# A classification scheme for the inverse problem in arbitrary dimension

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**Non-existence theorem**

Suppose that the matrix $\Phi$ is diagonalisable with distinct eigenvalues. If $\Phi$ has p non-integrable co-distributions and up to differential ideal step $p$ no differential ideal has been found, then there is no non-degenerate multiplier.

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- A differential ideal is generated at step 1, step 2,..., up to step $p$ $\Rightarrow$ the existence of non-degenerate multiplier depends on whether the Pfaffian system admits solutions.

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In Differential ideal step:

$$\Sigma^1 \supset \Sigma^2 \supset \ldots \supseteq \Sigma^f$$

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$\Rightarrow$ If there is 1 non-integrable co-distribution, necessary conditions for the existence of non-degenerate multiplier is that $\langle \Sigma^1 \rangle$ is a differential ideal.

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$\Rightarrow$ If there is 1 non-integrable co-distribution, necessary conditions for the existence of non-degenerate multiplier is that $\langle \Sigma^1 \rangle$ is a differential ideal.

$\Rightarrow$ If there is 2 non-integrable co-distributions, necessary conditions for the existence of non-degenerate multiplier is that $\langle \Sigma^1 \rangle$ or $\langle \Sigma^2 \rangle$ is a differential ideal.

# $\Phi$ is diagonalisable with distinct eigenvalues, $p \neq 0$ non-integrable, $n - p$ integrable co-distributions

**Theorem (Do, 2016)**

Assume that $\mathbf{\Phi}$ is diagonalisable with distinct (real) eigenvalues and with $p \neq 0$ non-integrable eigen co-distributions. Suppose that eigen co-distributions are ordered such that $Sp\{\phi^{AV}, \phi^{AH}\}$, $A = 1, \ldots, p$ are non-integrable. Suppose further that $rank(\mathbf{A}_1) = p - 1$. Then the necessary and sufficient conditions for the existence of a solution for the associated inverse problem are that the given conditions hold. Moreover, **the solution (if it exists) depends on $n - p$ arbitrary functions of $2$ variables each**.

# $\Phi$ is diagonalisable with distinct eigenvalues, $p \neq 0$ non-integrable, $n - p$ integrable co-distributions

In Differential ideal step:

$$\Sigma^1 \xrightarrow{\mathbf{A}_1} \Sigma^2 \xrightarrow{\mathbf{A}_2} \dots \xrightarrow{\mathbf{A}_{p-1}} \Sigma^p \xrightarrow{\mathbf{A}_p} \Sigma^{p+1}.$$

- $\mathbf{A}_i$ is system of homogeneous linear algebraic equations required for $d\omega \in \langle \Sigma^i \rangle$ where $\omega \in \Sigma^i$
- $rank(\mathbf{A}_1) \in \{0, 1, ..., p\}$ and $rank(\mathbf{A}_{i+1}) \leq p - \sum_{k=1}^{i} rank(\mathbf{A}_k)$,
- if $rank(\mathbf{A}_1) = p \Rightarrow$ no non-degenerate solutions.

# N=3: $\Phi$ is diagonalisable with distinct eigenvalues with $p$ non-integrable co-distributions

## A classification scheme

CASE 1. $p = 0$ (known).

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CASE 3. $p = 2$.

$rank(A_1) = 0 \Rightarrow \langle \Sigma^1 \rangle$ is DI.
$(rank(A_1) = 1$ and $rank(A_2) = 0 \Rightarrow \langle \Sigma^2 \rangle$ is DI) or no solutions.
$rank(A_1) = 2 \Rightarrow$ no solutions.

# N=3: $\Phi$ is diagonalisable with distinct eigenvalues with $p$ non-integrable co-distributions

## A classification scheme

CASE 1. $p = 0$ (known).
CASE 2. $p = 1$. $(rank(A_1) = 0 \Rightarrow \langle\Sigma^1\rangle$ is DI$)$ or no solutions.
CASE 3. $p = 2$.
$rank(A_1) = 0 \Rightarrow \langle\Sigma^1\rangle$ is DI.
$(rank(A_1) = 1$ and $rank(A_2) = 0 \Rightarrow \langle\Sigma^2\rangle$ is DI$)$ or no solutions.
$rank(A_1) = 2 \Rightarrow$ no solutions.
CASE 4. $p = 3$.
$rank(A_1) = 0 \Rightarrow \langle\Sigma^1\rangle$ is DI.
$rank(A_1) = 1$.
$rank(A_2) = 0 \Rightarrow \langle\Sigma^2\rangle$ is DI
$(rank(A_2) = 1$ and $rank(A_3) = 0 \Rightarrow \langle\Sigma^3\rangle$ is DI$)$ or no solutions.
$(rank(A_1) = 2$ and $rank(A_2) = 0 \Rightarrow \langle\Sigma^2\rangle$ is DI$)$ or no solutions.
$rank(A_1) = 3 \Rightarrow$ no solutions.

