Classification of the Lorentzian holonomy algebras and its applications

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Holonomy

Let (M, g) be a pseudo-Riemannian manifold of signature (r, s).
Any smooth curve $\gamma : [a, b] \to M$ defines the parallel transport

$$\tau_\gamma : T_{\gamma(a)}M \to T_{\gamma(b)}M.$$

Let $x \in M$. The holonomy group

$$H_x \subset O(T_xM, g_x) \simeq O(r, s)$$

at the point x is the group that consists of parallel transport along all loops at the point x.
The corresponding Lie subalgebra

$$\mathfrak{h}_x \subset \mathfrak{so}(T_xM, g_x) \simeq \mathfrak{so}(r, s)$$

is called the holonomy algebra.

Theorem. (Ambrose-Singer, 1952)

$$\mathfrak{hol}_x = \{(\tau_\gamma)^{-1} \circ R_{\gamma(b)}(\tau_\gamma(X), \tau_\gamma(Y)) \circ \tau_\gamma \Big| \gamma(a) = x,\ X, Y \in T_x M\}.$$

Fundamental principle. *There exists a one-to-one correspondence between parallel tensor fields* $A \in \Gamma(T^{p,q}(M))$ $(\nabla A = 0)$ *and tensors* $A_x \in T_x^{p,q}M$ *preserved by the holonomy group.*

The case of Riemannian manifolds

Let (N, h) be an n-dimensional Riemannian manifold.
By the de Rham theorem, locally (N, h) can be decomposed into a product of a flat space and of some Riemannian manifolds that can not be further decomposed.
This corresponds to the decomposition

$$\mathbb{R}^n = \mathbb{R}^{n_0} \oplus \mathbb{R}^{n_1} \oplus \cdots \oplus \mathbb{R}^{n_r} \tag{1}$$

and the of the tangent space and the decomposition

$$\mathfrak{h} = \{0\} \oplus \mathfrak{h}_1 \oplus \cdots \oplus \mathfrak{h}_r \tag{2}$$

of the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ of (N, h) each $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ is an irreducible Riemannian holonomy algebra.

Irreducible holonomy algebras of Riemannian manifolds (Berger 1953,..., Bryant 1987)

The holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$ of a locally indecomposable not locally symmetric n-dimensional Riemannian manifold (M, g) coincides with one of the following subalgebras of $\mathfrak{so}(n)$:

$\mathfrak{so}(n)$, $\mathfrak{u}(\frac{n}{2})$, $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$, $\mathfrak{sp}(\frac{n}{4})$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}_7 \subset \mathfrak{so}(8)$.

Holonomy algebras of symmetric Riemannian manifolds different from $\mathfrak{so}(n)$, $\mathfrak{u}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$ are called symmetric Berger algebras.

Special geometries:

$\mathfrak{so}(n)$: ,,generic" Riemannian manifolds;

$\mathfrak{u}(\frac{n}{2})$: Kählerian manifolds;

$\mathfrak{su}(\frac{n}{2})$: Calabi-Yau manifolds
or special Kählerian manifolds ($\mathrm{Ric} = 0$);

$\mathfrak{sp}(\frac{n}{4})$: hyper-Kählerian manifolds;

$\mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$: quaternionic-Kählerian manifolds;

$\mathfrak{spin}(7)$: 8-dimensional manifolds with a parallel 4-form;

G_2: 7-dimensional manifolds with a parallel 3-form.

Berger classified irreducible subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ linearly generated by the images of the maps from the space

$$\mathcal{R}(\mathfrak{h}) = \{R \in \Lambda^2(\mathbb{R}^n)^* \otimes \mathfrak{h} | R(X,Y)Z + R(Y,Z)X + R(Z,X)Y = 0\}$$

of algebraic curvature tensors of type $\mathfrak{h}$, under the condition that the space

$$\mathcal{R}^{\nabla}(\mathfrak{h}) = \{S \in (\mathbb{R}^n)^* \otimes \mathcal{R}(\mathfrak{h}) | S_X(Y,Z) + S_Y(Z,X) + S_Z(X,Y) = 0\}$$

is not trivial.

The Berger result can be reformulated in the following way:

If the connected holonomy group of an indecomposable Riemannian manifold does not act transitively on the unite sphere of the tangent space, then the manifold is locally symmetric.

Direct proofs: J. Simens 1962, C. Olmos 2005.

The spaces $\mathcal{R}(\mathfrak{h})$ for subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$ computed D.V. Alekseevsky,

$$\mathcal{R}(\mathfrak{h}) = \mathcal{R}_0(\mathfrak{h}) \oplus \mathcal{R}_1(\mathfrak{h}) \oplus \mathcal{R}'(\mathfrak{h}),$$

where $\mathcal{R}_0(\mathfrak{h})$ consists of tensors with zero Ricci tensor, $\mathcal{R}_1(\mathfrak{h})$, consists of tensors annihilated by $\mathfrak{h}$.

For example, $\mathcal{R}(\mathfrak{h}) = \mathcal{R}_0(\mathfrak{h})$ for $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4})$, $G_2 \subset \mathfrak{so}(7)$ and $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$, whence the corresponding manifolds are Ricci-flat.

Similarly, for $\mathfrak{h} = \mathfrak{sp}(\frac{n}{4}) \oplus \mathfrak{sp}(1)$, $\mathcal{R}(\mathfrak{h}) = \mathcal{R}_0(\mathfrak{h}) \oplus \mathcal{R}_1(\mathfrak{h})$, and the corresponding manifolds are Einstein ($\mathrm{Ric} = \Lambda g$, $\Lambda \neq 0$).

In M-theory and String theories:
our universe is locally a product:

$$\mathbb{R}^{1,3} \times M,$$

where M is a compact Riemannian manifold of dimension 6,7 or 8 (depending on the theory) with the holonomy algebra $\mathfrak{su}(3)$, G_2, or $\mathfrak{spin}(7)$.

