Matrix models describing the quantum bosonic membrane in the large N limit

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Outline

Extended objects = minimal surfaces in Minkowski space

Motivation: Theory of Everything, String Theory, M-theory, dimensionally reduced Yang-Mills theory

Problems : exact solvability, non-linearity, quantisation of systems with infinitely many degrees of freedom

Supersymmetric Matrix Models: continuous spectrum and embedded eigenvalues

Extended objects

Let us consider an M-dimensional compact orientable manifold Σ moving in Minkowski space $\mathbb{R}^{1,D}$. The world-volume is given by

$$S[x] = -\int_{\mathbb{R}\times\Sigma} \sqrt{G} d^{M+1}\varphi. \tag{1.1}$$

- φ^a, $a = 0, 1, ..., M$ - local coordinates on S
- $x^\mu(\varphi^0, \varphi^1, ..., \varphi^M)$ - embedding functions, $\mu = 0, 1, ..., D$
- $G = det[G_{\alpha\beta}]$, $G_{\alpha\beta} := \partial_\alpha x^\mu \partial_\beta x^\nu \eta_{\mu\nu}$

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$\Delta x^\mu = 0$, vanishing mean curvature

Extended objects: light-cone formulation

Light-cone variables

$$\tau := \frac{x^0 + x^D}{2}, \quad \zeta := x^0 - x^D, \tag{1.2}$$

Diffeomorphism invariance of S allows to set $\varphi^0 = \tau$ and $G_{0r} = 0, \quad r = 1, ..., D-1 =: d$ (**light-cone gauge**)

$$G_{\alpha\beta} = \begin{pmatrix} 2\dot{\zeta} - \dot{\vec{x}}^2 & 0 \\ 0 & -g_{rs} \end{pmatrix}$$

Light-cone Hamiltonian

$$H_-[\vec{x}, \vec{p}, \zeta_0, P_+] := P_- = \frac{1}{2P_+} \int \frac{p^2 + g}{\rho} d^M\varphi, \quad g = det(g_{rs}) \tag{1.3}$$

polynomial of order 2M!

Extended objects: light-cone formulation, M=2

Lorentz-invariant mass squared for membranes ($M = 2$)

$$\mathbb{M}_d^2 := P^\mu P_\mu = 2P_- P_+ - \vec{P}^2 \tag{1.4}$$

$$= \int_\Sigma \left(\frac{\vec{p}^2}{\rho} + \frac{\rho}{2} \sum_{i,j} \{x_i, x_j\}_\Sigma \right) d^2\varphi - \vec{P}^2, \tag{1.5}$$

where $\{.,.\}_\Sigma$ denotes the Poisson bracket on Σ

$$\{x_i, x_j\}_\Sigma = \frac{1}{\rho} \epsilon^{ab} \frac{\delta x_i}{\delta \varphi^a} \frac{\delta x_j}{\delta \varphi^b}. \tag{1.6}$$

Mode expansion

Expand the 'internal' phase-space variables into Fourier series

$$x_i(\varphi) = x_{i\alpha} Y_\alpha(\varphi) \quad p_i(\varphi) = p_{i\alpha} Y_\alpha(\varphi), \tag{1.7}$$

$\{Y_\alpha\}_{\alpha=1}^{\infty}$ eigenfunctions of Δ

$$\mathbb{M}_d^2 = p_{i\alpha} p_{i\alpha} + \frac{1}{2} g_{\alpha\beta\gamma} g_{\alpha\beta'\gamma'} x_{i\beta} x_{i\beta'} x_{j\gamma} x_{j\gamma'}, \tag{1.8}$$

where $g_{\alpha\beta\gamma} := \int Y_\beta \epsilon^{ab} \partial_a Y_\alpha \partial_b Y_\gamma d^2\varphi$ are the structure constants of the Lie algebra of VPD on Σ generating reparametrisations of Σ,

$$\{x_{i\alpha}, p_{j\beta}\}_{PS} = \delta_{ij}\delta_{\alpha\beta}, \tag{1.9}$$

constraints

$$\phi_\alpha := g_{\alpha\beta\gamma} x_{i\beta} p_{i\gamma} = 0, \tag{1.10}$$

Matrix regularization

Let $N_n \nearrow \infty$, $\hbar_n \searrow 0 \quad n = 1, 2, ...$
Let $T_n : C^\infty(\Sigma) \to \mathcal{M}(N_n \times N_n)$ such that $\lim_{n\to\infty} N_n \hbar_n$ finite and

$$\lim_{n\to\infty} ||T_n(f)|| < \infty, \tag{1.11}$$

$$\lim_{n\to\infty} ||T_n(f)T_n(g) - T_n(fg)|| = 0, \tag{1.12}$$

$$\lim_{n\to\infty} ||\frac{1}{i\hbar_n}[T_n(f), T_n(g)] - T_n(\{f, g\}_\Sigma)|| = 0, \tag{1.13}$$

$$\lim_{n\to\infty} 2\pi\hbar_n Tr(T_n(f)) = \int f\rho d^2\varphi, \tag{1.14}$$

where $[.,.]$ denotes the matrix commutator and $||.||$ is the operator norm. The family of maps T_n is called a matrix regularisation of Σ.
Theorem. T_n exist for Riemannian manifolds. (Hoppe; Klimek and Lesniewski)