# $\Phi$ is diagonalisable with distinct eigenvalues, $p = 2$ non-integrable co-distributions and $rank(\mathbf{A}_1) = 1$

Assume $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable

STEP 1:Differential ideal step

$\Sigma^1 := Sp\{\omega^c := \phi^{cV} \wedge \phi^{cH} : c = 1, \dots, n\}$

$d\omega \in \langle \Sigma^1 \rangle$, where $\omega := r_c \omega^c \in \Sigma^1$ is equivalent to system $\mathbf{A}_1$:

$$
\begin{aligned}
& r_1 \tau_2^{1\Gamma} + r_2 \tau_1^{2\Gamma} = 0, \\
& r_1 \tau_\alpha^{1\Gamma} = 0, \\
& r_2 \tau_\alpha^{2\Gamma} = 0, \\
& r_1 (\tau_{2\alpha}^{1V} - \tau_{\alpha 2}^{1V}) - r_2 \tau_{\alpha 1}^{2V} = 0, \\
& r_1 \tau_{2\alpha}^{1V} - r_2 \tau_{1\alpha}^{2V} = 0, \\
& r_1 (\tau_{2\alpha}^{1H} - \tau_{\alpha 2}^{1H}) - r_2 \tau_{\alpha 1}^{2H} = 0, \\
& r_1 \tau_{2\alpha}^{1H} - r_2 \tau_{1\alpha}^{2H} = 0, \\
& r_1 \phi^{1V}(R(X_2^H, X_\alpha^H)) - r_2 \phi^{2V}(R(X_1^H, X_\alpha^H)) = 0,
\end{aligned}
$$

# $\Phi$ is diagonalisable with distinct eigenvalues, $p = 2$ non-integrable co-distributions and $rank(\mathbf{A}_1) = 1$

STEP 1: Differential ideal step

We have conditions

$$\tau_\alpha^{1\Gamma} = 0, \qquad \tau_\alpha^{2\Gamma} = 0, \text{ for all } \alpha = 3, \ldots, n \tag{1}$$

And at least one of the expressions

$$-\frac{\tau_2^{1\Gamma}}{\tau_1^{2\Gamma}}, \ \frac{\tau_{2\alpha}^{1V} - \tau_{\alpha 2}^{1V}}{\tau_{\alpha 1}^{2V}}, \ \frac{\tau_{2\alpha}^{1V}}{\tau_{1\alpha}^{2V}}, \ \frac{\tau_{2\alpha}^{1H} - \tau_{\alpha 2}^{1H}}{\tau_{\alpha 1}^{2H}}, \ \frac{\tau_{2\alpha}^{1H}}{\tau_{1\alpha}^{2H}}, \ \frac{\phi^{1V}(R(X_2^H, X_\alpha^H))}{\phi^{2V}(R(X_1^H, X_\alpha^H))} \tag{2}$$

for all $\alpha = 3, \ldots, n$, must be non-zero and when they are non-zero, they must be equal. $\Rightarrow$ $h_2$ equals the non-zero expressions.

$\Sigma^2 := Sp\{\tilde{\omega}^1, \omega^\alpha : \alpha = 3, \ldots, n\}$, with $\tilde{\omega}^1 = \omega^1 + h_2\omega^2$

# $\Phi$ is diagonalisable with distinct eigenvalues, $p = 2$ non-integrable co-distributions and $rank(\mathbf{A}_1) = 1$

STEP 1: Differential ideal step

$\Sigma^2 := Sp\{\tilde{\omega}^1, \omega^\alpha : \alpha = 3, \ldots, n\}$, with $\tilde{\omega}^1 = \omega^1 + h_2\omega^2$

$d\omega \in \langle \Sigma^2 \rangle$, where $\omega := \tilde{r}_1\tilde{\omega}^1 + r_\alpha\omega^\alpha \in \Sigma^2$
if and only if

$$d\tilde{\omega}^1 \in \langle \Sigma^2 \rangle$$

equivalently

$$dh_2 + \xi^1_2 + h_2(\xi^2_2 - \xi^1_1) \equiv 0 \pmod{\phi^{2V}, \phi^{2H}} \tag{3}$$

## STEP 2: Pfaffian system step

Find closed 2-forms $\Omega \in \Sigma^2$

$d\Omega = 0$ gives the system of Pfaffian equations

$$
\begin{aligned}
&d\tilde{r}_1 + \tilde{r}_1 \tilde{\xi}_1^1 = 0 \\
&dr_\alpha + \tilde{r}_1 \tilde{\xi}_\alpha^1 + r_\alpha \xi_\alpha^\alpha = -P_\alpha \phi^{\alpha V} - Q_\alpha \phi^{\alpha H} \text{ (no sum on } \alpha)
\end{aligned}
$$

where $\alpha = p+1, ..., n$ and $P_\alpha, Q_\alpha$ are arbitrary functions.