The case of Lorentzian manifolds

By the Wu theorem, any Lorentzian manifold is either locally indecomposable, or it is locally a product of a Riemannian manifold and of a locally indecomposable Lorentzian manifold.

If (M, g) is locally indecomposable, then its holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ is weakly irreducible, i.e. it does not preserve any non-degenerate subspace of the tangent space. But $\mathfrak{g}$ may preserve a degenerate subspace, e.g. an isotropic line. Then $\mathfrak{g}$ is not reductive anymore.

Consider locally indecomposable Lorentzian manifold (M, g) of dimension $n+2$ with the holonomy algebra $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$.

If $\mathfrak{g} \subset \mathfrak{so}(1, n+1)$ is irreducible, then $\mathfrak{g} = \mathfrak{so}(1, n+1)$.

If $\mathfrak{g} \neq \mathfrak{so}(1, n+1)$ then $\mathfrak{g}$ preserves an isotropic line $l \subset \mathbb{R}^{1,n+1}$.

Then (M, g) admits a parallel distribution of isotropic lines and local recurrent lightlike vector fields p, i.e.

$$g(p, p) = 0, \qquad \nabla_X p = \theta(X) p$$

for a 1-form θ.

On such manifolds there exist the Walker coordinates $v, x^1, ..., x^n, u$ such that $\partial_v = p$ and

$$g = 2\mathrm{d}v\mathrm{d}u + h + 2A\mathrm{d}u + H(\mathrm{d}u)^2, \tag{3}$$

$h = h_{ij}(x^1, ..., x^n, u)\mathrm{d}x^i\mathrm{d}x^j$ is an u-dependent family of Riemannian metrics,

$A = A_i(x^1, \ldots, x^n, u)\ \mathrm{d}x^i$ is an u-dependent family of one-forms,

$H = H(v, x^1, ..., x^n, u)$ is a local function on M

Let (M, g) be a locally indecomposable Lorentzian manifold with a parallel distribution I of isotropic lines.

Let $p, e_1, ..., e_n, q$ be a Witt basis $((p, q) = (e_i, e_i) = 1)$ of $\mathbb{R}^{1,n+1} \simeq T_x M$ such that $\mathbb{R}p$ corresponds to the distribution I.

The holonomy algebra $\mathfrak{g}$ of (M, g) is contained in the maximal subalgebra of $\mathfrak{so}(1, n+1)$ preserving $\mathbb{R}p$,

$$\mathfrak{g} \subset \mathfrak{sim}(n) = \left\{ \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{R}, \\ X \in \mathbb{R}^n, \\ A \in \mathfrak{so}(n) \end{array} \right\} = (\mathbb{R} \oplus \mathfrak{so}(n)) \ltimes \mathbb{R}^n.$$

Example. A Lorentzian manifold (M, g) is a pp-wave, i.e. its metric is of the form

$$g = dvdu + \sum_{i=1}^{n} (dx^i)^2 + H(du)^2, \quad \partial_v H = 0,$$

if and only if the holonomy algebra of (M, g) is contained in $\mathbb{R}^n \subset \mathfrak{sim}(n)$.

Theorem. (L. Berard-Bergery, A. Ikemakhen 1993)
Weakly-irreducible subalgebras $\mathfrak{g} \subset \mathfrak{sim}(n)$:

Type I. $\mathfrak{g}^{1,\mathfrak{h}} = \left\{ \begin{pmatrix} a & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -a \end{pmatrix} \middle| \begin{array}{c} a \in \mathbb{R}, \\ A \in \mathfrak{h}, \\ X \in \mathbb{R}^n \end{array} \right\} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n,$

Type II. $\mathfrak{g}^{2,\mathfrak{h}} = \mathfrak{h} \ltimes \mathbb{R}^n,$

Type III. $\mathfrak{g}^{3,\mathfrak{h},\varphi} = \left\{ \begin{pmatrix} \varphi(A) & X^t & 0 \\ 0 & A & -X \\ 0 & 0 & -\varphi(A) \end{pmatrix} \middle| \begin{array}{c} A \in \mathfrak{h}, \\ X \in \mathbb{R}^n \end{array} \right\},$

Type IV. $\mathfrak{g}^{4,\mathfrak{h},m,\psi} = \left\{ \begin{pmatrix} 0 & X^t & \psi(A)^t & 0 \\ 0 & A & 0 & -X \\ 0 & 0 & 0 & -\psi(A) \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \begin{array}{c} A \in \mathfrak{h}, \\ X \in \mathbb{R}^m \end{array} \right\},$

where $\mathfrak{h} \subset \mathfrak{so}(n)$ is a subalgebra; $\varphi : \mathfrak{h} \to \mathbb{R}$ is a non-zero linear map, $\varphi|_{[\mathfrak{h},\mathfrak{h}]} = 0$;
for the last algebra $\mathbb{R}^n = \mathbb{R}^m \oplus \mathbb{R}^{n-m}$, $\mathfrak{h} \subset \mathfrak{so}(m)$, and
$\psi : \mathfrak{h} \to \mathbb{R}^{n-m}$ is a surjective linear map, $\psi|_{[\mathfrak{h},\mathfrak{h}]} = 0$.
(Recall that $\mathfrak{h} = [\mathfrak{h}, \mathfrak{h}] \oplus \mathfrak{z}(\mathfrak{h})$)

Geometric proof: a weakly irreducible not irreducible Lie subgroup $G \subset \mathrm{SO}(1, n+1)$ acts on the boundary of the hyperbolic space $\partial H^{n+1} \simeq S^n$ and preserves the point $\mathbb{R}p \in \partial H^{n+1}$, i.e. G acts on the Euclidean space $\mathbb{R}^n = S^n \backslash \{\mathrm{point}\}$.

This action is by similarity transformations

$$G \subset \mathrm{Sim}^0(n) = (\mathbb{R}^+ \times \mathrm{SO}(n)) \leftthreetimes \mathbb{R}^n$$

and transitive. All such groups are known.