Matrix regularization

Let us consider the family of n-dimensional matrix regularizations of the bosonic membrane

$$H_N = Tr(\vec{P}^2) - (2\pi n)^2 n \sum_{i<j}^{d} Tr([X_i, X_j]^2) \tag{1.15}$$

with the constraints

$$\sum_i [X_i, P_i] = 0, \tag{1.16}$$

where $P_i, X_i,\ i = 1, ..., d$ are hermitian traceless matrices of dimension n, approximating/regularizing the full field theoretic Hamiltonian

$$\mathbb{M}^2 = P^\mu P_\mu = \int_\Sigma \left(\frac{\vec{p}^2}{\rho} + \rho \sum_{i<j} \{x_i, x_j\}_\Sigma \right) d^2\varphi \tag{1.17}$$

where $g_{\alpha\beta\gamma} := \int Y_\beta \epsilon^{ab} \partial_a Y_\alpha \partial_b Y_\gamma d^2\varphi$

Classical description - matrix regularization

Using a basis T_a, $a = 1, ..., n^2 - 1 := N$ (with $Tr(T_a T_b) = \delta_{ab}$ and $[T_a, T_b] = i\hbar_n \frac{1}{\sqrt{n}} f^{(n)}_{abc} T_c$, $\hbar_n = \frac{1}{2\pi n}$, $f^{(n)}_{abc} = \frac{2\pi n^{\frac{3}{2}}}{i} Tr(T_a [T_b, T_c]))$ we can rewrite H_N (and the constraints) in terms of $d(n^2 - 1)$ canonical pairs p_{ia}, q_{ia} $(X_i = q_{ia} T_a, P_i = p_{ia} T_a)$

$$H_N(p, q) = p_{ia} p_{ia} + \frac{1}{2} f^{(n)}_{abc} f^{(n)}_{ab'c'} q_{ib} q_{ib'} q_{jc} q_{jc'}, \tag{1.18}$$

$$f^{(n)}_{abc} x_{ib} p_{jc} = 0. \tag{1.19}$$

while the continuum expression is

$$H_\infty = p_{i\alpha} p_{i\alpha} + \frac{1}{2} g_{\alpha\beta\gamma} g_{\alpha\beta'\gamma'} x_{i\beta} x_{i\beta'} x_{j\gamma} x_{j\gamma'} \tag{1.20}$$

$$g_{\alpha\beta\gamma} x_{i\beta} p_{j\gamma} = 0. \tag{1.21}$$

Quantum description - matrix regularization

Canonical quantisation - CQ

$$x_{i\alpha} \to CQ(x_{i\alpha}) = x_{i\alpha}, \tag{1.22}$$

$$p_{i\alpha} \to CQ(p_{i\alpha}) = -i\frac{\partial}{\partial x_{i\alpha}}, \tag{1.23}$$

Membrane Matrix Models:

$$H_N := CQ(\mathbb{M}^2_{n,d}) = -\Delta_{d(n^2-1)} + \frac{1}{2} f^{(n)}_{abc} f^{(n)}_{ab'c'} x_{ib} x_{jc} x_{ib'} x_{jc'}, \tag{1.24}$$

acting on wave-functions $\psi \in L^2(\mathbb{R}^{d(n^2-1)})$ constrained by

$$\phi_a \psi = f^{(n)}_{abc} x_{ib} \partial_{ic} \psi = 0. \tag{1.25}$$

essentially self-adjoint, positive, purely discrete spectrum (Simon, L üscher)

Matrix regularization - summary

$$
\begin{tikzcd}
\mathbb{M}^2_{n,d} \arrow[r, "CQ"] \arrow[d, bend right, "n \to \infty"'] & H_{n,d} \arrow[d, "n \to \infty"] \\
\mathbb{M}^2_d \arrow[u, bend right, "T_n"'] \arrow[r, dotted, "?"] & H_{\infty,d}
\end{tikzcd}
$$

Relations between the 'classical continuum mass-squared' of the bosonic membrane $\mathbb{M}^2_d$, its classical finite n regularisation $\mathbb{M}^2_{n,d}$, the quantum finite n Hamiltonian $H_{n,d}$ and the (still rather elusive) quantum continuum Hamiltonian $H_{\infty,d}$.

Towards a quantum description - rescaling

Rescaling of the classical phase-space variables ($N := n^2 - 1$)

$$q_{ia} \to N^{-\alpha} q_{ia}, p_{ia} \to N^{\alpha} p_{ia}$$

leads to the following classical energy ($N^{-2\alpha} H_{nd} \leftrightarrow H_{nd}$)

$$H_N(p,q) = p_{ia} p_{ia} + N^{-6\alpha} \frac{1}{2} f^{(n)}_{abc} f^{(n)}_{ab'c'} q_{ib} q_{ib'} q_{jc} q_{jc'}. \tag{1.26}$$

More general model

$$H_n(p,q) = \sum_{I \in \mathcal{J}_N} p_I p_I + \sum_{I \in \mathcal{J}_N} \omega_{0I}^2 q_I q_I + n^{-6\alpha} \sum_{I,J,K,L \in \mathcal{J}_N} c^{(N)}_{IJKL} q_I q_J q_K q_L \tag{1.27}$$