Set

$$
\begin{aligned}
\sigma_1 :=& d\tilde{r}_1 + \tilde{r}_1 \tilde{\xi}_1^1 \\
\sigma_\alpha :=& dr_\alpha + \tilde{r}_1 \tilde{\xi}_\alpha^1 + r_\alpha \xi_\alpha^\alpha + P_\alpha \phi^{\alpha V} + Q_\alpha \phi^{\alpha H}
\end{aligned}
$$

Then

$$
\begin{aligned}
d\sigma_1 \equiv& 0 \qquad (\text{mod } \sigma_1), \\
d\sigma_\alpha \equiv& \tilde{\pi}_\alpha^P \wedge \phi^{\alpha V} + \tilde{\pi}_\alpha^Q \wedge \phi^{\alpha H} \ (\text{mod } \sigma_1, \sigma_\alpha)
\end{aligned}
$$

STEP 2: Pfaffian system step

Find closed 2-forms $\Omega \in \Sigma^2$

$$\begin{aligned} d\sigma_1 &\equiv 0 \qquad (\text{mod } \sigma_1), \\ d\sigma_\alpha &\equiv \tilde{\pi}^P_\alpha \wedge \phi^{\alpha V} + \tilde{\pi}^Q_\alpha \wedge \phi^{\alpha H} \ (\text{mod } \sigma_1, \sigma_\alpha) \end{aligned}$$

if and only if (torsion is absorbed)

$$d\tilde{\xi}^1_1 = d(\xi^1_1 + h_2\xi^2_1) = 0, \tag{4}$$

$$(\xi^\alpha_\alpha - \tilde{\xi}^1_1) \wedge \tilde{\xi}^1_\alpha + d\tilde{\xi}^1_\alpha \equiv 0 \quad (\text{mod } \phi^{\alpha V}, \phi^{\alpha H}), \alpha = 3, \ldots, n. \tag{5}$$

# $\Phi$ is diagonalisable with distinct eigenvalues, 2 non-integrable, $n - p$ integrable co-distributions

STEP 3: Cartan-Kähler theorem

$$\begin{aligned} d\sigma_1 \equiv& 0 \qquad (\text{mod } \sigma_1), \\ d\sigma_\alpha \equiv& \tilde{\pi}_\alpha^P \wedge \phi^{\alpha V} + \tilde{\pi}_\alpha^Q \wedge \phi^{\alpha H} \ (\text{mod } \sigma_1, \sigma_\alpha) \end{aligned}$$

Cartan characters $s_1 = n - 2, s_2 = n - 2, s_k = 0$ for all $k > 2$ The system pass Cartan-Kähler test and so solution depend on $n - 2$ functions of 2 variables.

# $\Phi$ is diagonalisable with distinct eigenvalues, 2 non-integrable co-distributions and $rank(\mathbf{A}_1) = 1$

Examples:

Non-existence example:

$$\ddot{x} = x\dot{y}, \;\; \ddot{y} = \dot{x}, \;\; \ddot{z} = 0, \tag{6}$$

on an appropriate domain. Denoting the derivatives by $u, v, w$, we find that $\mathbf{\Phi}$ is diagonalisable with distinct eigenvalues and corresponding re-scaled eigenvectors $X_a$ as follows,

$$\begin{aligned} \lambda_1 &= -\frac{x}{4} && \text{and} && X_1 = \left(\frac{u}{\sqrt[4]{v^3}}, \sqrt[4]{v}, 0\right), \\ \lambda_2 &= -\frac{4v + x}{4} && \text{and} && X_2 = \left(\frac{1}{\sqrt[4]{v}}, 0, 0\right), \\ \lambda_3 &= 0 && \text{and} && X_3 = (0, 0, 1). \end{aligned}$$

# $\Phi$ is diagonalisable with distinct eigenvalues, $p = 2$ non-integrable co-distributions and $rank(\mathbf{A}_1) = 1$

Non-existence example:

The structure functions $\tau$'s are zero except for

$$\tau_2^{1\Gamma} = -\frac{\sqrt{v}}{4v},\ \tau_1^{2\Gamma} = -\frac{3u^2}{4v\sqrt{v}},\ \tau_{11}^{1H} = \frac{u}{8v\sqrt[4]{v^3}},\ \tau_{11}^{1V} = \frac{1}{2\sqrt[4]{v^3}},$$

$$\tau_{21}^{1H} = \frac{1}{8v\sqrt[4]{v}},\ \tau_{11}^{2H} = \frac{2xv^2 - u^2}{2v^2\sqrt[4]{v}},\ \tau_{11}^{2V} = \frac{-u}{v\sqrt[4]{v}},\ \tau_{12}^{2H} = \frac{-u}{8v\sqrt[4]{v^3}},$$

$$\tau_{12}^{2V} = -\frac{1}{2\sqrt[4]{v^3}},\ \tau_{22}^{2V} = -\frac{1}{8v\sqrt[4]{v}},\ \phi^{2V}(R(X_1^H, X_2^H)) = -\sqrt[4]{v}$$

These results show that the eigen co-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable and the third one is integrable that the conditions (1) and (2) hold with

$$h_2 = -\frac{\tau_2^{1\Gamma}}{\tau_1^{2\Gamma}} = -\frac{v}{3u^2}.$$

But the condition (3) does not hold $\Rightarrow$ no non-degenerate

# $\Phi$ is diagonalisable with distinct eigenvalues, $p = 2$ non-integrable co-distributions and $rank(\mathbf{A}_1) = 1$

Existence example:

$$\ddot{x} = zt, \ \ddot{y} = 0, \ \ddot{z} = x, \tag{7}$$

on an appropriate domain. Denoting the derivatives by $u, v, w$, we find that $\mathbf{\Phi}$ is diagonalisable with distinct eigenvalues and corresponding eigenvectors $X_a$ as follows,

$$\begin{aligned} \lambda_1 &= \sqrt{t} \quad && \text{and} \quad && X_1 = (-\sqrt{t}, 0, 1), \\ \lambda_2 &= -\sqrt{t} \quad && \text{and} \quad && X_2 = (\sqrt{t}, 0, 1), \\ \lambda_3 &= 0 \quad && \text{and} \quad && X_3 = (0, 1, 0). \end{aligned}$$

## $\Phi$ is diagonalisable with distinct eigenvalues, $p = 2$ non-integrable, $n - p$ integrable co-distributions

Existence example:

The structure functions $\tau$'s are zero except for

$$\tau_1^{1\Gamma} = \tau_2^{2\Gamma} = -\tau_2^{1\Gamma} = -\tau_1^{2\Gamma} = \frac{1}{4t}.$$

These results show:

- the eigen co-distributions $Sp\{\phi^{1V}, \phi^{1H}\}$ and $Sp\{\phi^{2V}, \phi^{2H}\}$ are non-integrable and the third one is integrable.

$\langle \Sigma^1 \rangle$ is not a differential ideal.

-The conditions (1) and (2) hold with $h_2 = -1$.

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Existence example:

Further examination gives

$$d\tilde{\omega}^1 = -\frac{1}{2t} dt \wedge \tilde{\omega}^1, \quad \tilde{\omega}^1 = \omega^1 - \omega^2,$$

$\Rightarrow$ the condition (3) holds with $\tilde{\xi}^1_1 = -\frac{1}{2t} dt$ and $\tilde{\xi}^1_3 = 0$.

$\Rightarrow \Sigma^2 := Sp\{\tilde{\omega}^1, \omega^3\}$ generates a differential ideal.

The remaining conditions (4) and (5)also hold for solution.

$\Rightarrow$ this system is variational and the solution depends on one arbitrary function of two variables.

# $\Phi$ is diagonalisable with distinct eigenvalues, $p = 2$ non-integrable co-distributions and $rank(\mathbf{A}_1) = 1$

Existence example:

To determine the explicit expression of the Cartan two-form, we examine the Pfaffian equations:

$$d\tilde{r}_1 + \tilde{r}_1\tilde{\xi}_1^1 = 0,$$
$$dr_3 + P_3\phi^{3V} + Q_3\phi^{3H} = 0$$

$\Rightarrow \tilde{r}_1 = G\sqrt{t}$ where $G$ is a constant,
and $r_3 = r_3(u_3^1, u_3^2)$ is an arbitrary function of two variables $u_3^1 = y - vt$ and $u_3^2 = v$.

$$\Rightarrow \Omega = G\sqrt{t}(\omega^1 - \omega^2) + \tilde{r}_3(u_3^1, u_3^2)\omega^3.$$

Thank you for your attention