T. Leistner, 2003: the subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ associated to a Lorentzian holonomy algebra $\mathfrak{g} \subset \mathfrak{sim}(n)$ is spanned by the images of the maps from the space

$$\mathcal{P}(\mathfrak{h}) = \{P \in \mathrm{Hom}(\mathbb{R}^n, \mathfrak{h}) | \\ g(P(X)Y, Z) + g(P(Y)Z, X) + g(P(Z)X, Y) = 0\}.$$

Each such subalgebra is the holonomy algebra of a Riemannian manifold.

$(R \in \mathcal{R}(\mathfrak{h}),\ X_0 \in \mathbb{R}^n \Longrightarrow R(\cdot, X_0) \in \mathcal{P}(\mathfrak{h}))$

About the proof:

If $\mathfrak{h} \subset \mathfrak{u}(m) \subset \mathfrak{so}(2m)$, then $\mathcal{P}(\mathfrak{h}) \simeq (\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{gl}(m, \mathbb{C}))^{(1)}$
$= \{\varphi : \mathbb{C}^m \to \mathfrak{h} \otimes \mathbb{C} | \varphi(X)Y = \varphi(Y)X,\ X, Y \in \mathbb{C}^m\}$.

If $\mathfrak{h} \not\subset \mathfrak{u}(m)$, then $\mathfrak{h} \otimes \mathbb{C} \subset \mathfrak{so}(n, \mathbb{C})$ is irreducible and it is necessary to use the classification of irreducible representations of complex semisimple Lie algebras.

Construction of metrics with each possible holonomy algebra:

L. Berard-Bergery, A. Ikemakhen, 1993

$$g = 2dvdu + h + H(du)^2$$

if $\partial_v H = 0$ and H is quite general, then $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^n$

if $H = v^2 + H_0$, $\partial_v H_0 = 0$ and H_0 is quite general, then $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n$

Lemma. For each subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$ there exists $P \in \mathcal{P}(\mathfrak{h})$ such that its image generates $\mathfrak{h}$.

Construction

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + 2A_i dx^i du + H \cdot (du)^2,$$

where $A_i = \frac{1}{3}(P^i_{jk} + P^i_{kj})x^j x^k$, $P(e_i)e_j = P^k_{ji} e_k$
For $\mathfrak{g}^{3,\mathfrak{h},\varphi}$ let $\varphi_i = \varphi(P(e_i))$. For $\mathfrak{g}^{4,\mathfrak{h},m,\psi}$ let ψ_{ij}, $j = m+1, ..., n$, be such that $\psi(P(e_i)) = -\sum_{j=m+1}^{n} \psi_{ij} e_j$.

H	$\mathfrak{g}$
$v^2 + \sum_{i=1}^{n}(x^i)^2$	$\mathfrak{g}^{1,\mathfrak{h}}$
$\sum_{i=1}^{n}(x^i)^2$	$\mathfrak{g}^{2,\mathfrak{h}}$
$2v\varphi_i x^i + \sum_{i=1}^{n}(x^i)^2$	$\mathfrak{g}^{3,\mathfrak{h},\varphi}$
$2\sum_{j=m+1}^{n} \psi_{ij} x^i x^j + \sum_{i=1}^{m}(x^i)^2$	$\mathfrak{g}^{4,\mathfrak{h},m,\psi}$

Curvature tensors

Each curvature tensor $R \in \mathcal{R}(\mathfrak{g}^{1,\mathfrak{h}})$ is uniquely given by

$$\lambda \in \mathbb{R}, \quad \vec{v} \in \mathbb{R}^n, \quad R_0 \in \mathcal{R}(\mathfrak{h}), \quad P \in \mathcal{P}(\mathfrak{h}), \quad T \in \odot^2\mathbb{R}^n :$$

$$\begin{aligned} R(p,q) =& (\lambda p, 0, \vec{v}), \\ R(X,Y) =& (0, R_0(X,Y), P(Y)X - P(X)Y), \\ R(X,q) =& (g(\vec{v},X), P(X), T(X)), \quad R(p,X) = 0. \end{aligned}$$

In particular, there exists an isomorphism of $\mathfrak{h}$-modules

$$\mathcal{R}(\mathfrak{g}^{1,\mathfrak{h}}) \simeq \mathbb{R} \oplus \mathbb{R}^n \oplus \odot^2\mathbb{R}^n \oplus \mathcal{R}(\mathfrak{h}) \oplus \mathcal{P}(\mathfrak{h}).$$

Next,

$$\begin{aligned} \mathcal{R}(\mathfrak{g}^{2,\mathfrak{h}}) &= \{R \in \mathcal{R}(\mathfrak{g}^{1,\mathfrak{h}}) | \lambda = 0, \vec{v} = 0\}, \\ \mathcal{R}(\mathfrak{g}^{3,\mathfrak{h},\varphi}) &= \{R \in \mathcal{R}(\mathfrak{g}^{1,\mathfrak{h}}) | \lambda = 0,\ R_0 \in \mathcal{R}(\ker\varphi),\ g(\vec{v},\cdot) = \varphi(P(\cdot))\}, \\ \mathcal{R}(\mathfrak{g}^{4,\mathfrak{h},m,\psi}) &= \{R \in \mathcal{R}(\mathfrak{g}^{2,\mathfrak{h}}) | R_0 \in \mathcal{R}(\ker\psi),\ \mathrm{pr}_{\mathbb{R}^{n-m}} \circ T = \psi \circ P\}. \end{aligned}$$

$$g = 2dvdu + h_{ij}dx^i dx^j + 2A_i dx^i du + H(du)^2,$$

Consider the frame $p = \partial_v$, $X_i = \partial_{x^i} - A_i\partial_v$, $q = \partial_u - \frac{1}{2}H\partial_v$.
R is given by the tensor fields λ, $\vec{v}$, R_0, P and T,

$$\lambda = \frac{1}{2}\partial_v^2 H, \quad \vec{v} = \frac{1}{2}\left(\partial_i\partial_v H - A_i\partial_v^2 H\right) h^{ij} X_j,$$

