Discrete eigenvalues asymptotics in two models with delta interaction

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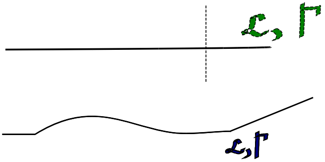
Ostrava, 27 September 2016

Geometrically induced bound states - preliminaries

- Two dimensional system $L^2(R^2)$, potential supported by a curve Γ.

$$"H = -\Delta - \alpha\delta_\Gamma" \qquad \alpha > 0\,.$$



Exner, Ichinose 2001 for 2D , Exner, SK, 2002 for 3D.

Definition of Hamiltonian

$$\mathcal{E}[f] = \int_{R^2} |\nabla f|^2 - \alpha \int_\Gamma |f|^2\,, \quad f \in W^{1,2}(R^2)\,, \quad W^{1,2}(R^2) \hookrightarrow L^2(\Gamma)\,.$$

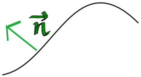
The Hamiltonian H: $\qquad (Hf, f)_{L^2(R^2)} = \mathcal{E}[f]\,.$

Boundary conditions

More precisely, we have

$$H\psi = -\Delta\psi \qquad \text{a.e. in } R^d\,,$$
$$D(H) = \left\{ \psi \in W^{1,2}(R^d) \cap W^{2,2}\left(R^d \setminus \Gamma\right) : \psi \text{ satisfies (2)} \right\}. \tag{1}$$

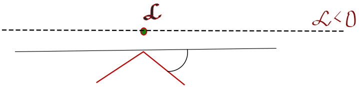
$$\partial_n^+\psi|_\Gamma - \partial_n^-\psi|_\Gamma = -\alpha\psi|_\Gamma\,. \tag{2}$$



Point interaction in D1

$$H = -\Delta - \alpha\delta(x)\,.$$

$$D(H) := \{ f \in W^{1,2}(R) \cap W^{2,2}(R \setminus \{0\})\,, \quad f(0^+)' - f(0^-)' = -\alpha f(0) \}\,.$$



Spectrum of Hamiltonian

Straight line interaction

$$"H = -\Delta - \alpha\delta_\Gamma"$$

Transversal component: $-\Delta^{(1)} - \alpha\delta(x)$

$$-\frac{\alpha^2}{4} \cup [0, \infty)\,,$$

Spectrum of H

$$\sigma(H) = [-\frac{\alpha^2}{4}\,, \infty)$$

Curved wire. 2D: Exner, Ichinose; 3D: Exner, SK.

Bending acts as an attractive potential for quantum wires

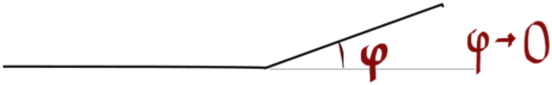
Γ- infinite asymptotically straight, C^2 piecewise,
$(|\gamma(s) - \gamma(s')| \geq C|s - s'|$ no intersections, near-selfintersection).

Spectrum

$$\sigma_{ess}(H) = [-\frac{\alpha^2}{4}, \infty), \quad \sigma_d(H) \neq \emptyset$$

Formulation of the problem

Γ- angle.



What is an asymptotics of the discrete spectrum is we approach to a straight line?

Some inspirations...

Strong coupling constant asymptotics. Curvature k as an effective potential:

$$\lambda(H) = -\frac{\alpha^2}{4} + \lambda\Big(-\frac{d^2}{ds^2} - \frac{k^2}{4}\Big) + o(1)\,.$$

for $\alpha \to \infty$.
Consider $k \leftrightarrow \beta k$ where $\beta \to 0$. Then

$$\lambda\Big(-\frac{d^2}{ds^2} - \frac{(\beta k)^2}{4}\Big) = \mathcal{O}(\beta^4)\,.$$

Total curvature

$\int_s^{s'} k(t)$ measures the angle between tangential vectors $t(s)$ and $t(s')$.

I class of deformations and the result

Theorem [P.Exner, SK '15].

For φ small enough there is a unique discrete eigenvalue which admits the asymptotics

$$\lambda = -\frac{\alpha^2}{4} - A\varphi^4 + o(\varphi^4)\,, \quad A > 0\,.$$

Parameterization of Γ_φ: $s \mapsto \gamma_\varphi(s)$

A is defined by the first perturbation term of

$$K_0(|\gamma_\varphi(s) - \gamma_\varphi(s')|) - K_0(|s - s'|)$$

Generalization

Curvature of Γ_β is defined by βk.

