Singularities of relativistic membranes

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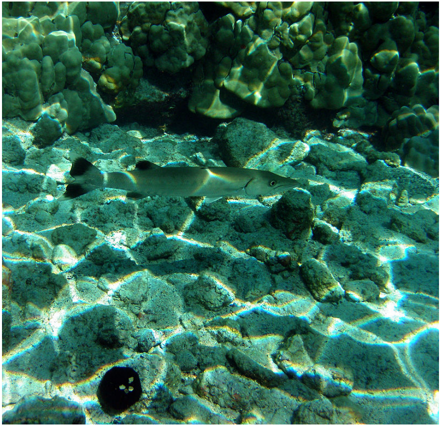
Outline

Laplace-Beltrami equation

$$\Delta x^{\mu} = 0$$
$$\updownarrow$$



Extended objects

Let us consider an M-dimensional compact orientable manifold Σ moving in Minkowski space $\mathbb{R}^{1,D}$. The world-volume is given by

$$S[x] = -\int_{\mathbb{R}\times\Sigma} \sqrt{G} d^{M+1}\varphi. \tag{1.1}$$

- φ^a, $a = 0, 1, ..., M$ - local coordinates on S
- $x^\mu(\varphi^0, \varphi^1, ..., \varphi^M)$ - embedding functions, $\mu = 0, 1, ..., D$
- $G = det[G_{ab}]$, $G_{ab} := \partial_a x^\mu \partial_b x^\nu \eta_{\mu\nu}$

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$$\frac{\delta S}{\delta x} = 0 \implies \frac{1}{\sqrt{G}} \partial_\alpha \sqrt{G} G^{\alpha\beta} \partial_\beta x^\mu = 0, \;\; \mu = 0, ..., D$$

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$$\Delta x^\mu = 0, \quad \text{vanishing mean curvature}$$

Extended objects: orthogonal gauge

- $$t := x^0 = \varphi_0, \quad \dot{\vec{x}} \cdot \partial_a \vec{x} = 0, a = 1, 2, ..., M \tag{1.2}$$
$$G_{\alpha\beta} = \begin{pmatrix} 1 - \dot{\vec{x}}^2 & 0 \\ 0 & -g_{rs} \end{pmatrix}$$
$$g_{rs} := \partial_r \vec{x} \cdot \partial_s \vec{x}, \quad g := det(g_{rs}), \quad \dot{\vec{x}} := \partial_t \vec{x}$$
- time-dependent surface $(x^\mu) = (t, \vec{x}(t, \vec{\varphi}))$
- $\mu = 0$ component $\implies$ local conservation law

$$\partial_t \sqrt{\frac{g}{1 - \dot{\vec{x}}^2}} \equiv \partial_\tau \rho(\varphi) = 0 \tag{1.3}$$

- first order system implied

$$\dot{\vec{x}}^2 + \frac{g}{\rho^2} = 1 \tag{1.4}$$
$$\dot{\vec{x}} \cdot \partial_a \vec{x} = 0 \tag{1.5}$$

Co-dimension 1

- $\vec{x} = (x^1, ..., x^D)$, $\vec{\varphi}^M = (\varphi^1, ..., \varphi^M)$, $co - dim = 1 \leftrightarrow D = M + 1$
- assume $\dot{\vec{x}}, \partial_a \vec{x}$ are linearly independent. Then

Lemma 1. Zero mean curvature in co-dim 1

$$\Delta x^\mu = 0 \iff \begin{cases} \dot{\vec{x}}^2 + \frac{g}{\rho^2} = 1 \\ \dot{\vec{x}} \cdot \partial_a \vec{x} = 0 \end{cases}$$

$\mu = 0, 1, ..., M + 1, \quad a = 1, ..., M$

- examples: closed strings (curves) in the plane, membranes (compact surfaces) in $\mathbb{R}^3$