$$h_{il}P^l_{jk} = -\frac{1}{2}\nabla_k F_{ij} + \frac{1}{2}\nabla_k \dot{h}_{ij} - \dot{\Gamma}^l_{kj} h_{li}, \quad F_{ij} = \partial_i A_j - \partial_j A_i$$

$$\begin{aligned} T_{ij} = &\frac{1}{2}\nabla_i\nabla_j H - \frac{1}{4}(F_{ik} + \dot{h}_{ik})(F_{jl} + \dot{h}_{jl})h^{kl} - \frac{1}{4}(\partial_v H)(\nabla_i A_j + \nabla_j A_i) \\ &- \frac{1}{2}(A_i\partial_j\partial_v H + A_j\partial_i\partial_v H) - \frac{1}{2}(\nabla_i \dot{A}_j + \nabla_j \dot{A}_i) \\ &+ \frac{1}{2}A_i A_j \partial_v^2 H + \frac{1}{2}\ddot{h}_{ij} + \frac{1}{4}\dot{h}_{ij}\partial_v^2 H. \end{aligned}$$

The spaces $\mathcal{P}(\mathfrak{h})$

Let $\mathfrak{h} \subset \mathfrak{so}(n)$ be irreducible. Consider the $\mathfrak{h}$-equivariant map

$$\widetilde{\mathrm{Ric}} : \mathcal{P}(\mathfrak{h}) \to \mathbb{R}^n, \qquad \widetilde{\mathrm{Ric}}(P) = \sum_{i=1}^{n} P(e_i)e_i.$$

Let $\mathcal{P}_0(\mathfrak{h}) = \ker \widetilde{\mathrm{Ric}}$, $\mathcal{P}_1(\mathfrak{h}) = \mathcal{P}_0(\mathfrak{h})^{\perp}$. Then

$$\mathcal{P}(\mathfrak{h}) = \mathcal{P}_0(\mathfrak{h}) \oplus \mathcal{P}_1(\mathfrak{h}).$$

$\mathfrak{h} \subset \mathfrak{so}(n)$	$\mathcal{P}_1(\mathfrak{h})$	$\mathcal{P}_0(\mathfrak{h})$	$\dim \mathcal{P}_0(\mathfrak{h})$
$\mathfrak{so}(2)$	$\mathbb{R}^2$	0	0
$\mathfrak{so}(3)$	$\mathbb{R}^3$	$V_{4\pi_1}$	5
$\mathfrak{so}(4)$	$\mathbb{R}^4$	$V_{3\pi_1+\pi_1'} \oplus V_{\pi_1+3\pi_1'}$	16
$\mathfrak{so}(n),\ n \geq 5$	$\mathbb{R}^n$	$V_{\pi_1+\pi_2}$	$\frac{(n-2)n(n+2)}{3}$
$\mathfrak{u}(m),\ n = 2m \geq 4$	$\mathbb{R}^n$	$(\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$	$m^2(m-1)$
$\mathfrak{su}(m),\ n = 2m \geq 4$	0	$(\odot^2(\mathbb{C}^m)^* \otimes \mathbb{C}^m)_0$	$m^2(m-1)$
$\mathfrak{sp}(m) \oplus \mathfrak{sp}(1),\ n = 4m \geq 8$	$\mathbb{R}^n$	$\odot^3(\mathbb{C}^{2m})^*$	$\frac{m(m+1)(m+2)}{3}$
$\mathfrak{sp}(m),\ n = 4m \geq 8$	0	$\odot^3(\mathbb{C}^{2m})^*$	$\frac{m(m+1)(m+2)}{3}$
$G_2 \subset \mathfrak{so}(7)$	0	$V_{\pi_1+\pi_2}$	64
$\mathfrak{spin}(7) \subset \mathfrak{so}(8)$	0	$V_{\pi_2+\pi_3}$	112
$\mathfrak{h} \subset \mathfrak{so}(n),\ n \geq 4,$ is a symmetric Berger alg.	$\mathbb{R}^n$	0	0

Corollary. $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n \Leftrightarrow \exists$ symmetric space with the holonomy algebra $\mathfrak{h}$;
$\mathcal{P}_0(\mathfrak{h}) \neq 0 \Leftrightarrow \exists$ non-locally symmetric space with the holonomy algebra $\mathfrak{h}$;

A direct proof of Leistner's theorem for semisimple non-simple irreducible subalgebras $\mathfrak{h} \subset \mathfrak{so}(n)$.

If $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{so}(n_1, \mathbb{C}) \oplus \mathfrak{so}(n_2, \mathbb{C})$ or $\mathfrak{h} \otimes \mathbb{C} = \mathfrak{sp}(n_1, \mathbb{C}) \oplus \mathfrak{sp}(n_2, \mathbb{C})$ ($n_i \geq 3$), then it holds $\mathcal{P}(\mathfrak{h} \otimes \mathbb{C}) \simeq \mathbb{C}^n$

Thus it is enough to consider the case
$\mathfrak{h} \otimes \mathbb{C} = \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{k} \subset \mathfrak{so}(4m, \mathbb{C}), \ \ \mathfrak{k} \subsetneq \mathfrak{sp}(2m, \mathbb{C})$.

It holds $\mathcal{P}(\mathfrak{h}) \otimes \mathbb{C} \simeq \mathbb{C}^2 \otimes \mathfrak{g}_1$, where $\mathfrak{g}_1$ is the first Tanaka prolongation of the Lie algebra $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0$, where
$\mathfrak{g}_{-2} = \mathbb{C}, \quad \mathfrak{g}_{-1} = \mathbb{C}^{2m}, \quad \mathfrak{g}_0 = \mathfrak{k} \oplus \mathbb{C}\mathrm{id}_{\mathbb{C}^{2m}};$

$$\mathfrak{g}_1 = \{\varphi : \mathbb{C}^{2m} \to \mathfrak{g}_0 | \exists Z \in \mathbb{C}^{2m},\ \varphi(X)Y - \varphi(Y)X = \omega(X, Y)Z\}.$$

If $\mathcal{P}(\mathfrak{h}) \neq 0$, then the total Tanaka prolongation defines the simple $|2|$-graded complex Lie algebra

$$\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2.$$

Such Lie algebra defines a simply connected quaternionic-Kählerian manifold with the holonomy algebra $\mathfrak{h}$.

