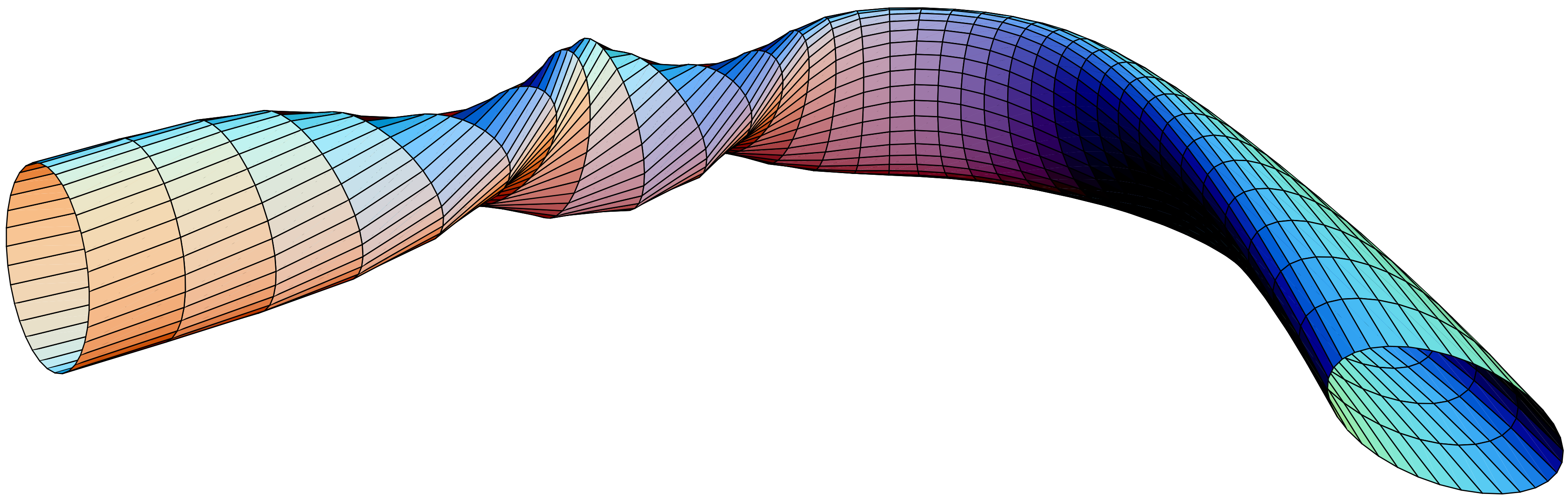
Spectral geometry of tubes

David KREJČIŘÍK

http://people.fjfi.cvut.cz/krejcirik

Czech Technical University in Prague



Spectral geometry in physics

Why is the spectrum (of the Laplacian) so important ?

Spectral geometry in physics

Why is the spectrum (of the Laplacian) so important ?

$$\frac{\partial^2 \Psi}{\partial t^2} - \Delta \Psi = 0 \qquad \frac{\partial \Psi}{\partial t} - \Delta \Psi = 0 \qquad i\frac{\partial \Psi}{\partial t} = -\Delta \Psi$$

Spectral geometry in physics

Why is the spectrum (of the Laplacian) so important ?

$$\frac{\partial^2 \Psi}{\partial t^2} - \Delta\Psi = 0 \qquad \frac{\partial \Psi}{\partial t} - \Delta\Psi = 0 \qquad i\frac{\partial \Psi}{\partial t} = -\Delta\Psi \qquad \rightsquigarrow \qquad \boxed{-\Delta\psi = \lambda\psi}$$

Spectral geometry in physics

Why is the spectrum (of the Laplacian) so important ?