Closed curves in the plane, M=1, D=2

Setting

$$\vec{x}(t,\varphi) = (x(t,\varphi), y(t,\varphi))$$
$$g = \vec{x}'^2 := (\partial_\varphi \vec{x})^2$$

Zero mean curvature condition:

$$\begin{cases} \dot{\vec{x}}^2 + \frac{x'^2}{\rho^2} = 1 \\ \dot{\vec{x}} \cdot \vec{x}' = 0 \end{cases} \tag{1.6}$$

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Solution:

$$\vec{x}' = \rho \cos(f-h) \begin{pmatrix} -\sin(f+h) \\ \cos(f+h) \end{pmatrix}$$

$$\dot{\vec{x}} = -\sin(f-h) \begin{pmatrix} \cos(f+h) \\ \sin(f+h) \end{pmatrix}$$

where $f = f(\varphi + \frac{t}{\rho}), h = h(\varphi - \frac{t}{\rho})$ (implied by the integrability conditions: $\partial_\varphi \dot{\vec{y}} = \partial_t \vec{y}')$

Closed strings in the plane, M=1, D=2

Curvature:

$$k(t,\varphi) = \frac{f' + g'}{\cos(f - g)}$$

Theorem. k becomes singular in finite time
Proof (idea): closedness $\implies$ Range$(f + h) \geq 2\pi$ and thus
$\exists t_s, 0 < t_s < \infty$ s.t.
$|f - h| = \frac{\pi}{2}$
at $t = t_s$ and some $\varphi = \varphi_s$ (rigorous proof: L. Nguyen and G. Tian, 2013)

Example. shrinking circle (the only self-similar solution)

$$\vec{x} = \cos(t) \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \end{pmatrix}$$

$f(\varphi) = h(\varphi) = \frac{\varphi}{2}, \quad k \propto \frac{1}{\cos(t)}$

Comparison with the curve shortening flow

$$\dot{\vec{x}} = \kappa \vec{n}$$

Theorem (Gage-Hamilton-Grayson). Smooth simple closed curves shrink to points

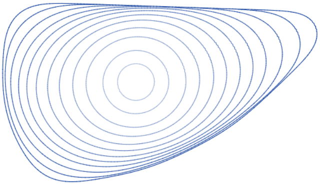


Figure: From Wikipedia

Extended objects 1. $\exists$ solutions with this property
2. $\exists$ counter-examples

Harmonic perturbations of the shrinking circle (Hoppe, 1995)

Take

$$f(\varphi) = h(\varphi) = \frac{1}{2}(\varphi + \epsilon \sin(m\varphi)), \quad m \in \mathbb{Z}, \quad |m\epsilon| << 1$$

Then

$$\vec{x}' = \rho \cos(t + \epsilon \sin(mt) \cos(m\varphi)) \begin{pmatrix} -\sin(t + \epsilon \cos(mt) \sin(m\varphi)) \\ \cos(t + \epsilon \cos(mt) \sin(m\varphi)) \end{pmatrix}$$

Curvature

$$k(t, \varphi) = \frac{1 + m\epsilon(\cos(mt) + \cos(m\varphi))}{|\cos(t + \epsilon \sin(mt) \cos(m\varphi))|}$$

- $m = 2n$ curve smoothly shrinks to a point
- $m = 2n + 1$ then m discrete singular points before it has shrunk completely