Farther remarks.

Suppose now that $\mathfrak{h} \subset \mathfrak{so}(n)$ is simple and irreducible of non-complex type. Prove the following facts:

• If $\mathcal{P}_1(\mathfrak{h}) \neq 0$ (i.e. $\mathcal{P}_1(\mathfrak{h}) \simeq \mathbb{R}^n$), then the $\mathfrak{h}$-equivariant map $R : \mathbb{R}^n \to \mathcal{P}(\mathfrak{h})$ satisfies $R(X)(Y) = -R(Y)X$, i.e. it belongs to $\mathcal{R}(\mathfrak{h})$ and it is the curvature tensor of a symmetric space with the holonomy algebra $\mathfrak{h} \subset \mathfrak{so}(n)$.
(Such proof is obtained under the assumption that there are not more then two non-zero labels on the Dynkin diagram of the representation)

• If the connected Lie subgroup $H \subset \mathrm{SO}(n)$ corresponding to $\mathfrak{h} \subset \mathfrak{so}(n)$ does not act transitively on the unite sphere, then $\mathcal{P}_0(\mathfrak{h}) = 0$.

Remark on other signatures

Consider the Witt basis $p_1, p_1, e_1, ..., e_n, q_2, q_1$
$(g(p_a, q_a) = g(e_i, e_i) = 1)$

$$\mathfrak{g}^{\mathfrak{h}} = \left\{ \begin{pmatrix} 0 & 0 & -Y^t & -c & 0 \\ 0 & 0 & -X^t & 0 & c \\ 0 & 0 & A & X & Y \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \middle| \; A \in \mathfrak{h}, X, Y \in \mathbb{R}^n, c \in \mathbb{R} \right\}$$

$\mathfrak{g}^{\mathfrak{h}} \subset \mathfrak{so}(2, n+2)$ is the holonomy algebra for an arbitrary subalgebra $\mathfrak{h} \subset \mathfrak{so}(n)$

APPLICATIONS

Applications to the Einstein Equation

Recently the Einstein Equation on Lorentzian manifolds with special holonomy is considered in

G. W. Gibbons, C. N. Pope, *Time-Dependent Multi-Centre Solutions from New Metrics with Holonomy* $\mathrm{Sim}(n-2)$, Class. Quantum Grav. 25 (2008) 125015 (21pp).

(M, g) is called Einstein if

$$\mathrm{Ric} = \Lambda g, \quad \Lambda \in \mathbb{R}.$$

$$\mathrm{Ric} = \Lambda g \iff \lambda = -\Lambda,\ \mathrm{Ric}_0 = \Lambda h,\ \widetilde{\mathrm{Ric}}\, P = \vec{v},\ \mathrm{tr}\, T = 0$$

Theorem

If (M, g) is Ricci-flat, then one of the following holds:

(1) *The holonomy algebra of (M, g) coincides with*

$$(\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n,$$

and in the decomposition (2) of $\mathfrak{h} \subset \mathfrak{so}(n)$ at least one subalgebra $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{u}(\frac{n_i}{2})$, $\mathfrak{sp}(\frac{n_i}{4}) \oplus \mathfrak{sp}(1)$ or with a symmetric Berger algebra.

(2) *The holonomy algebra of (M, g) coincides with*

$$\mathfrak{h} \ltimes \mathbb{R}^n,$$

and in the decomposition (2) of $\mathfrak{h} \subset \mathfrak{so}(n)$ each subalgebra $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{su}(\frac{n_i}{2})$, $\mathfrak{sp}(\frac{n_i}{4})$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$.

Theorem

If (M, g) is Einstein and not Ricci-flat, then the holonomy algebra of (M, g) coincides with

$$(\mathbb{R} \oplus \mathfrak{h}) \ltimes \mathbb{R}^n,$$

and in the decomposition (2) *of $\mathfrak{h} \subset \mathfrak{so}(n)$ each subalgebras $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras $\mathfrak{so}(n_i)$, $\mathfrak{u}(\frac{n_i}{2})$, $\mathfrak{sp}(\frac{n_i}{4}) \oplus \mathfrak{sp}(1)$ or with a symmetric Berger algebra. Moreover, in the decomposition* (1) *it holds $n_{s+1} = 0$.*

Let $n = 2$, i.e. $\dim M = 4$

If (M, g) is Ricci-flat, then either
$\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{so}(2)) \ltimes \mathbb{R}^2$, or $\mathfrak{g} = \mathbb{R}^2$
(the last case corresponds to pp-waves).

If (M, g) is Einstein with $\Lambda \neq 0$, then $\mathfrak{g} = (\mathbb{R} \oplus \mathfrak{so}(2)) \ltimes \mathbb{R}^2$.
These statements are already proved in the papers of Schell, Hall, Lonie.

Unlike to the case of Riemannian manifolds, it can not be stated that a Lorentzian manifold with some holonomy algebra is automatically an Einstein manifold, but there is a weaker statement.
(M, g) is called *totally Ricci-isotropic* if the image of its Ricci operator is isotropic. If (M, g) is a spin Lorentzian manifold and it admits a parallel spinor, then it is totally Ricci-isotropic (but not necessary Ricci-flat, unlike in the Riemannian case)

Theorem

The holonomy algebras of totally Ricci-isotropic (M, g) are the same as for Ricci-flat (M, g).
Conversely, if the holonomy algebra of (M, g) is $\mathfrak{h} \ltimes \mathbb{R}^n$ and in the decomposition (2) of $\mathfrak{h} \subset \mathfrak{so}(n)$ each subalgebra $\mathfrak{h}_i \subset \mathfrak{so}(n_i)$ coincides with one of the Lie algebras
$\mathfrak{su}(\frac{n_i}{2})$, $\mathfrak{sp}(\frac{n_i}{4})$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}(7) \subset \mathfrak{so}(8)$,
then (M, g) is totally Ricci-isotropic.