Quantum description for finite N in Fock space

- Annihilation-creation operators

$$a_I(\omega_I) = \frac{1}{\sqrt{2}}\left(\frac{\partial_I}{\sqrt{\omega_I}} + \sqrt{\omega_I}x_I\right)$$
$$a_I^\dagger(\omega_I) = \frac{1}{\sqrt{2}}\left(-\frac{\partial_I}{\sqrt{\omega_I}} + \sqrt{\omega_I}x_J\right) \tag{2.1}$$

with

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- vacuum $\Psi_0(\omega) := \Pi_{I=1}^{\infty}\psi_{\omega_I}(x_I)$ with $\psi_{\omega_I}(x_I) := \sqrt[4]{\frac{\omega_I}{\pi}}e^{-\frac{1}{2}\omega_I x_I^2}$, $a_I(\omega_I)\Psi_0(\omega) = 0\ \forall I$

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- standard bosonic Fock space $\mathcal{H}_\omega$, i.e. the Hilbert space generated by

$$\psi_{\{I_1,\ldots,I_k\}} := a_{I_1}^\dagger(\omega_{I_1})\ldots a_{I_k}^\dagger(\omega_{I_k})\Psi_0(\omega), \quad k \text{ finite}, \tag{2.3}$$

(linear combinations of the Hermite wave-functions)

$$p_I(\omega_I) = -\frac{i\sqrt{\omega_I}}{\sqrt{2}}(a_I - a_I^\dagger)$$
$$x_I(\omega_I) = \frac{1}{\sqrt{2\omega_I}}(a_I + a_I^\dagger). \tag{2.4}$$

H_N rewritten in terms of a_I and $a_I^\dagger$ becomes

$$\begin{aligned} H_N \equiv T_N + V_N^{(2)} + V_N^{(4)} = \frac{1}{2}\sum_I \omega_I(2a_I^\dagger a_I - a_I^\dagger a_I^\dagger - a_I a_I + \mathbb{I}) + \frac{1}{2}\sum_I \frac{\omega_{0I}^2}{\omega_I}(2a_I^\dagger a_I + a_I^\dagger a_I^\dagger + a_I a_I + \mathbb{I}) \\ + \frac{N^{-6\alpha}}{4}\sum_{IJKL}\frac{c_{IJKL}}{\sqrt{\omega_I\omega_J\omega_K\omega_L}}(a_I a_J a_K a_L + a_I^\dagger a_J a_K a_L + a_I a_J^\dagger a_K a_L + a_I a_J a_K^\dagger a_L + a_I a_J a_K a_L^\dagger \\ + a_I^\dagger a_J^\dagger a_K a_L + a_I^\dagger a_J a_K^\dagger a_L + a_I^\dagger a_J a_K a_L^\dagger + a_I a_J^\dagger a_K^\dagger a_L + a_I a_J^\dagger a_K a_L^\dagger + a_I a_J a_K^\dagger a_L^\dagger \\ a_I^\dagger a_J^\dagger a_K^\dagger a_L + a_I^\dagger a_J^\dagger a_K a_L^\dagger + a_I^\dagger a_J a_K^\dagger a_L^\dagger + a_I a_J^\dagger a_K^\dagger a_L^\dagger \\ + a_I^\dagger a_J^\dagger a_K^\dagger a_L^\dagger) \end{aligned} \tag{2.5}$$

We rewrite the quartic potential

$$V_N^{(4)} = N^{-6\alpha}\frac{1}{4}\sum_{IJK}\frac{1}{\omega_K\sqrt{\omega_I\omega_J}}\left(\frac{1}{2}a_I^\dagger a_J c_{(IJKK)} + \frac{1}{4}(a_I a_J + a_I^\dagger a_J^\dagger)c_{(IJKK)}\right) \quad (2.6)$$

$$+\frac{N^{-6\alpha}}{4}\sum_{I,J}\frac{1}{\omega_I\omega_J}(c_{IIJJ} + c_{IJIJ} + c_{IJJI})\mathbb{I} + N^{-6\alpha} : V_N : \quad (2.7)$$

$$\equiv N^{-6\alpha}A_{IJ}^{(N)}a_I^\dagger a_J + N^{-6\alpha}\frac{1}{2}A_{IJ}^{(N)}(a_I a_J + a_I^\dagger a_J^\dagger) + \quad (2.8)$$

$$+N^{-6\alpha}f(N)\mathbb{I} + N^{-6\alpha} : V_N : \quad (2.9)$$

where we have defined $A_{IJ}^{(N)} := \sum_K \frac{c_{(IJKK)}^{(N)}}{8\omega_K\sqrt{\omega_I\omega_J}}$ and $f(N) := \sum_{I,J}\frac{c_{IIJJ}^{(N)}+c_{IJIJ}^{(N)}+c_{IJJI}^{(N)}}{4\omega_I\omega_J}$, $\{(I_1, ..., I_k)\} := \sum_{\pi\in S_k}\{I_{\pi(1)}, ..., I_{\pi(k)}\}$ :: normal ordering with respect to $\Psi_0(\omega)$.