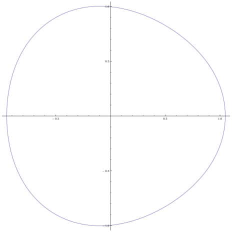


Figure: m=3,$\epsilon = 0.1$, t=0

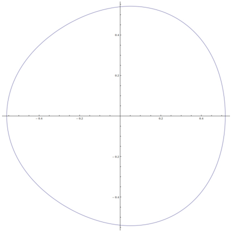


Figure: m=3,$\epsilon = 0.1$, t=1

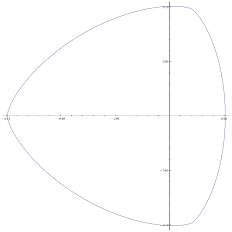


Figure: m=3,$\epsilon = 0.1$, t=1.47

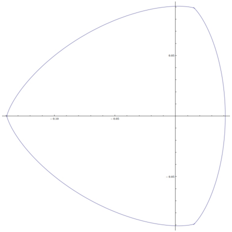


Figure: m=3,$\epsilon = 0.1$, t=1.48

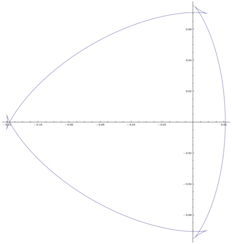


Figure: m=3,$\epsilon = 0.1$, t=1.5

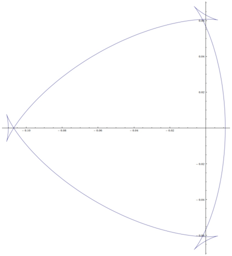


Figure: m=3,$\epsilon = 0.1$, t=1.51

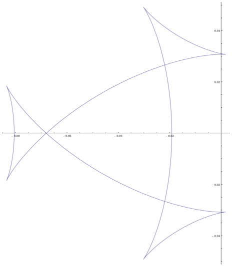
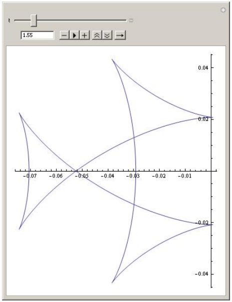


Figure: m=3,$\epsilon = 0.1$, t=1.54



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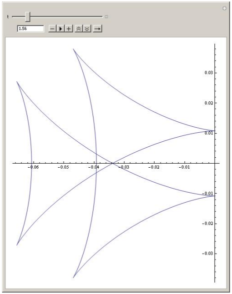


Figure: m=3,$\epsilon = 0.1$, t=1.56

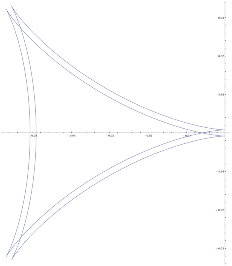


Figure: m=3,$\epsilon = 0.1$, t=1.57

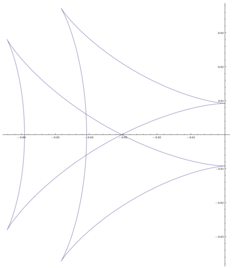


Figure: m=3,$\epsilon = 0.1$, t=1.58

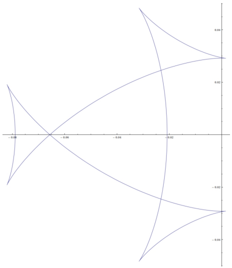


Figure: m=3,$\epsilon = 0.1$, t=1.6

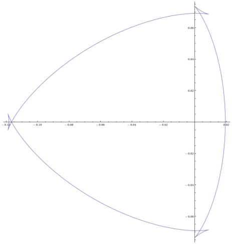


Figure: m=3,$\epsilon = 0.1$, t=1.64

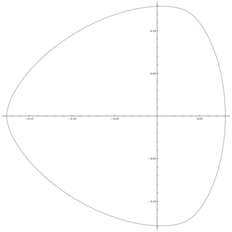


Figure: m=3,$\epsilon = 0.1$, t=1.7

Axially symmetric membranes: M=2, D=3

$$\vec{x}(t,\varphi,\psi) = \begin{pmatrix} r(t,\varphi)\cos\psi \\ r(t,\varphi)\sin\psi \\ z(t,\varphi) \end{pmatrix} \tag{1.7}$$

