# LIE ALGEBRAS THAT CAN BE WRITTEN AS THE SUM OF TWO NILPOTENT SUBALGEBRAS

PASHA ZUSMANOVICH

This is a short survey about the current state of affairs with Lie algebras $L$ that can be written as the sum of two nilpotent subalgebras $A, B$: $L = A + B$. The sum is understood in the sense of vector spaces and is not (necessarily) direct.

Motivated by similar questions from Group Theory and Ring Theory, Kegel [Ke] asked in 1963 whether a Lie ring that can be written as the sum of two nilpotent subrings, is solvable.

If we restrict our attention to algebras over a field, it is easy to see that without loss of generality one can assume that the ground field is algebraically closed. If, further, we restrict our attention to the finite-dimensional algebras, the existing powerful arsenal of structure theory immediately yields a positive answer in the zero characteristic case (see [G] or [Kos]). As it always happens, in characteristic $p > 0$ the situation becomes more complicated, and the attention to this question was renewed in 1982 by Kostrikin [Kos].

After that, Petravchuk [Pe1] gave an example providing a negative answer to the question in characteristic 2. In positive direction, a few particular results (with restrictions on the index of nilpotency of one of the summands) were obtained (including [Pe1]), and at the beginning of 1990s Panyukov [Pa1] (for characteristic $p > 2$) and myself [Z] (for characteristic $p > 5$), using different approaches, provided a positive answer in the general case.

A further question arise of precise description of a class of such algebras inside the class of all solvable algebras (see [Z], [Pa2], and [BTT]).

Numerous results about finite groups that can be decomposed into a product of two groups with different properties, may be found in [CS], [F], [H, Chap. VI], and references therein.

## DEGREE OF SOLVABILITY

So, $L$ is solvable. What can be said about its degree of solvability $s(L)$ in terms of degrees of nilpotency $n(A)$ and $n(B)$ of $A$ and $B$?

In [Kol, Proposition 1.5], following the well-known result of Ito for groups, it is proved that if $n(A) = n(B) = 1$ (i.e., both $A$ and $B$ are abelian), then $s(L) = 2$ (i.e., $L$ is metabelian).

In [Pe5] it is proved that if $n(A) = 1$ (i.e., $A$ is abelian) and $n(B) = 2$ (i.e., $B$ is 2-step nilpotent), then $s(L) \le 10$. This bound is, probably, too rough. I do not know an example of such algebra with $s(L) > 3$.

**Question 1** (Dietrich Burde)**.** *Is it true that if $n(A) = 1$ and $n(B) = 2$, then $s(L) \le 3$?*

Examples of groups for which $s(L) > n(A) + n(B)$, are given in [CS].

Form the results of [Pe2] follows, that if $n(A) = 1$, then $s(L)$ is bounded by a function of $n(B)$.

**Question 2.** *Is it possible to bound $s(L)$ by a linear function of $n(A)$ and $n(B)$?*

The same question for groups is asked in Kourovka notebook, question 14.43.

## INFINITE-DIMENSIONAL ALGEBRAS

What happens in the infinite-dimensional situation?

**Question 3.** *Is it true, that an infinite-dimensional Lie algebra over a field of characteristic $\neq 2$ that can be written as the sum of two nilpotent subalgebras, is solvable?*

*Date*: first written October 28, 2003; last minor revison May 27, 2018.

This was (re)asked, in particular, in [BTT] and by Rutwig Campoamor-Stursberg.

In [Pe2] a positive answer is obtained in the case when one of the summands is abelian.

In [Pe3], a positive answer to this question is provided in the two cases: when one of the summands is finite-dimensional, and when commutants of both summands are finite-dimensional, and in [HS] - for the class of locally-finite Lie algebras (i.e. Lie algebras all whose finitely-generated algebras are finite-dimensional). Both results are obtained by a quick and easy reduction to the finite-dimensional case.

Similar results for infinite groups (with, again, that or another finiteness conditions) were obtained by N.S. Chernikov (see [C] and references in [Pe3]).

A weaker question is also open:

**Question 4.** *Is it true, that an (infinite-dimensional) associative algebra that can be written as the sum of two Lie-nilpotent associative algebras, is Lie-solvable?*

An affirmative answer is known in the cases when one of the summands is commutative ([Pe4]), or is an one-sided ideal ([LP, Corollary 1]).

## References


[BTT] Y. Bahturin, M. Tvalavadze, and T. Tvalavadze, *Sums of simple and nilpotent Lie algebras*, Comm. Algebra **30** (2002), 4455–4471.

[C] N.S. Chernikov, *Infinite groups that are products of nilpotent subgroups*, Soviet Math. Dokl. **21** (1980), N3, 701–703.

[CS] J. Cossey and S. Stonehewer, *On the derived length of finite dinilpotent groups*, Bull. London Math. Soc. **30** (1998), 247–250.

[F] E. Fisman, *Finite factorizable groups*, PhD Thesis, Bar-Ilan University, 1982.

[G] M. Goto, *Note on a characterization of solvable Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I **26** (1962), 1–2.

[HS] M. Honda and T. Sakamoto, *Lie algebras represented as a sum of two subalgebras*, Math. J. Okayama Univ. **42** (2000), 73–81.

[H] B. Huppert, *Endliche Gruppen. I*, Springer, 1967.

[Ke] O.H. Kegel, *Zur Nilpotenz gewisser assoziativer Ringe*, Math. Ann. **149** (1963), 258–260.

[Kol] B. Kolman, *Semi-modular Lie algebras*, J. Sci. Hiroshima Univ. Ser. A-I **29** (1965), 149–163.

[Kos] A.I. Kostrikin, *A solvability criterion for a finite-dimensional Lie algebra*, Moscow Univ. Math. Bull. **37** (1982), N2, 21–26.

[LP] V.S. Luchko and A.P. Petravchuk, *On one-sided Lie nilpotent ideals of associative rings*, Algebra Discrete Math. 2007, N4, 102–107; arXiv:0803.0968.

[Pa1] V.V. Panyukov, *On the solvability of Lie algebras of positive characteristic*, Russ. Math. Surv. **45** (1990), N4, 181–182.

[Pa2] ———, *On solvable Lie algebras decomposable into a sum of two nilpotent subalgebras*, Sbornik Math. **83** (1995), 221-235.

[Pe1] A.P. Petravchuk, *Lie algebras which can be decomposed into the sum of an abelian subalgebra and a nilpotent subalgebra*, Ukrain. Math. J. **40** (1988), 331–334.

[Pe2] ———, *The solubility of a Lie algebra which decomposes into a direct sum of an abelian and a nilpotent subalgebra*, Ukrain. Math. J. **43** (1991), 920–924.

[Pe3] ———, *On the sum of two Lie algebras with finite dimensional commutants*, Ukrain. Math. J. **47** (1995), N8, 1244–1252.

[Pe4] ———, *On the Lie solvability of sums of some associative rings*, Visn. Kyïv. Univ. Ser. Fiz.-Mat. Nauki 1999, N1, 78–81 (in Russian).†

[Pe5] ———, *On derived length of the sum of two nilpotent Lie algebras*, Visn. Mat. Mekh. Kyïv. Univ. 2001, No.6, 53–56 (in Ukrainian).

[Z] P. Zusmanovich, *A Lie algebra that can be written as a sum of two nilpotent subalgebras, is solvable*, Math. Notes **50** (1991), 909–912; arXiv:0911.5418.


*Email address*: `pasha.zusmanovich@gmail.com`

†I have not seen that paper and judging it solely by MR2003a:16043.