$$H_N = \sum_{I,J}\{((\omega_I + \frac{\omega_{0I}^2}{\omega_I})\delta_{IJ} + N^{-6\alpha}A_{IJ})a_I^\dagger a_J + \frac{1}{2}(N^{-6\alpha}A_{IJ} - (\omega_I + \frac{\omega_{0I}^2}{\omega_I})\delta_{IJ})(a_I a_J + a_I^\dagger a_J^\dagger)\} \tag{2.10}$$

$$+(\frac{1}{2}\sum_I(\omega_I + \frac{\omega_{0I}^2}{\omega_I}) + N^{-6\alpha}f(N) + \beta_N)\mathbb{I} + N^{-6\alpha} : V_N : -\beta_N\mathbb{I} \tag{2.11}$$

$$\equiv \sum_{I,J}\left(A_{IJ}^{(N+)}a_I^\dagger a_J + \frac{1}{2}A_{IJ}^{(N-)}(a_I a_J + a_I^\dagger a_J^\dagger)\right) + N^{-6\alpha} : V_N : -\beta_N\mathbb{I} \tag{2.12}$$

$$\beta_N := -\tfrac{1}{2}\textstyle\sum_I(\omega_I + \frac{\omega_{0I}^2}{\omega_I}) - N^{-6\alpha}f(N)$$
$$A_{IJ}^{(N+)} := N^{-6\alpha}A_{IJ}^{(N)} + (\omega_I + \frac{\omega_{0I}^2}{\omega_I})\delta_{IJ}$$
$$A_{IJ}^{(N-)} := N^{-6\alpha}A_{IJ}^{(N)} - (\omega_I - \frac{\omega_{0I}^2}{\omega_I})\delta_{IJ}$$

$$H_N = \sum_{I,J} \left(A_{IJ}^{(N+)} a_I^\dagger a_J + \frac{1}{2} \underline{A_{IJ}^{(N-)} (a_I a_J + a_I^\dagger a_J^\dagger)} \right) + N^{-6\alpha} : V_N : - \underline{\beta_N \mathbb{I}} \tag{2.13}$$

$$\beta_N := -\tfrac{1}{2} \textstyle\sum_I (\omega_I + \frac{\omega_{0I}^2}{\omega_I}) - N^{-6\alpha} f(N) \to \infty$$

$$A_{IJ}^{(N+)} := N^{-6\alpha} A_{IJ}^{(N)} + (\omega_I + \tfrac{\omega_{0I}^2}{\omega_I}) \delta_{IJ}$$

$$A_{IJ}^{(N-)} := N^{-6\alpha} A_{IJ}^{(N)} - (\omega_I - \tfrac{\omega_{0I}^2}{\omega_I}) \delta_{IJ}.$$

$$||\underline{\cdots} \psi|| \to \infty, \quad \psi \in \mathcal{H}_\omega \text{ (generically)}$$

Bad operators!

Can we get rid of the problematic terms? We need

$$\lim_{N\to\infty} A_{IJ}^{(N-)} = \lim_{N\to\infty} N^{-6\alpha} A_{IJ}^{(N)} - (\omega_I - \frac{\omega_{0I}^2}{\omega_I})\delta_{IJ} \simeq \frac{const}{N}, \quad \forall I, J, \quad (2.14)$$

The smallest α for which it is possible we call α_{crit}. The condition $\lim_{N\to\infty} A_{IJ}^{(N-)} = 0$ implies

$$N^{-6\alpha} \sum_K \frac{c_{(IJKK)}^{(N)}}{8\omega_K \sqrt{\omega_I \omega_J}} - (\omega_I - \frac{\omega_{0I}^2}{\omega_I})\delta_{IJ} \simeq 0 \quad (2.15)$$

(condition for the ω_I's), which turns out to be equivalent to $0 = \frac{\partial \beta_N}{\partial \omega_I}$.

Let $\tilde{\omega}$ be the real solution of $\lim_{N\to\infty} A_{IJ}^{(N-)} = 0$

Lemma 1. Optimized Fock space decomposition

$$H_N = \left(2\sum_{I=1}^{N} \tilde{\omega}_I a_I^\dagger a_I + \frac{1}{N^{-6\alpha}} : V_N : + N e_0^{(0)} \mathbb{I} \right) \otimes \mathbb{I}_{\mathcal{H}_{\tilde{\omega},N}^{\perp}} + R_N, \qquad (2.16)$$

$e_0^{(0)} = -\lim_{n\to\infty} \frac{\beta_N}{N} = \lim_{n\to\infty} e_{0,N}^{(0)}$ is the qaussian variational upper bound for the ground state energy, and $\lim_{n\to\infty} ||R_N\psi||_{\tilde{\omega}} = 0,\ \forall \psi \in \mathcal{H}_{\tilde{\omega}}$

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- the "best" Fock representation of the CCR's among infinitely many inequivalent representations
- $\frac{1}{N^{-6\alpha}} : V_N :$ may be still ill-defined in the limit $N \to \infty$

$U(N)$-invariant anharmonic oscillator

- 1-matrix model

$$2H_N = Tr(\dot{M}^2) + Tr(M^2 + \frac{2g}{n}M^4) \tag{3.1}$$

where M is a hermitian $n \times n$ matrix and $P := \dot{M}$.