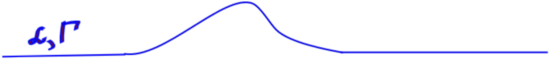
$$\Gamma_\beta(s) = \left(\int_0^s \left(\cos \int_0^u \beta k du'\right) du, \int_0^s \left(\sin \int_0^u \beta k du'\right) du\right)$$

Then

$$\lambda = -\frac{\alpha^2}{4} + B_\Gamma \beta^4 + o(\beta^4)$$

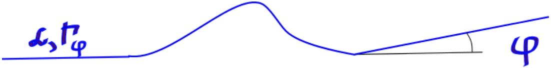
II class of deformations

Γ_φ- infinite, asymptotically straight - with the same straight line at infinity.



Hamiltonian has the discrete eigenvalues: λ_k, $k \in N$

Introduce deformation



Theorem [P.Exner, SK '15].

For φ small enough eigenvalues of H_{Γ_φ} admit the asymptotics

$$\lambda_k + A_k \varphi + o(\varphi)$$

Some ideas of the proof

Birman–Schwinger argument.
Analysis of the resolvent, $H = -\Delta + V$

$$(H - z)^{-1} = R(z) - R(z)V^{1/2}[\mathbf{I} + |\mathbf{V}|^{\mathbf{1/2}}\mathbf{R(z)V^{1/2}}]^{-1}|V|^{1/2}R(z)\,,$$

where $R(z) = (-\Delta - z)^{-1}$

Analysis of "poles" of $I + |V|^{1/2}R(z)V^{1/2}$

i.e. z such that

$$\ker(I + |V|^{1/2}R(z)V^{1/2}) \neq \emptyset$$

Embedding to $L^2(\Gamma)$.

Modification of the Birman-Schwinger argument

B-S principle

$$-\kappa^2 \in \sigma_{\mathrm{d}}(H_{\Gamma_\beta}) \quad \Leftrightarrow \quad \ker(I - \alpha Q_{\Gamma_\varphi}(\kappa)) \neq \emptyset\,, \qquad (*)$$

where

$$\mathcal{Q}_{\Gamma_\varphi}(\kappa; s, s') = \frac{1}{2\pi} K_0(\kappa|\gamma_\varphi(s) - \gamma_\varphi(s')|) = \frac{1}{2\pi} K_0(\kappa|s - s'|) + D_\varphi(\kappa; s, s')\,.$$

Modification of the Birman-Schwinger argument, cont.

$$\ker(I-\alpha Q_{\Gamma_\varphi}(\kappa_\delta)) \neq \emptyset \;\Leftrightarrow\; \ker(I - B_\delta \check{D}_\varphi) \neq \emptyset where\text{-}\kappa^2 = -\alpha^2/4 - \delta^2,$$

$$B_\delta = \frac{1}{\delta} L + M_\delta\,, \quad \|M_\delta\|_{HS} \leq const$$

where L is rank one,

$$\check{D}_\varphi = D^1 \varphi^2 + o(\varphi^2)\,.$$

$$\delta(I - B_\delta \check{D}_\varphi) \quad = \quad \delta - LD^1\varphi^2 + s.t. then \delta = \lambda(LD^1)\varphi^2 + s.t.$$

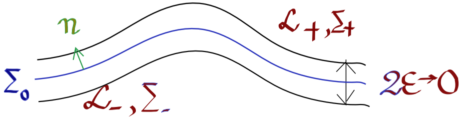
$$-\kappa^2 = -\alpha^2/4 - \delta^2 = -\alpha^2/4 - \lambda(LD^1)^2\varphi^4 + s.t.$$

Colliding quantum wires; based on a common research with D. Krejčiřík

Let $\Sigma_0 := \partial\Omega$ the boundary of Ω (bounded smooth open set in R^2) For all sufficiently small positive ϵ, we consider parallel curves (hypersurfaces)

$$\Sigma_{\pm} := \{q \pm \epsilon n(q) : q \in \Sigma_0\}\,, \tag{3}$$

- n - pointed outward of Σ_0, in the direction of Σ_+.



Hamiltonian

$$H_\epsilon := -\Delta + \alpha_+ \, \delta_{\Sigma_+} + \alpha_- \, \delta_{\Sigma_-} \,, \tag{4}$$

Analysis of spectral asymptotics for $\epsilon \to 0$.

