \( \alpha \).

3.4. Thin decompositions. Subalgebras. Prove that any simple Lie algebra of dimension \( > 3 \) over a field of characteristic 2 admitting a thin decomposition, has:
   a) a proper simple graded subalgebra (with respect to this decomposition);
   b) a graded subalgebra isomorphic either to \( W_1 \) or to \( H_2 \).

3.5. Thin decompositions. Modules. Conjecture. Let \( L \) be a simple Lie algebra of dimension \( 2^n - 1 \) with a thin decomposition, \( V \) an irreducible \( 2^n \)-dimensional \( L \)-module. Assume that there is a simple Lie algebra with thin decomposition of the form \( L \oplus V \), where \( L \) is a subalgebra, the multiplication between \( L \) and \( V \) is given by the action of \( L \) on \( V \), and the multiplication between elements of \( V \) is given by the map \( f : V \times V \rightarrow L \). The the map \( f \) is determined uniquely. Study this situation for \( L = W_1 \) or \( H_2 \) (the Skryabin algebra arises in this way from \( H_2 \)).

3.6. Variety of algebras with a thin decomposition. Let \( L \) be a Lie algebra with thin decomposition, with multiplication defined by \([e_g, e_h] = f(g, h)e_{g+h}\). Study the variety of all possible functions \( f \) defining a thin Lie algebra, from the algebro-geometric viewpoint. What are irreducible components? Which simple Lie algebras are “generic” in this sense?

REFERENCES


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