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- mode expansion $\{t_a\}_{a=1}^{n^2}$, $Tr(t_a t_b) = \delta_{ab}$
completeness relation

$$(t_a)_{ij}(t_a)_{kl} = \delta_{jk}\delta_{il}, \tag{3.2}$$

$$2H_N = p_a p_a + q_a q_a + \frac{1}{n}c_{abcd}q_a q_b q_c q_d \tag{3.3}$$

with $c_{abcd} = 2gTr(t_a t_b t_c t_d)$.

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1-matrix model: Optimized Fock space decomposition

Assume $\omega_a = \omega \ \forall a$:

$$2H_N = 2\sum_a \tilde{\omega} a_a^\dagger a_a + \frac{2g}{n} : TrM^4 : + n^2 e_0^{(0)} \mathbb{I} + R_N \tag{3.5}$$

for $\tilde{\omega}$ solving

$$\omega^3 = \omega + 4g \tag{3.6}$$

R_N is of order $O(\frac{1}{n^2})$ and thus $||R_N\psi||_{\tilde{\omega}} \to 0 \ \ \forall \psi \in \mathcal{H}_{\tilde{\omega}}$.

$$\frac{e_0^{(0)}(g)}{2} = -\lim_{N\to\infty} \frac{\beta_N}{N} = \frac{\tilde{\omega}}{4} + \frac{1}{4\tilde{\omega}} + \frac{g}{2\tilde{\omega}^2} \simeq 0.59527 g^{\frac{1}{3}} + o(g^{-\frac{1}{3}}) \tag{3.7}$$

Exact answer due to Brezin et al: $e_0(g) \simeq 0.58993 g^{\frac{1}{3}}$.

Multi-matrix model

- Membrane Matrix Models

$$H_N(p,q) = p_{ia}p_{ia} + N^{-6\alpha}\frac{1}{2}f^{(n)}_{abc}f^{(n)}_{ab'c'}q_{ib}q_{ib'}q_{jc}q_{jc'}. \tag{4.1}$$

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- CQ + optimized Fock space decomposition

$$H_N = 2\tilde{\omega}\sum_{i,a} a^\dagger_{ia}a_{ia} - \beta_N\mathbb{I} + n^{-4} : V_N : + R_N \tag{4.2}$$

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- optimal Fock space frequency and Gaussian upper bound

$$\tilde{\omega} = \sqrt[3]{4\pi^2(d-1)} \tag{4.3}$$

$$\beta_N = -\frac{1}{2}d(n^2-1)\tilde{\omega} - \frac{\pi^2 d(d-1)(n^2-1)}{\tilde{\omega}^2} \tag{4.4}$$

$$e_0^{(0)} = -\lim_{n\to\infty}\frac{\beta_N}{n^2} = \frac{\tilde{\omega}d}{2} + \frac{\pi^2 d(d-1)}{\tilde{\omega}^2} = \frac{3d(d-1)^{\frac{1}{3}}\pi^{\frac{2}{3}}}{2^{\frac{4}{3}}} \tag{4.5}$$

SU(N)/U(N) invariant wave functions: Partition Basis

- 1-matrix model

$$\psi_\lambda := \mathcal{N}_\lambda (a^\dagger)^\lambda \Psi_0(\omega) \equiv \mathcal{N}_\lambda \, Tr(a^{\dagger\lambda_1}) \, Tr(a^{\dagger\lambda_2}) ... Tr(a^{\dagger\lambda_m}) \Psi_0(\omega), \quad (5.1)$$

$$\mathcal{N}_\lambda = n^{-\frac{|\lambda|}{2}} (\Pi_i i^{\lambda_i} \lambda_i!)^{-\frac{1}{2}} \quad (5.2)$$

where $Tr(a^{\dagger\lambda_i}) := Tr(T_{b_1} ... T_{b_{\lambda_i}}) a^\dagger_{b_1} ... a^\dagger_{b_{\lambda_i}}$ and $\lambda = (\lambda_1, \lambda_2, ..., \lambda_m)$ is an m-partition of a certain natural number $k = |\lambda| := \sum_i i\lambda_i$.

$$\lim_{n\to\infty} \langle \psi_\lambda, \psi_{\lambda'} \rangle = \delta_{\lambda\lambda'}$$

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$$\lim_{n\to\infty} \langle \psi_\lambda, \psi_{\lambda'} \rangle = \delta_{\lambda\lambda'}$$

- multi-matrix model: similar, but extra indices $\Lambda = \{\lambda_\Lambda, I_\Lambda\}$, where λ_Λ - partition; I_Λ - $SO(d)$ structure

$$\lim_{n\to\infty} \langle \psi_\Lambda, \psi_{\Lambda'} \rangle = \delta_{\lambda_\Lambda \lambda_{\Lambda'}} M_{I_\Lambda I_{\Lambda'}}$$

Divergent matrix elements

$$H_N = 2\tilde{\omega}\sum_a a_a^\dagger a_a - \beta_N \mathbb{I} + \frac{1}{n} : V_N : \; + const. Tr(a^2 + a^{\dagger 2}) \tag{5.3}$$

- diagonal elements $\langle \psi_\lambda, -\beta_N \mathbb{I} \psi_\lambda \rangle \propto n^2$
- $\langle \psi_\lambda, (Tr(a^2 + a^{\dagger 2}))\psi_\delta \rangle \propto n$, where $\lambda_2 = \delta_2 \pm 1$
- $\langle \psi_\lambda, \frac{1}{n}(Tr(a^4 + a^{\dagger 4}))\psi_\delta \rangle \propto n$, where $\lambda_4 = \delta_4 \pm 1$

What to do with the remaining divergent matrix elements?
$\hat{H}_N := H_N + \beta_N \mathbb{I}$, but what about $\frac{1}{n}(Tr(a^4 + a^{\dagger 4}))$?