The form of the Einstein equation

Consider the Walker metric

$$g = 2\mathrm{d}v\mathrm{d}u + h + 2A\mathrm{d}u + H(\mathrm{d}u)^2,$$

$h = h_{ij}(x^1, ..., x^n, u)\mathrm{d}x^i\mathrm{d}x^j$ is an u-dependent family of Riemannian metrics,

$A = A_i(x^1, \ldots, x^n, u)\ \mathrm{d}x^i$ is an u-dependent family of one-forms,

$H = H(v, x^1, ..., x^n, u)$ is a local function on M

(M, g) is Einstein iff

$$H = \Lambda v^2 + vH_1 + H_0, \qquad \partial_v H_1 = \partial_v H_0 = 0, \tag{4}$$

$$\Delta H_0 - \frac{1}{2} F^{ij} F_{ij} - 2A^i \partial_i H_1 - H_1 \nabla^i A_i + 2\Lambda A^i A_i$$

$$-2\nabla^i \dot{A}_i + \frac{1}{2} \dot{h}^{ij} \dot{h}_{ij} + h^{ij} \ddot{h}_{ij} + \frac{1}{2} h^{ij} \dot{h}_{ij} H_1 = 0, \tag{5}$$

$$\nabla^j F_{ij} + \partial_i H_1 - 2\Lambda A_i + \nabla^j \dot{h}_{ij} - \partial_i (h^{jk} \dot{h}_{jk}) = 0, \tag{6}$$

$$\Delta H_1 - 2\Lambda \nabla^i A_i + \Lambda h^{ij} \dot{h}_{ij} = 0, \tag{7}$$

$$\mathrm{Ric}_{ij} = \Lambda h_{ij}, \tag{8}$$

where $F_{ij} = \partial_i A_j - \partial_j A_i$.

A special example of the Walker metric is the metric of a pp-wave

$$g = 2\mathrm{d}v\mathrm{d}u + \sum_{i=1}^{n}(\mathrm{d}x^i)^2 + H\cdot(\mathrm{d}u)^2, \quad \partial_v H = 0. \tag{9}$$

If such metric is Einstein, then it is Ricci-flat, and it is Ricci-flat if and only if $\sum_{i=1}^{n}\partial_i^2 H = 0$.

Simplification of the Einstein equation

The Walker coordinates are not defined canonically!

And any other Walker coordinates $\tilde{v}, \tilde{x}^1, \ldots, \tilde{x}^n, \tilde{u}$ such that $\partial_{\tilde{v}} = \partial_v$ are given by

$$\tilde{v} = v + \varphi(x^1, ..., x^n, u), \quad \tilde{x}^i = \psi^i(x^1, ..., x^n, u), \quad \tilde{u} = u + c.$$

The aim: find new coordinates in order to simplify the Einstein Equation.

Theorem [Galaev, Leistner 2010] (M, g) *is an Einstein manifold with* $\Lambda \neq 0$ *iff there exist Walker coordinates* $v, x^1, ..., x^n, u$ *such that* $A = 0$ *and* $H_1 = 0$*, i.e.*

$$g = 2\mathrm{d}v\mathrm{d}u + h + (\Lambda v^2 + H_0)(\mathrm{d}u)^2,$$

and

$$\Delta H_0 + \frac{1}{2} h^{ij} \ddot{h}_{ij} = 0, \tag{10}$$

$$\nabla^j \dot{h}_{ij} = 0, \tag{11}$$

$$h^{ij} \dot{h}_{ij} = 0, \tag{12}$$

$$\mathrm{Ric}_{ij} = \Lambda h_{ij}, \tag{13}$$

where $\dot{h}_{ij} = \partial_u h_{ij}$.

Proof.

At a point: choose other vector q'.

There exists a unique vector $w \in E$ such that

$q' = -\frac{1}{2}g(w,w)p + w + q$ and $E' = \{-g(x,w)p + x | x \in E\}$.

Consider the map $x \in E \mapsto x' = -g(x,w)p + x \in E'$.

$R = R(\lambda', \tilde{v}, R_0', P', T')$.

$\lambda' = \lambda$, $\tilde{v} = (\vec{v} - \lambda w)'$, $P'(x') = (P(x) - R_0(x,w))'$,

$R_0'(x', y') = (R_0(x,y))'$

On the manifold:

Proposition

Let x^a be Walker coordinates. For any $W \in \Gamma(E)$ such that $\nabla_{\partial_v} W = 0$ there exist new Walker coordinates $\tilde{x}^a$ such that the corresponding vector field q' has the form
$q' = -\frac{1}{2} g(W, W)p + W + q$.

Proof. Find $x^i = x^i(\tilde{x}^1, ..., \tilde{x}^n, \tilde{u})$, $u = \tilde{u}$.
$W = W^i X_i$
We need only to solve the system of ODE:

$$\frac{\partial x^i(\tilde{x}^1, ..., \tilde{x}^n, \tilde{u})}{\partial \tilde{u}} = W^i(x^1(\tilde{x}^1, ..., \tilde{x}^n, \tilde{u}), ..., x^n(\tilde{x}^1, ..., \tilde{x}^n, \tilde{u}), \tilde{u}).$$

□

Proof of the theorem

It holds

$$\vec{v} = \left(\frac{1}{2}\partial_i H_1 - \lambda A_i\right) h^{ij} X_j.$$

Find new coordinates such that $\tilde{v} = 0$
(recall that $\tilde{v} = \vec{v} - \lambda W$, $\lambda = -\Lambda$)