Vanishing mean curvature ($' := \partial_\varphi, \quad \psi \in (0,2\pi), \varphi \in (0,\pi)$):

$$r^{'}\dot{r} + z^{'}\dot{z} = 0 \tag{1.8}$$

$$\dot{r}^2 + \dot{z}^2 + \frac{r^2(z^{'2} + r^{'2})}{\rho^2} = 1 \tag{1.9}$$

General solution can be again parametrized by two functions F and G
$\vec{y} := (r, z)$

$$\frac{r}{\rho}\vec{y}' = \cos(F(\varphi,t)) \begin{pmatrix} -\sin(G(\varphi,t)) \\ \cos(G(\varphi,t) \end{pmatrix} \tag{1.10}$$

$$\dot{\vec{y}} = -\sin(F(\varphi,t)) \begin{pmatrix} \cos(G(\varphi,t)) \\ \sin(G(\varphi,t) \end{pmatrix} \tag{1.11}$$

Axially symmetric membranes: M=2, D=3

- "Solution"

$$\frac{r}{\rho}\vec{y}' = \cos(F(\varphi,t))\begin{pmatrix}-\sin(G(\varphi,t)) \\ \cos(G(\varphi,t)\end{pmatrix} \tag{1.12}$$

$$\dot{\vec{y}} = -\sin(F(\varphi,t))\begin{pmatrix}\cos(G(\varphi,t)) \\ \sin(G(\varphi,t)\end{pmatrix} \tag{1.13}$$

with $y := (r,z)$.

- integrability conditions: $\partial_\varphi \dot{\vec{y}} = \partial_t \vec{y}'$ unsolved
- curvature of the curve (giving rise to the solid of revolution)

$$k(\varphi,t) = \frac{r}{\rho}\frac{G'}{\cos F}. \tag{1.14}$$

- **Conjecture**. The motion becomes singular in finite time.

Example: shrinking sphere

The only known exact compact solution for $M = 2$
Ansatz:

$$r(t,\varphi) = R(t)\sin(\varphi), \qquad z(t,\varphi) = R(t)\cos(\varphi)$$

Then

$$\dot{R}^2 + \frac{R^4}{\rho^2} = 1, \quad \rho = \frac{R(0)^4}{1 - \dot{R}(0)^2} = const.$$

(compare

$$\dot{R}^2 + \frac{R^2}{\rho^2} = 1)$$

for the string)
Curvature $k \to \infty$ as $t \to t_s$.

Zero-mean curvature

$$\begin{cases} \dot{\vec{x}}^2 + \frac{g}{\rho^2} = 1 \\ \dot{\vec{x}} \cdot \partial_a \vec{x} = 0 \end{cases}, \quad a = 1, ..., M$$

For real solutions and smooth initial conditions (ρ smooth) we get, for all times

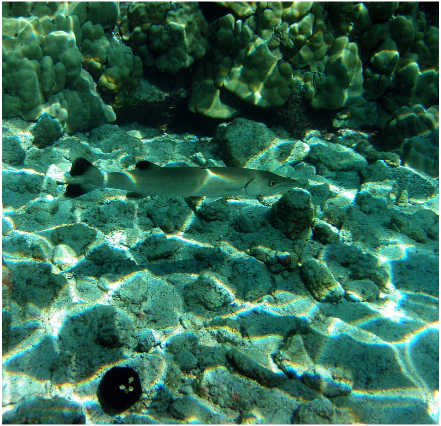
$$|\dot{\vec{x}}| \leq 1, \qquad |g| \leq \max_{\vec{\varphi}} \rho(\vec{\varphi})$$

- strings ($M = 1$): $g = \vec{x}'^2 \implies |\vec{x}'| < \infty$, actually $\vec{x} \in C^\infty$
- $M > 1$ numerical evidence: $\vec{x} \in C^\infty$

