QUESTIONS ABOUT SIMPLE LIE ALGEBRAS IN CHARACTERISTIC 2 BY ALEXANDER GRISHKOV

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The ground field is assumed algebraically closed of characteristic 2, unless stated otherwise, and all algebras and modules assumed to be finite-dimensional. W_1 and H_2 are 7-dimensional simple Lie algebras, the Zassenhaus algebra and the Hamiltonian algebra, respectively (see [GG2] and [GGA]), the Skryabin algebra is a certain simple 15-dimensional algebra (see [GGRZ]).

1. Lie algebras of low dimension

1.1. (asked in [GGA]). Describe irreducible representations of W_1 and H_2.

1.2. Elements in the 15-dimensional Skryabin algebra. For the 2-envelope of the 15-dimensional Skryabin algebra, describe conjugacy classes of the following elements:

(1) toral elements;
(2) elements satisfying $x^{[2]} \in \langle x \rangle$;
(3) elements satisfying $x^{[2^n]} = 0$, for all n;

1.3. Gradings. Describe gradings of the 7-dimensional Hamiltonian algebra and the 15-dimensional Skryabin algebra (the latter question is asked in [GGRZ]). The $\mathbb{Z}/2\mathbb{Z}$-gradings probably could be handled with the approach from [KL].

2. Classification of simple Lie algebras

2.1. Describe simple Lie algebras of dimension 7. Conjecture: they are isomorphic either to W_1, or to H_2. For algebras of absolute toral rank 3, this is proved in [GGA].

2.2. For each known simple Lie algebra, compute its absolute toral rank and the automorphism group.

2.3. (posed in [GG1]). Let L be a Lie algebra over a field of characteristic $p > 0$ such that $\mathrm{Der}(L)$ contains a subalgebra S isomorphic to sl_2 (note that if $p = 2$, then sl_2 is nilpotent). Let us call the pair (S,L) semisimple, if L decomposed as an S-module as

$$L = \bigoplus_i S_i \oplus \bigoplus_j V_j,$$

where each S_i is a submodule of $M_2(K)$, and each V_j is an irreducible S-module.

Conjecture: L is of classical type if and only if there exists a semisimple pair (L,S).

2.4. Simple Lie algebras of toral rank 1. Describe simple Lie algebras L having a Cartan subalgebra of toral rank 1 in L. According to [S, Theorem 6.3], they are either Zassenhaus, or Hamiltonian, or filtered deformations of semisimple Lie algebras G of the form

$$S \otimes \mathcal{O}_1(n) \subset G \subseteq \mathrm{Der}(S) \otimes \mathcal{O}_1(n) + K\partial,$$

where either $n = 2$ and $S \simeq W_1'(m)$, or $n = 1$ and $S \simeq H_2''(m_1,m_2)$. In [GZ] those deformations are computed in the simplest case when $S = W_1'(2)$, the 3-dimensional simple algebra. So the question is reduced to computation of those deformations in general case.

Which of those simple algebras admit a thin decomposition?

2.5. Classify finite-dimensional simple Lie 2-algebras.

2.6. (asked in [GGRZ]). Classify finite-dimensional simple Lie algebras having a $\mathbb{Z}$-grading with all homogeneous components of dimension < 3.

3. Root space decompositions

3.1. Conjecture. Let L be a simple Lie algebra, T a torus of the maximal dimension in the 2-envelope of L, and $T \cap L = 0$. Then in the root space decomposition $L = \oplus_\alpha L_\alpha$, $\dim L_\alpha$ is a constant (i.e., does not depend on α).

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3.2. Thin decompositions. Derivations. (asked in [GGRZ]). Conjecture. If L is a simple Lie algebra admitting a thin decomposition, then $\mathrm{Der}(L) \simeq T \oplus L$.

3.3. Thin decompositions. Classification. (asked in [GGRZ]). To classify simple Lie algebras over an algebraically closed field admitting a *thin decomposition*, i.e., when $\dim L = 2^n - 1$, $\dim T = n$, the roots are exactly $GF(2)^n \backslash (0, \dots, 0)$, and $\dim L_\alpha = 1$ for any root α.

3.4. Thin decompositions. Subalgebras. Prove that any simple Lie algebra of dimension > 3 over a field of characteristic 2 admitting a thin decomposition, has:
a) a proper simple graded subalgebra (with respect to this decomposition);
b) a graded subalgebra isomorphic either to W_1 or to H_2.

3.5. Thin decompositions. Modules. Conjecture. Let L be a simple Lie algebra of dimension $2^n - 1$ with a thin decomposition, V an irreducible 2^n-dimensional L-module. Assume that there is a simple Lie algebra with thin decomposition of the form $L \oplus V$, where L is a subalgebra, the multiplication between L and V is given by the action of L on V, and the multiplication between elements of V is given by the map $f : V \times V \to L$. The the map f is determined uniquely. Study this situation for $L = W_1$ or H_2 (the Skryabin algebra arises in this way from H_2).

3.6. Variety of algebras with a thin decomposition. Let L be a Lie algebra with thin decomposotion, with multiplication defined by $[e_g, e_h] = f(g,h)e_{g+h}$. Study the variety of all possible functions f defining a thin Lie algebra, from the algebro-geometric viewpoint. What are irreducible components? Which simple Lie algebras are "generic" in this sense?

References

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