For this take in the above proposition $W = -\frac{1}{\Lambda}\vec{v}$

In the new coordinates get

$$A_i = \frac{1}{2\Lambda}\partial_i H_1.$$

under the transformation $\tilde{v} = v - f(x^1, ..., x^n, u)$, $\tilde{x}^i = x^i$, $\tilde{u} = u$
it holds

$$A_i \mapsto A_i + \partial_i f, \quad H_1 \mapsto H_1 + 2\Lambda f.$$

Take $f = -\frac{1}{2\Lambda}H_1$, then with respect to the new coordinates
$A_i = H_1 = 0$. $\square$

Theorem (Schimming 1974; Galaev, Leistner 2010). *On any* (M, g) *there exist Walker coordinates such that* $A_i = 0$ *(and* $H_0 = 0$ *if the manifold is Einstein).*

Then the Einstein Equation is equivalent to the system

$$\frac{1}{2}\dot{h}^{ij}\dot{h}_{ij} + h^{ij}\ddot{h}_{ij} + \frac{1}{2}h^{ij}\dot{h}_{ij}H_1 = 0, \tag{14}$$

$$\partial_i H_1 + \nabla^j \dot{h}_{ij} - \partial_i(h^{jk}\dot{h}_{jk}) = 0, \tag{15}$$

$$\Delta H_1 + \Lambda h^{ij}\dot{h}_{ij} = 0, \tag{16}$$

$$\mathrm{Ric}_{ij} = \Lambda h_{ij}. \tag{17}$$

Lorentzian spin-manifolds with recurrent spinor fields

(M, g), S, ∇^S

$s \in \Gamma(S)$ is recurrent if

$$\nabla^S_X s = \theta(X)s$$

for all vector fields $X \in \Gamma(TM)$, here θ is a complex valued 1-form.

If $\theta = 0$, then s is parallel.

Theorem (Wang 1989) A simply connected locally indecomposable Riemannian spin-manifold admits a parallel spinor if and only if its holonomy algebra is one of $\mathfrak{su}(\frac{n}{2})$, $\mathfrak{sp}(\frac{n}{4})$, $G_2 \subset \mathfrak{so}(7)$, $\mathfrak{spin}_7 \subset \mathfrak{so}(8)$.

Theorem If a simply connected locally indecomposable Riemannian spin-manifold (M, g) admits a recurrent spinor and does not admit a parallel spinor, then (M, g) is Kählerian and not Ricci flat.

Theorem (Leistner 2002) A locally indecomposable simply connected Lorentzian spin-manifold admits a parallel spinor if and only if its holonomy algebra coincides with

$$\mathfrak{h} \ltimes \mathbb{R}^n,$$

where $\mathfrak{h} \subset \mathfrak{so}(n)$ is the holonomy algebra of a Riemannian manifold admitting a parallel spinor.

Theorem If a locally indecomposable simply connected Lorentzian spin-manifold admits a recurrent spinor, then the orthogonal part $\mathfrak{h} \subset \mathfrak{so}(n)$ of the holonomy algebra is the holonomy algebra of a Riemannian manifold admitting a recurrent spinor.

Conformally flat Lorentzian manifolds with special holonomy

Assume that (M, g) is locally indecomposable, conformally flat $(W = 0)$ and the holonomy algebra is contained in $\mathfrak{sim}(n)$

Problem: find all such metrics g!

Theorem. Let (M, g) be a conformally flat Walker Lorentzian manifold. Then locally

$$g = 2dvdu + \Psi \sum_{i=1}^{n} (dx^i)^2 + 2Adu + (\lambda(u)v^2 + vH_1 + H_0)(du)^2,$$

where

$$\Psi = \frac{4}{(1 - \lambda(u) \sum_{k=1}^{n} (x^k)^2)^2},$$

$$A = A_i dx^i, \quad A_i = \Psi \left(-4C_k(u)x^k x^i + 2C_i(u) \sum_{k=1}^{n} (x^k)^2 \right),$$

$$H_1 = -4C_k(u)x^k \sqrt{\Psi} - \partial_u \ln \Psi + K(u),$$

$$s = -(n-2)(n+1)\lambda(u)$$

Theorem.
If the function λ is non-vanishing at a point, then in a neighborhood of this point there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \Psi \sum_{i=1}^{n} (dx^i)^2 + (\lambda(u)v^2 + vH_1 + H_0)(du)^2,$$

where

$$\Psi = \frac{4}{(1 - \lambda(u) \sum_{k=1}^{n} (x^k)^2)^2},$$

$$H_1 = -\partial_u \ln \Psi, \quad H_0 = \sqrt{\Psi} \left(a(u) \sum_{k=1}^{n} (x^k)^2 + D_k(u) x^k + D(u) \right).$$

Theorem. If $\lambda \equiv 0$ in a neighborhood of a point, then in a neighborhood of this point there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + 2Adu + (vH_1 + H_0)(du)^2,$$

where

$$A = A_i dx^i, \quad A_i = C_i(u)\sum_{k=1}^{n}(x^k)^2, \quad H_1 = -2C_k(u)x^k$$

$$H_0 = \sum_{k=1}^{n}(x^k)^2\left(\frac{1}{4}\sum_{k=1}^{n}(x^k)^2\sum_{k=1}^{n}C_k^2(u) - (C_k(u)x^k)^2 + \dot{C}_k(u)x^k + a(u)\right)$$

In particular, if all $C_i \equiv 0$, then the metric can be rewritten in the form

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + a(u)\sum_{k=1}^{n}(x^k)^2(du)^2. \qquad (18)$$

Remarks.
The field equations of Nordström's theory of gravitation, which appeared before Einstein's theory, are the following:

$$W = 0, \quad s = 0.$$

Thus we have found all solutions to Nordström's gravity with holonomy algebras contained in $\mathfrak{sim}(n)$.

The case of dimension 4.
Possible holonomy algebras of conformally flat 4-dimensional Lorentzian manifolds were classified also in

G. S. Hall, D. P. Lonie, *Holonomy groups and spacetimes*, Class. Quantum Grav. 17 (2000), 1369–1382.

It is stated that it is an open problem to construct a conformally flat metric with the holonomy algebra $\mathfrak{sim}(2)$ (which is denoted in by R_{14}).