From now on consider only metric singularities $g \to 0$

We are still aiming for discussing the connection:

$$\Delta x^{\mu} = 0$$
$$\updownarrow$$



Metric singularities

Zero-mean curvature

$$\begin{cases} \dot{\vec{x}}^2 + \frac{g}{\rho^2} = 1 \\ \dot{\vec{x}} \cdot \partial_a \vec{x} = 0 \end{cases}$$

Conserved quantity

$$\rho(\vec{\varphi}) = \sqrt{\frac{g}{1 - \dot{\vec{x}}^2}}, \quad g = det(g_{rs}) \equiv det(\partial_r \vec{x} \cdot \partial_s \vec{x})$$

Observation: Singularities move with the speed of light

1-1 correspondence between $g \to 0$ and $|\dot{\vec{x}}| \to 1$

Thus around singularities:

$$\begin{cases} \dot{\vec{x}}^2 = 1 \\ \dot{\vec{x}} \cdot \partial_a \vec{x} = 0 \end{cases}$$

Metric singularities

Around singularities:

$$\begin{cases} \dot{\vec{x}}^2 = 1 \\ \dot{\vec{x}} \cdot \partial_a \vec{x} = 0 \end{cases}$$

Assume that all points move either outwards or all inwards. Then:

$$\dot{\vec{x}} = \vec{n}$$

i.e. solutions around singularities satisfy **the Eikonal equation** (solvability around singularities!).

Catastrophe theory

Solutions of the eikonal equation

Huygens solution

$$\vec{x}(\mathbf{u}, t) = \vec{x}(\mathbf{u}, 0) + \vec{n}(\mathbf{u}, 0)t, \tag{1.15}$$

Let $\vec{x} = (\vec{x}_{||}(\mathbf{u}, t), z(\mathbf{u}, t)))$ and $z(\mathbf{u}, 0) = f(\mathbf{u})$. Then

$$\vec{n}(\mathbf{u}, 0) = \frac{(-\nabla f, 1)}{\sqrt{1 + \nabla f^2}},$$

and thus the solution cast in a **graph form** :
$\vec{x}(\mathbf{u}, t) = \vec{x}(\mathbf{u}, 0) + \frac{(-\nabla f, 1)}{\sqrt{1+\nabla f^2}} t$. or

$$\vec{x}_{||} = \mathbf{u} - \frac{\nabla f}{\sqrt{1 + \nabla f^2}} t, \tag{1.16}$$

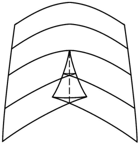
$$z = f(\mathbf{u}) + \frac{t}{\sqrt{1 + \nabla f^2}}. \tag{1.17}$$

Let $M = 2$, $\mathbf{u} = (\varphi, \psi)$ and $\vec{x} = (\vec{x}_{\|}, z) = (x, y, z)$. Singularities of the EE $\iff$ $rank(D)$ is no longer maximal

$$D = \begin{pmatrix} x_\varphi & y_\varphi & z_\varphi \\ x_\psi & y_\psi & z_\psi \end{pmatrix}$$

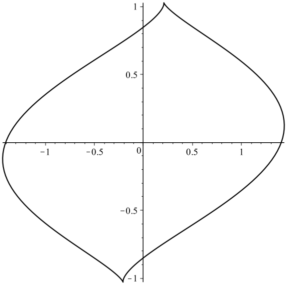
$g = D_1^2 + D_2^2 + D_3^2 = 0$ leads to the shape of singularities.
Elementary catastrophes (Thom's theorem)

- fold - 1 control variable
- **cusp** - 2 control variables (t, φ)
- **swallow tail** - 3 control variables (t, φ, ψ)



More rigorous analysis: self-similar solutions, blow-ups.