Divergent matrix elements

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- $\cancel{\langle \psi_\lambda, (Tr(a^2 + a^{\dagger 2}))\psi_\delta \rangle \propto n, \text{ where } \lambda_2 = \delta_2 \pm 1}$ optimized Fock space
- $\langle \psi_\lambda, \frac{1}{n}(Tr(a^4 + a^{\dagger 4}))\psi_\delta \rangle \propto n$, where $\lambda_4 = \delta_4 \pm 1$

What to do with the remaining divergent matrix elements?
$\hat{H}_N := H_N + \beta_N \mathbb{I}$, but what about $\frac{1}{n}(Tr(a^4 + a^{\dagger 4}))$?

Composite creation/annihilation operators

$$A := Tr(T_a T_b T_c T_d) a_a a_b a_c a_d \tag{5.4}$$

$$\left[A, A^\dagger\right] = 4n^4 \mathbb{I} + O(n^2) \tag{5.5}$$

Lemma 2. Negative energy shift

$$2\tilde{\omega} \sum_a a_a^\dagger a_a + \frac{1}{4\tilde{\omega} n}(A + A^\dagger + 4\, Tr(a^{\dagger 2} a^2)) \tag{5.6}$$

$$= \frac{\Omega}{n^4} B^\dagger B + \sum_{\lambda, \lambda_4 = 0} G(\lambda) P_\lambda + \tilde{e}_0 n^2 \mathbb{I} + O(\frac{1}{n}) \tag{5.7}$$

for some $\Omega > 0$, $\tilde{e}_0 < 0$ and $B := A + \alpha n^3 \mathbb{I}$

$\tilde{e}_0 < 0 \implies$ the divergent terms decrease the ground state energy

Hamiltonian

$$H_N = 2\tilde{\omega} a_a^\dagger a_a + \frac{g}{4n\tilde{\omega}^2}(A + A^\dagger + 4\,Tr(a^\dagger a^\dagger aa)) \tag{5.8}$$

$$+\frac{g}{n\tilde{\omega}^2}(\,Tr(a^\dagger a^\dagger a^\dagger a) + \,Tr(a^\dagger aaa)) \tag{5.9}$$

$$+\frac{g}{2n\tilde{\omega}^2}\,:\,Tr(a^\dagger aa^\dagger a)\,: + e_0^{(0)} n^2 \mathbb{I} \tag{5.10}$$

Hamiltonian

$$H_N = 2\tilde{\omega}(1-\epsilon)a_a^\dagger a_a + \frac{g}{4n\tilde{\omega}^2}(A + A^\dagger + 4\,Tr(a^\dagger a^\dagger aa)) \tag{5.11}$$

$$+\underbrace{2\epsilon\tilde{\omega}a_a^\dagger a_a + \frac{g}{n\tilde{\omega}^2}(\,Tr(a^\dagger a^\dagger a^\dagger a) + \,Tr(a^\dagger aaa))}_{>0 \quad \text{for } \epsilon > \epsilon_0} \tag{5.12}$$

$$+\underbrace{\frac{g}{2n\tilde{\omega}^2} : \,Tr(a^\dagger aa^\dagger a) :}_{\propto O(\frac{1}{n})} + e_0^{(0)} n^2 \mathbb{I} \tag{5.13}$$

for some $0 < \epsilon < 1$.

U(N) invariant anharmonic oscillator - lower bound

Theorem 1. 'Absorption of the composites for the 1-matrix model'

$$H_N \geq \frac{\Omega}{n^4} B^\dagger B + (e_0^{(0)} + \tilde{e}_0) n^2 \mathbb{I} + \sum_{\lambda, \lambda_4 = 0} P_\lambda G(\lambda) + O(\frac{1}{n}) \tag{5.14}$$

for some $\Omega \geq 0, \quad \tilde{e}_0 < 0$

Corollary 1. Lower bound for the ground state energy

$$e_0 \geq e_0^{(0)} + \tilde{e}_0^{(-)} \tag{5.15}$$

Membrane Matrix Models

Two composite annihilation operators

$$A := Tr(abcd)a_{ai}a_{bi}a_{cj}a_{dj} \equiv (iijj), \quad B := \frac{1}{d+1}(iijj) - \frac{1}{2}(ijij) \tag{5.16}$$

$$\left[A, A^\dagger\right] = c_1 n^4 + O(n^2) \tag{5.17}$$

$$\left[B, B^\dagger\right] = c_2 n^4 + O(n^2) \tag{5.18}$$

$$\left[A, B^\dagger\right] = \left[B, A^\dagger\right] = O(n^2) \tag{5.19}$$

Theorem 2. 'Absorption of the composites for the multi-matrix model'

$$H_N \geq \Omega_1 \tilde{A}^\dagger \tilde{A} + \Omega_2 \tilde{B}^\dagger \tilde{B} + \sum_{\lambda, \lambda_4 = 0} P_\lambda G(\lambda) + (e_0^{(0)} + \tilde{e}_0) n^2 \mathbb{I} \tag{5.20}$$

for some $\Omega_1, \Omega_2 \geq 0, \quad \tilde{e}_0 < 0$.