$$\frac{\partial^2 \Psi}{\partial t^2} - \Delta\Psi = 0 \qquad \frac{\partial \Psi}{\partial t} - \Delta\Psi = 0 \qquad i\frac{\partial \Psi}{\partial t} = -\Delta\Psi \qquad \rightsquigarrow \qquad \boxed{-\Delta\psi = \lambda\psi}$$

$$e^{t\left(\begin{smallmatrix} 0 & 1 \\ \Delta & 0 \end{smallmatrix}\right)} \qquad e^{t\Delta} \qquad e^{it\Delta} \qquad \boxed{e^{-tH} = \int_{\sigma(H)} e^{-t\lambda}\,\mathrm{d}E_H(\lambda)}$$

Spectral geometry in physics

Why is the spectrum (of the Laplacian) so important ?

$$\frac{\partial^2\Psi}{\partial t^2}-\Delta\Psi=0 \qquad \frac{\partial\Psi}{\partial t}-\Delta\Psi=0 \qquad i\frac{\partial\Psi}{\partial t}=-\Delta\Psi \qquad \rightsquigarrow \qquad \boxed{-\Delta\psi=\lambda\psi}$$

$$e^{t\left(\begin{smallmatrix}0&1\\ \Delta&0\end{smallmatrix}\right)} \qquad e^{t\Delta} \qquad e^{it\Delta} \qquad \boxed{e^{-tH}=\int_{\sigma(H)} e^{-t\lambda}\,\mathrm{d}E_H(\lambda)}$$

We are thus lead to the study of the spectral-geometric problem for the Laplacian :

$$\begin{cases} -\Delta\psi=\lambda\psi & \text{in} \quad \Omega, \\ \dfrac{\partial\psi}{\partial n}+\alpha\psi=0 & \text{on} \quad \partial\Omega. \end{cases}$$

(Neumann $\alpha=0$, Dirichlet $\alpha=\infty$)

geometry of Ω $\quad\leftrightarrows\quad$ spectrum of $-\Delta$

Spectral geometry in physics

Why is the spectrum (of the Laplacian) so important ?

$$\frac{\partial^2\Psi}{\partial t^2} - \Delta\Psi = 0 \qquad \frac{\partial\Psi}{\partial t} - \Delta\Psi = 0 \qquad i\frac{\partial\Psi}{\partial t} = -\Delta\Psi \qquad \rightsquigarrow \qquad \boxed{-\Delta\psi = \lambda\psi}$$

$$e^{t\left(\begin{smallmatrix}0 & 1\\ \Delta & 0\end{smallmatrix}\right)} \qquad e^{t\Delta} \qquad e^{it\Delta} \qquad \boxed{e^{-tH} = \int_{\sigma(H)} e^{-t\lambda}\,\mathrm{d}E_H(\lambda)}$$

We are thus lead to the study of the spectral-geometric problem for the Laplacian :

$$\begin{cases} -\Delta\psi = \lambda\psi & \text{in} \quad \Omega, \\ \dfrac{\partial\psi}{\partial n} + \alpha\psi = 0 & \text{on} \quad \partial\Omega. \end{cases}$$

(Neumann $\alpha = 0$, Dirichlet $\alpha = \infty$)

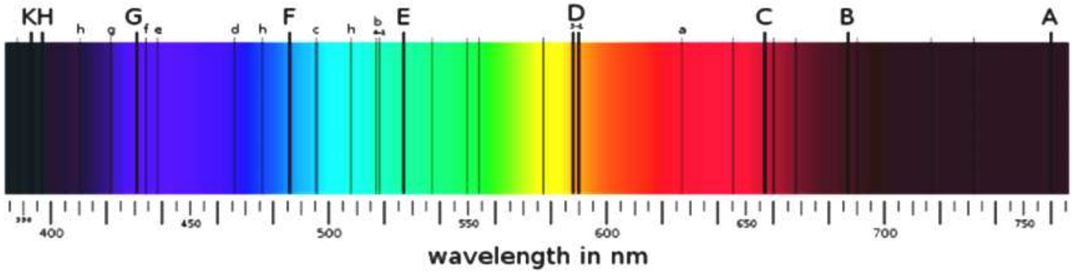
geometry of Ω $\leftrightarrows$ spectrum of $-\Delta$