- new time $t' = t_0 - t$
- "lip" - singular wave-front projected onto $z = 0$, height $\propto |t'|^{\frac{1}{2}}$, width $\propto |t'|^{\frac{3}{2}}$



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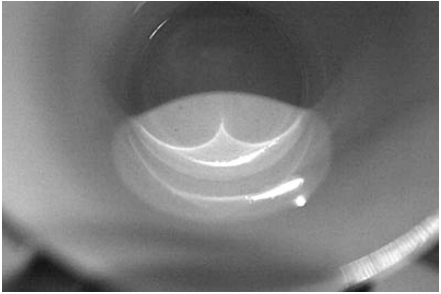


- "Optical distance" ℓ
- assume that $Z \gg f$, as well as $Z \gg X - x$ and $Z \gg Y - y$:

$$\ell = Z - f + \frac{(X-x)^2 + (Y-y)^2}{2Z}. \tag{1.18}$$

- a parboloid of revolution (up to a rescaling of the axes):

$$f(x,y) \approx \frac{x^2+y^2}{2Z_0}.$$

  This means that a first focus occurs at $Z = Z_0$
- subtract the leading-order quadratic behaviour and introduce a new function g:

$$g(x,y) = f(x,y) - \frac{x^2+y^2}{2Z_0}. \tag{1.19}$$

- control variables:

$$\xi = \frac{X}{Z}, \quad \eta = \frac{Y}{Z}, \quad \zeta = \frac{1}{2}\left(\frac{1}{Z_0} - \frac{1}{Z}\right), \tag{1.20}$$

- ‘potential” ϕ by $\phi = Z_0 - \ell$, we have

$$\phi = g(x,y) + \zeta(x^2+y^2) + \xi x + \eta y - 2\zeta Z_0^2,$$

  where everything has been expanded to linear order in the control variables
- $g(x,y)$ determines the catastrophe:**fold** :$g = x^3 + y^2$, **cusp**: $g = -x^4 + y^2$
- ray conditions: $t' = \phi, \phi_x = 0 = \phi_y$
- focusing conditions $\phi_{xx}\phi_{yy} - \phi xy^2 = 0$

Lip for the cusp catastrophe

- Choose $g = -x^4 + y^2$
- the ray conditions give

$$\xi = -8x^3, \quad \eta = -2y, \quad \zeta = 6x^2 \tag{1.21}$$

- the focusing conditions imply

$$-t' = 12x^2 + y^2.$$

- self-similar form ($x = \frac{X}{|t'|^{1/2}}, \quad y = \frac{Y}{|t'|^{1/2}}$

$$1 = 12X^2 + Y^2 \tag{1.22}$$

$$\xi = -8|t'|^{3/2}X^3, \quad \eta = -2|t'|^{1/2}Y, \tag{1.23}$$

Some numerics

We solve the following

$$\dot{r} = -z' \sqrt{\frac{1}{z'^2 + r'^2} - \frac{r^2}{\rho(\varphi)^2}} \tag{1.24}$$

$$\dot{z} = r' \sqrt{\frac{1}{z'^2 + r'^2} - \frac{r^2}{\rho(\varphi)^2}}. \tag{1.25}$$

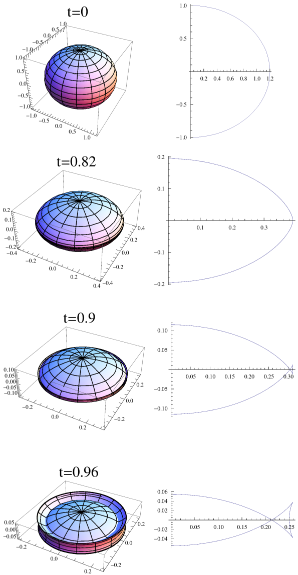
subject to (rotationally symmetric ellipsoid :

$$r(0,\varphi) = a\sin(\varphi) \quad z(0,\varphi) = -\cos(\varphi)$$

with a homogeneous initial velocity distribution, i.e. $\dot{\mathbf{x}}^2 = v^2 = const$

$$\rho = \frac{a\sin(\varphi)\sqrt{a^2\cos(\varphi)^2 + \sin(\varphi)^2}}{\sqrt{1 - v^2}}$$