Corollary 2. Lower bound for the ground state energy

$$e_0 \geq e_0^{(0)} + \tilde{e}_0 \tag{5.21}$$

Formal perturbative expansion

- Lemma 1 provides a solvable part $\hat{H}_{0,N} = 2\tilde{\omega} a_a^\dagger a_a + e_0^{(0)} n^2 \mathbb{I}$ even if the original quadratic potential is absent

$$H_N = \hat{H}_{0,N} + \frac{1}{N^{6\alpha}} : V_N : \tag{6.1}$$

- Rayleigh-Schrödinger series

$$E_{k,N} = E_{k,N}^{(0)} + \epsilon_N E_{k,N}^{(1)} + \epsilon_N^2 E_{k,N}^{(2)} + \epsilon_N^3 E_{k,N}^{(3)} + O(\epsilon^4), \tag{6.2}$$

$$E_{k,N}^{(1)} = \langle \psi_k^{(0)}, : V_N : \psi_k^{(0)} \rangle, \tag{6.3}$$

$$E_{k,N}^{(2)} = \langle \psi_k^{(0)}, : V_N : \frac{Q_0}{E_{k,N}^{(0)} - \hat{H}_{0,N}} : V_N : \psi_k^{(0)} \rangle, \tag{6.4}$$

$$E_{k,N}^{(3)} = \langle \psi_k^{(0)}, : V_N : \frac{Q_0}{E_{k,N}^{(0)} - \hat{H}_{0,N}} : V_N : \frac{Q_0}{E_{k,N}^{(0)} - \hat{H}_{0,N}} : V_N : \psi_k^{(0)} \rangle \tag{6.5}$$

$$- E_{k,N}^{(1)} \langle \psi_k^{(0)}, : V_N : \frac{Q_0}{(E_{k,N}^{(0)} - \hat{H}_{0,N})^2} : V_N : \psi_k^{(0)} \rangle, \tag{6.6}$$

Formal perturbative expansion

- all terms of the R-S series diverge at large N
- "Linked-cluster theorem" - only connected diagrams involved (to be proved)
- ground state energy corrections

$$E_0^{(k)} \propto n^2$$

- renormalized energies of excited states

$$E_l^{(k),R} := \lim_{N\to\infty} (E_l^{(k)} - E_0^{(k)}) < \infty$$

- convergence? asymptotic series? Padé, Borel summability?

Some numerical values

Table: Exact ground state energies $E_0 = e_0 n^2$ vs the upper variational bound (the optimized Fock space approximation) $e_0^{(0)}$ and the lower bound $e^{(lower)}$ for the 1-matrix model

g	$e_0^{(0)}$	$e_0^{(lower)}$	e_0
0.01	0.505	0.505	0.505
0.1	0.543	0.542	0.542
0.5	0.653	0.651	0.651
1.0	0.743	0.740	0.740
50	2.235	2.214	2.217
1000	5.968	5.907	5.915
$g \to \infty$	$0.59527\ g^{\frac{1}{3}}$	$0.589075\ g^{\frac{1}{3}}$	$0.58993\ g^{\frac{1}{3}}$

Some numerical values 2

Table: Exact ground state energies $E_0 = e_0 n^2$ vs the R-S series in the optimized Fock space for the 1-matrix model

g	$e_0^{(0)}$	$e_0^{(2)}$	$e_0^{(3)}$	e_0
0.01	0.505	0.505	0.505	0.505
0.1	0.543	0.542	0.542	0.542
0.5	0.653	0.651	0.651	0.651
1.0	0.743	0.740	0.740	0.740
50	2.235	2.214	2.219	2.217
1000	5.968	5.907	5.922	5.915
$g \to \infty$	$0.59527\ g^{\frac{1}{3}}$	$0.589075\ g^{\frac{1}{3}}$	$0.59062\ g^{\frac{1}{3}}$	$0.58993\ g^{\frac{1}{3}}$

Some numerical values 3

Table: Exact spectral gap $\omega(g)$ vs the R-S series in the optimized Fock space (renormalized energies of the first excited state) for the 1-matrix model

g	$\omega(g)$	$\omega^{(0)}$	$\omega^{(1)}$	$\omega^{(2)}$	$\omega^{(3)}$
2	2.45	2.17	2.59	2.43	2.39
50	6.81	5.91	7.34	6.64	6.47
200	10.76	9.32	11.62	10.48	10.20
1000	18.37	15.90	19.85	17.88	17.39

Some numerical values 4

Table: R-S series in the optimized Fock space and the lower bound for the ground state energy of the multi-matrix model

d	2	3	4	5
$e_0^{(0)}$	5.107	9.653	14.732	20.269
$e_0^{(2)}$	4.682	9.351	14.460	20.005
$e_0^{(3)}$	4.735	9.364	14.466	20.009
$e_0^{(lower)}$	4.834	9.349	14.410	19.931

Some numerical values 5

Table: R-S sereis for the renormalized energy (the vacuum energy subtracted) of the first $SO(d) \times SU(n)$ invariant excited state for the multi-matrix model at large n

d	3	9	15	25	35
$E^{(0)}_{\Lambda,R}$	17.16	27.24	32.82	39.29	44.12
$E^{(1)}_{\Lambda,R}$	21.45	34.05	41.03	49.11	55.15
$E^{(2)}_{\Lambda,R}$	16.09	31.92	39.57	48.09	54.34

$$H_N = 2\tilde{\omega}\left[(i^\dagger i) + \frac{\pi^2}{\tilde{\omega}^3}(\text{interaction terms})\right] \tag{6.7}$$

Therefore the effective coupling constant is $\frac{\pi^2}{\tilde{\omega}^3} = \frac{1}{4(d-1)}$.

