LEITES' (SUPER)QUESTIONS

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This is an account of (some of the) questions posed by Dimitry Leites, compiled by me mainly during few years at the beginning of 2000s. While the essence belongs to him, all wording, (mis)interpretations, as well as possible errors are mine. More thorough and inspirational exposition belonging to Leites himself may be found in his and his collaborators (numerous) writings ([Be, Appendices], [GL2], [GLS], [LLS], [Lei], [LS1], [LS2], [LS3], [LS4], [S] and references therein). Needless to say, there are much more questions there, I account here merely for what I was able to grasp.

The questions, being concentrated around one topic – Lie superalgebras – may be roughly divided into two categories: first, the very particular questions about the very particular algebras (though nobody will beat you, except may be Leites himself under very particular circumstances, for a "proper" generalizations, like, say, replacing the Laurent polynomials ring by an arbitrary commutative associative ring), and second, more "generic" or "theoretical" questions (like describing superalgebra structures on the whole cohomology space, elucidation in the supercase of certain issues well-known in the plain even setting, etc.).

1. Cohomology in all dimensions

1.1. **Cohomology of Hamiltonian (super)algebras.** Cohomology of Poisson and Hamiltonian Lie (super)algebras, with emphasis and low-dimensional cohomology and deformations.

Particularly, compute cohomology with trivial coefficients of Hamiltonian Lie superalgebra $h(0|n)$ for $n > 4$ ($h(0|4) \simeq psl(2|2)$, so covered by Leites–Fuchs in their paper on cohomology of classical Lie superalgebras ([LF]; later Shapovalov discovered one cocycle missed there - see [LS4, §2.2.1] and [Kor1, §4.1]). Compute cohomology with trivial coefficients of (infinite-dimensional) Hamiltonian Lie algebra $H(n)$ ($n = 2$ considered by Gelfand–Kalinin–Fuchs in [GKF] with a very partial results). Study the behavior of cohomology of a (finite-dimensional) Hamiltonian Lie algebra in characteristic p as $p \to \infty$.

Can we utilize the fact that $H(2)$ is a "limit" of $sl(n)$'s in an appropriate basis? (See [AH, Example 2.12] and [Ze] and references therein).

1.2. **Computer calculations.** To develop novel approaches to computer calculations of Lie superalgebras cohomology. What kind of sparse matrices arises? What methods are most suitable for dealing with these sparse matrices? (Raised also by Kornyak in [Kor2, §2.3])

1.3. **Structure of cohomology algebra.** ([GL2, §4.1-4.2] and [LLS]) Describe the Nijenhuis-Richardson Lie superalgebra structure on $H^*(L, L)$ for some "interesting" (super)algebras L - not in terms of homogeneous components or generators and relations, like it is done traditionally, but from the structure theory viewpoint - i.e. determine its radical, semisimple part, etc. Say, for maximal nilpotent subalgebras of simple classical and vectorial Lie (super)algebras. The case of maximal nilpotent subalgebras of $sl(3)$ and G_2 was treated by Lebedev (see [Leb] and [LLS]).

Conjectures ([GL3]):

(i) $H^*(psl(2|2), psl(2|2))$ is generated by certain explicitly given cocycles of degrees 0,2 and 3.
(ii) $H^*(osp(4|2;\alpha), osp(4|2,\alpha))$ is exhausted by explicitly given cocycles in degrees 1 and 4.

Date: first written April 30, 2003; last revised August 5, 2018.

A similar question of study of an algebra of simplicial cohomology (in the context of computer calculations) was raised in [DHSW]. It could be that something similar was undertaken for the cohomology of associative algebras and groups.

2. Low-dimensional cohomology and related invariants

2.1. **Invariants of "classical" Lie superalgebras.** Describe (in all remaining cases) central extensions, derivations, automorphisms, invariant bilinear forms, deformations and forms (that is, over algebraically nonclosed fields, notably over $\mathbb{R}$) of all (remaining cases) of "interesting" Lie superalgebras: Lie superalgebras of vector fields, "stringy" superalgebras and (possibly twisted) current superalgebras (all with polynomial, formal and Laurent coefficients). This is too broad and vague, so let's start with some particular questions:

2.1.1. *Central extensions of vectorial superalgebras.* ([LS4, §6.2]) Prove that there are no central extensions of simple Lie superalgebras of vector fields with polynomial or formal coefficients, except the following:

(i) Poisson superalgebra $po(2n|m)$ extending Hamiltonian one $h(2n|m)$;
(ii) Deformed Buttin superalgebra $b_\lambda(n)$ extending $le(n)$ (probably values $\lambda = 1$ or -1 are exceptional in some sense);
(iii) Two amazing extensions of $sle^o(3)$ described by Shchepochkina and Post in [SP].

(Well, may be something else is missing, but the most nontrivial cases are certainly here).

2.1.2. *Central extension of Lie superalgebras of infinite matrices.* There are various superalgebras with different finiteness constraints. See [S, §3.2.1]. Probably some works of Feigin (?), cyclic cohomology, and "Japanese cocycle" are relevant.