Some numerics: ellipsoid, $a > 1$



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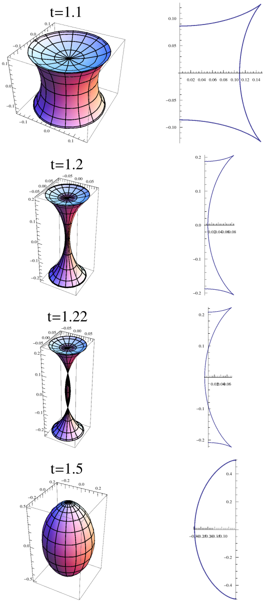

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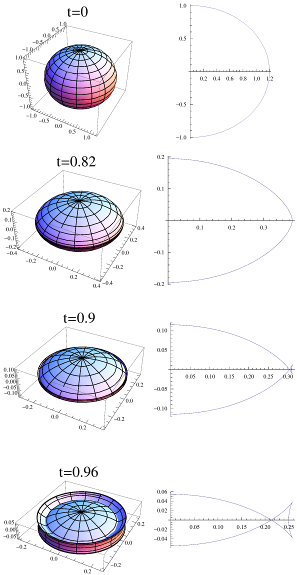

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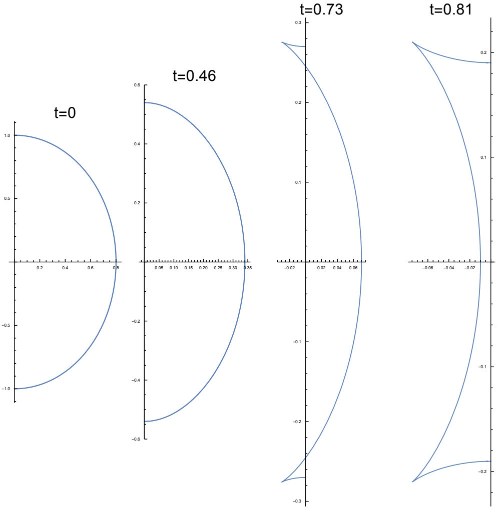


Some numerics: non-rotationally symmetric case

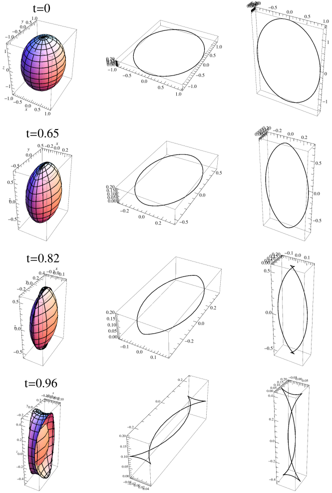
We solve

$$\dot{\mathbf{x}} = \pm \frac{\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}}{|\partial_1 \mathbf{x} \times \partial_2 \mathbf{x}|} \sqrt{1 - \frac{g}{\rho^2}} \tag{1.26}$$

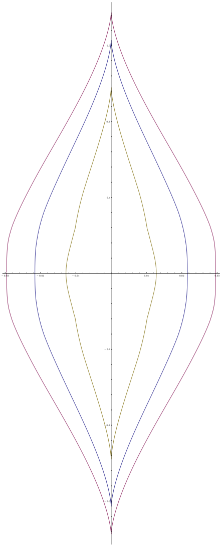
with a uniform initial velocity distribution i.e. $\dot{\mathbf{x}}^2 = v^2 = const.$ and an ellipsoid as the initial condition,

$$\begin{aligned} x(0) &= a \cos\varphi \cos\theta \\ y(0) &= b \sin\varphi \cos\theta \\ z(0) &= c \sin\theta, \end{aligned} \tag{1.27}$$

Some numerics: ellipsoid (non-rotationally symmetric)



Some numerics: "lip" for an ellipsoid (non-rotationally symmetric)



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