Supersymmetric matrix models

$$\tilde{H}_N = H_N \mathbb{I} + H_F = H_N \mathbb{I} + \frac{i}{2} x_{ja} f^{(n)}_{abc} \gamma^j_{\alpha\beta} \theta_{\alpha b} \theta_{\beta b}, \tag{7.1}$$

$$\tilde{\phi}_a := f^{(n)}_{abc} (x_{ib} p_{ic} - \frac{i}{4} \theta_{\alpha b} \theta_{\alpha c}) = 0. \tag{7.2}$$

- $\mathcal{H} = L^2(\mathbb{R}^{d(n^2-1)}) \otimes \mathcal{F}$, $\dim(\mathcal{F}) = 2^{\frac{\mathcal{N}_d}{2}(n^2-1)}$
- anti-commuting Clifford variables $\theta_{a\alpha}$,

$$[\theta_{\alpha a}, \theta_{\beta_b}]_+ = \delta_{\alpha\beta} \delta_{ab}, \tag{7.3}$$

- supersymmetry $\implies d = 2, 3, 5, 9$
- $\tilde{H}_N = Q^2_\alpha$, $\alpha = 1, ..., \mathcal{N}_d = 2(d-1)$
- continuous spectrum $spec(\tilde{H}_N) = [0, \infty)$
- BFSS conjecture, embedded eigenvalues?

Supersymmetric matrix models, d=3,5

- fermionic creation and annihilation operators $\lambda_{\hat{\alpha}a}, \lambda^{\dagger}_{\hat{\alpha}a}, \hat{\alpha} = 1, ..., \frac{\mathcal{N}_d}{2}$

$$[\lambda_{\hat{\alpha}a}, \lambda^{\dagger}_{\hat{\beta}b}]_+ = \delta_{\hat{\alpha}\hat{\beta}}\delta_{ab}\mathbb{I}, \tag{7.4}$$

$$[\lambda_{\hat{\alpha}a}, \lambda_{\hat{\beta}b}]_+ = 0 = [\lambda^{\dagger}_{\hat{\alpha}a}, \lambda^{\dagger}_{\hat{\beta}b}]_+ \tag{7.5}$$

Supersymmetric matrix models, d=3,5

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$$[\lambda_{\hat{\alpha}a}, \lambda^{\dagger}_{\hat{\beta}b}]_+ = \delta_{\hat{\alpha}\hat{\beta}}\delta_{ab}\mathbb{I}, \tag{7.4}$$

$$[\lambda_{\hat{\alpha}a}, \lambda_{\hat{\beta}b}]_+ = 0 = [\lambda^{\dagger}_{\hat{\alpha}a}, \lambda^{\dagger}_{\hat{\beta}b}]_+ \tag{7.5}$$

- for $d = 3, 5$ $\exists$ a canonical choice of $\lambda_{\hat{\alpha}a}, \lambda^{\dagger}_{\hat{\alpha}a}$, s.t.

$$H_F = 2ix_{ja}f^{(n)}_{abc}\gamma^{j}_{\hat{\alpha}\hat{\beta}}\lambda_{\hat{\alpha}b}\lambda^{\dagger}_{\hat{\beta}b}, \tag{7.6}$$

$$\tilde{\phi}_a := f^{(n)}_{abc}(x_{ib}p_{ic} - i\lambda_{\alpha b}\lambda^{\dagger}_{\alpha c}) = 0. \tag{7.7}$$

and thus

$$[\tilde{H}_N, F] = 0, \quad [\tilde{H}_N, \tilde{\phi}_a] = 0, \tag{7.8}$$

Supersymmetric matrix models, d=3,5

- natural grading of $\mathcal{H}$

$$\mathcal{H} = \bigoplus_{k=1}^{\frac{\mathcal{N}_d}{2}(n^2-1)} \mathcal{F}^{(k)}, \tag{7.9}$$

Supersymmetric matrix models, d=3,5

- natural grading of $\mathcal{H}$

$$\mathcal{H} = \bigoplus_{k=1}^{\frac{\mathcal{N}_d}{2}(n^2-1)} \mathcal{F}^{(k)}, \tag{7.9}$$

- $$\tilde{H}_{n,d}\Big|_{\mathcal{F}^{(0)}} = H_{n,d}\Big|_{\mathcal{F}^{(0)}}, \quad \forall n, d = 3, 5, \tag{7.10}$$

  so the purely discrete spectrum of $H_{N,d=3,5}$ is a subset of the full spectrum of $\tilde{H}_{N,d=3,5}$

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