An attempt to construct such metric is made in
R. Ghanam, G. Thompson, *Two special metrics with R_{14}-type holonomy*, Class. Quantum Grav. 18 (2001), 2007–2014
where the following metric was constructed:

$$g = 2dxdt + 4ydtdy - 4zdtdz + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2} + 2(x + y^2 - z^2)^2(dt)^2.$$

Making the transformation

$$x \mapsto x - y^2 + z^2, \quad y \mapsto y, \quad z \mapsto z, \quad t \mapsto t,$$

we obtain

$$g = 2dxdt + 2x^2(dt)^2 + \frac{(dy)^2}{2y^2} + \frac{(dz)^2}{2y^2}.$$

This metric is decomposible and its holonomy algebra coincides with $\mathfrak{so}(1,1) \oplus \mathfrak{so}(2)$, but not with $\mathfrak{sim}(2)$.

Weyl tensor

$$W = R + R_L,$$

where the tensor R_L is defined by

$$R_L(X, Y) = LX \wedge Y + X \wedge LY,$$

$$L = \frac{1}{d-2}\left(\mathrm{Ric} - \frac{s}{2(d-1)}\mathrm{Id}\right)$$

$d = n + 2$ is the dimension

Lemma The equation $W = 0$ is equivalent to the following system of equations:

$$s_0 = -n(n-1)\lambda, \quad R_0 = -\frac{1}{2}\lambda R_{\mathrm{Id}}, \quad P(X) = \vec{v} \wedge X, \quad T = f\mathrm{id}_E,$$

where X is any section of E and f is a function. In particular, $W = 0$ implies that $\widetilde{\mathrm{Ric}}\, P = -(n-1)\vec{v}$ and the Weyl tensor W_0 of h is zero.

From the lemma it follows that
$\partial_v \lambda = 0$, hence

$$H = \lambda v^2 + H_1 v + H_0, \quad \partial_v H_1 = \partial_v H_0 = 0.$$

Each metric in the family $h(u)$ is of constant sectional curvature with the scalar curvature $s_0 = -n(n-1)\lambda$.
The coordinates can be chosen in such a way that

$$h = \Psi \sum_{k=1}^{n} (dx^k)^2, \quad \Psi = \frac{4}{\left(1 - \lambda(u) \sum_{k=1}^{n} (x^k)^2\right)^2}.$$

Now we must find the 1-form A and the functions H_1 and H_0.

Part of the equations:

$$f_1\delta_{ij} = \frac{1}{2}\nabla_i\nabla_j H_1 - \lambda(u)\frac{1}{2}(\nabla_i A_j + \nabla_j A_i).$$

These equations are equivalent to

$$\nabla_i Z_i = \nabla_j Z_j, \quad \nabla_i Z_j + \nabla_j Z_i = 0, \qquad i \neq j,$$

where

$$Z_i = \lambda A_i - \frac{1}{2}\partial_i H_1$$

and to

$$\partial_i\left(\frac{Z_i}{\Psi}\right) = \partial_j\left(\frac{Z_j}{\Psi}\right), \quad \partial_i\left(\frac{Z_j}{\Psi}\right) + \partial_j\left(\frac{Z_i}{\Psi}\right) = 0, \quad i \neq j.$$

Two-symmetric Lorentzian manifolds

A Lorentzian manifold (M, g) is called two-symmetric if $\nabla^2 R = 0$ and $\nabla R \neq 0$.

First detailed investigation of two-symmetric Lorentzian spaces:
J. M. Senovilla, *Second-order symmetric Lorentzian manifolds. I. Characterization and general results,* Classical Quantum Gravity 25 (2008), no. 24, 245011, 25 pp.

It is proven that any two-symmetric Lorentzian space admits a parallel null vector field.

A classification of four-dimensional two-symmetric Lorentzian spaces is obtained in the paper
O. F. Blanco, M. Sánchez, J. M. Senovilla, *Complete classifcation of second-order symmetric spacetimes.* Journal of Physics: Conference Series 229 (2010), 012021, 5pp.

Theorem (Alekseevsky, Galaev 2010)

Let (M, g) be a locally indecomposable Lorentzian manifold of dimension $n+2$. Then (M, g) is two-symmetric if and only if locally there exist coordinates $v, x^1, ..., x^n, u$ such that

$$g = 2dvdu + \sum_{i=1}^{n}(dx^i)^2 + (H_{ij}u + F_{ij})x^i x^j (du)^2,$$

where H_{ij} is a nonzero diagonal real matrix with the diagonal elements $\lambda_1 \leq \cdots \leq \lambda_n$, and F_{ij} is a symmetric real matrix.

Theorem The holonomy algebra $\mathfrak{g}$ of an $(n+2)$-dimensional locally indecomposable two-symmetric Lorentzian manifold (M, g) is $\mathbb{R}^n \subset \mathfrak{sim}(n)$.

(any such manyfold is a pp-wave!)

Proof.

• reduction using the adapted coordinates of Ch. Boubel $\Rightarrow$ assume that either $\mathfrak{g} = \mathbb{R}^n$ or $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^n$, $\mathfrak{h} \subset \mathfrak{so}(n)$ is irreducible.

• assume that $\mathfrak{g} = \mathfrak{h} \ltimes \mathbb{R}^n$ and show that $\mathcal{R}^{\nabla}(\mathfrak{g})^{\mathfrak{g}}$ is one-dimensional. Then ∇R is defined up to a multiple. It holds $\nabla W = 0$.

• the results of A. Derdzinski and W. Roter show that (M, g) is a pp-wave.

Another proof:
O. F. Blanco, M. Sánchez, J. M. Senovilla, Structure of second-order symmetric Lorentzian manifolds, J. Eur. Math. Soc. 15 (2) (2013) 595–634.

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