2.1.3. *Deformations of current algebras.* ([LS3, Warning 3 on p. 6]). It is probably proved (Lecomte and Roger, see [R]) that current and Kac-Moody (super)algebras are rigid (though probably all the cases, including super ones, are formally not covered). This is, however, "ideologically wrong", as there are interesting examples which are by all possible means should count as deformations of corresponding current algebras: namely, Krichever-Novikov algebras, example of P. Golod (for the latter, see [G] or [LS3], p. 38), and examples of Fialowski–Schlichenmaier. So:

(i) Accurately compute all "classical" (= Gerstenhaber) deformations of current (super)algebras. Are all they rigid?
(ii) Build an "ideologically right" deformation theory which will include Krichever-Novikov and Golod algebras. Are there others?

Ideally, this should also include, as a special case, filtered deformations of modular $W_1(n) \otimes A$ (computed in [Zu2]). Probably the unification can be done by considering "Block (super)algebras" as defined (in the non-super case) in [Zu1].

2.1.4. *Deformations and outer derivations of stringy superalgebras.* ([Be, Appendix D3, §8.5]) Compute deformations and outer derivations of "stringy" superalgebras (i.e. exactly those described in [GLS] or [Be, Appendix D3, §8]); some cocycles formulae in [GLS] describing central extensions are incorrect). It is known ([HK]) that some of the initial algebras in the series, namely, $k^L(1|n)$ and $k^M(1|n)$ for $n = 0, 1$ have zero second degree cohomology in the adjoint module (and, consequently, are rigid), and that $svect^L(1|n)$ has at least one deformation (denoted as $svect^L_\lambda(1|n)$ in [GLS]). No new algebras are expected.

2.1.5. *Deformations of Hamiltonian algebras.* Accurately describe all deformations of Lie superalgebras of Hamiltonian vector fields $h(2n|m)$. Why the case of $h(2|2)$ is exceptional? This is discussed, based on the previous (partially unpublished) works of Kochetkov, in [LS1].

Kochetkov in [Koc] described deformations of a Hamiltonian Lie algebra $H(2)$ and Cheng and Kac in [CK] proved that there is no filtered (with respect to the standard grading) deformations of $h(2n|m)$ (though the latter probably should be taken with a caution).

2.1.6. *Filtered deformations.*

(i) Compute filtered deformations of Lie superalgebras of vector fields in all possible "Weisfeiler"gradings (Cheng and Kac [CK] considered only standard gradings with even deformation parameter).
(ii) Provide interesting (whatever it means) examples when different gradings of the same (super)algebra give rise to nonisomorphic filtered deformations. (This is, particularly, related to some gaps in Kac's classification(s); see [LS2], §I.4 and [LS4], §1.12).

2.1.7. *Modules of tensor fields.* Compute H^1 for "stringy" superalgebras in modules of "tensor fields" and their generalizations (see [GLS, §1.3], [Lei, §1.2-1.3] and [P1]). This sounds to vague, so let's start from the following:

(i) $H^1(L, F_{\lambda;\mu})$ for $L = k^L(1|n)$ and $k^M(1|n)$.
(ii) $H^1(L, F_{\lambda_1,\lambda_2;\mu})$ for $L = k^L(1|2)$ and $k^M(1|3)$.
(iii) for $k^{Lo}(1|4)$ and $k^{Mo}(1|5)$.

$H^1(L, F_\lambda)$ for $L = k^L(1|n)$ and $k^\alpha(1|4)$, and $H^1(L, T(\lambda, \mu))$ for $L = k^L(1|2)$ and $k^+(1|2)$ were computed by Poletaeva (unpublished M.Sc. thesis; see [P1] and [Lei, §3]).

Some additional papers possibly instrumental in understanding of the structure of algebras and modules involved: [P2] and [Kac]. Some low-dimensional cohomology related to tensor fields considered by Wagemann ([W]).

2.2. **Structure functions.** To compute structure functions associated with (twisted) current superalgebras (that is, "twist" and superize results of [Zu3]). Or better yet, consider a "nonholonomic" situation, when the underlying Lie (super)algebra is not abelian, but "negatively-graded" nilpotent, with corresponding generalized Cartan prolong and generalized Spencer cohomology (see e.g. [GL2, §4.1]).

3. Other

3.1. **Presentations of Lie algebras associated with Cartan matrix.** Find "reasonable" presentation for Kac-Moody-type Lie (super)algebras associated with arbitrary Cartan matrix (the question is open even for ordinary Lie algebras and matrices of size 3×3). See [GL1] and [GL2, §§1.8.4 and 1.9].

3.2. **Volichenko algebras.** Further study of Volichenko algebras (which are, roughly speaking, nonhomogeneous subalgebras of Lie superalgebras). See [S, Chapter 2]. In particular, to provide an intrinsic definition of Volichenko algebras (they do not form a variety, see [Ba], so such description is impossible solely in the language of identities).

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Addendum August 5, 2018

This was mainly written long time ago. It seems that all the questions described here largely remain relevant, though there may be some advancement concerning some particular superalgebras. Below is the list of references which may contain such advancements, though I did not attempt to go into details thoroughly.

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