- Limitting operator:

$$H_0 := -\Delta + (\alpha_+ + \alpha_-) \, \delta_{\Sigma_0} \,. \tag{5}$$

Essential spectrum

- Compact perturbation, essential spectrum:

$$\sigma_{ess}(H_\epsilon) = \sigma_{ess}(H_0) = \sigma_{ess}(-\Delta) = [0, \infty) \,.$$

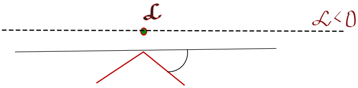
Hamiltonian

$$H = -\Delta + \alpha\delta(x)\,.$$

$$D(H) := \{f \in W^{1,2}(R) \cap W^{2,2}(R \setminus \{0\})\,, \quad f(0^+)' - f(0^-)' = \alpha f(0)\}\,.$$

- Assume $\alpha < 0$. Eigenvalue:

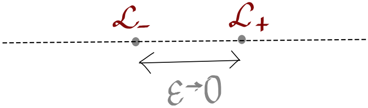
$$\lambda_0 = -\frac{\alpha^2}{4}\,, \quad \phi_0 = \mathrm{e}^{\frac{\alpha}{2}|x|} \cong 1 + \frac{\alpha}{2}|x| + \ldots$$



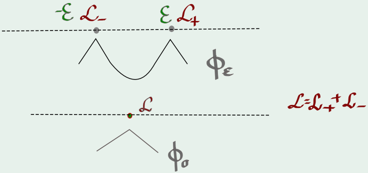
Point interactions in D1

Hamiltonian

$$H = -\Delta + \alpha_-\delta(x+\epsilon) + \alpha_+\delta(x-\epsilon)\,.$$



Singularity of eigenfunctions



$$\phi_\epsilon(x)' \nrightarrow \phi_0(x)'\,, \quad x \in (-\epsilon, \epsilon)\,.$$

$$\int_{-\epsilon}^{\epsilon} |(\phi_\epsilon - \phi_0)'|^2 = (\alpha_+^2 + \alpha_-^2)|\phi_0|^2\epsilon + s.t.$$

Theorem ($d = 1$)

Let $\alpha_+ + \alpha_- < 0$. For ε small enough operator H_ε has a unique simple discrete eigenvalue which admits the following asymptotics

$$\lambda_\varepsilon = \lambda_0 + \Big[\alpha_+ \, |\psi_0|^{2\prime}(0^+) - \alpha_- \, |\psi_0|^{2\prime}(0^-) - (\alpha_+^2 + \alpha_-^2)\, |\psi_0|^2(0)\Big]\, \varepsilon + O(\varepsilon^2)\,. \tag{6}$$

or, equivalently,

$$\lambda_\epsilon = \lambda_0 - (\alpha_+ + \alpha_-)\alpha_+\alpha_- \, \epsilon + O(\epsilon^2) \tag{7}$$

- $\alpha_+\alpha_- < 0$ pushing down the spectrum after separation;
- $\alpha_+\alpha_- > 0$ pushing up spectrum after separation;

Generalization: colliding curves. Preliminary statement.

Theorem

For any $z \in \rho(H_0)$, there exists a positive constant ϵ_0 such that, for all $\epsilon < \epsilon_0$, we have $z \in \rho(H_\epsilon)$ and

$$\left\| (H_\epsilon - z)^{-1} - (H_0 - z)^{-1} \right\|_{L^2(R^d) \to L^2(R^d)} = O(\epsilon) \qquad as \qquad \epsilon \to 0\,. \tag{8}$$

• ingradients of proof: first representation theorem, variational arguments, elliptic regularity (regularity of eigenfunctions of H_ϵ).
• Consequences: convergence of spectrum (discrete spectrum).

Generalization: colliding curves

Theorem

Let λ_0 be a simple discrete eigenvalue of H_0 and let ψ_0 be the corresponding eigenfunction. H_ϵ possesses precisely one simple eigenvalue with asymptotics:

$$\lambda_\epsilon = \lambda_0 + \lambda_0' \, \epsilon + O(\epsilon^2) \qquad \textit{as} \qquad \epsilon \to 0 \tag{9}$$

for ϵ small, with

$$\lambda_0' := \frac{1}{\int_{R^d} |\psi_0|^2} \left(\alpha_+ \int_{\Sigma_0} \partial_n^+ |\psi_0|^2 - \alpha_- \int_{\Sigma_0} \partial_n^- |\psi_0|^2 - \right. \tag{10}$$

$$\left. \int_{\Sigma_0} \left[\alpha_+^2 + \alpha_-^2 + (\alpha_+ - \alpha_-) K_1 \right] |\psi_0|^2 \right) , \tag{11}$$

where K_1 the sign curvature of Σ_0.

Generalizations

- Hypersurfaces in R^d.
- Complex coupling constants. Non-self-adjoint operators.
- Semisimple eigenvalues.

Thank you for your attention