But: spectrum known explicitly only for $\Omega = \mathbb{R}^d$, ball and parallelipiped

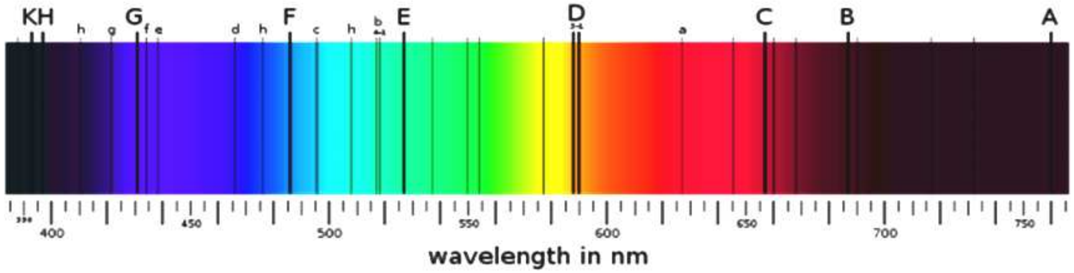
$\Longrightarrow$ functional-analytic tools have to be employed

What is the spectrum?

What is the spectrum?

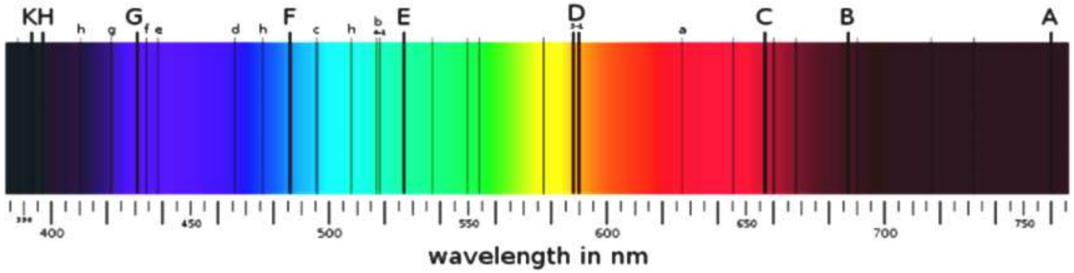


What is the spectrum?



¿ How to correctly understand $\begin{cases} -\Delta\psi = \lambda\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$?

What is the spectrum?



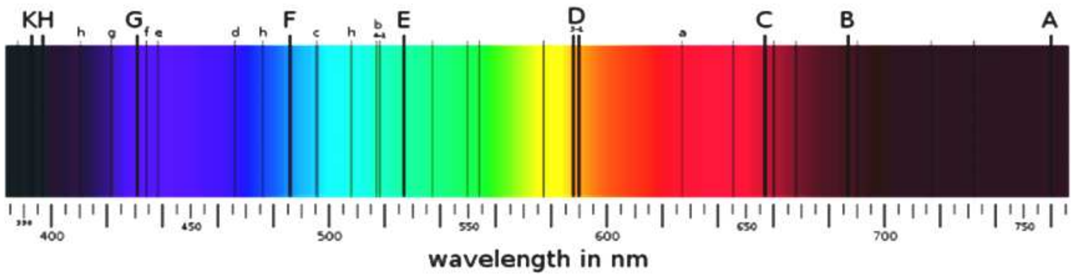
¿ How to correctly understand $\begin{cases} -\Delta\psi = \lambda\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$?

$\rightarrow$ Spectral problem for an unbounded (self-adjoint) operator in a Hilbert space:

$$-\Delta_D^\Omega : L^2(\Omega) \to L^2(\Omega) : \{\psi \mapsto -\Delta\psi\}$$

$$\mathrm{Dom}(-\Delta_D^\Omega) := \{\psi \in W_0^{1,2}(\Omega) \mid \Delta\psi \in L^2(\Omega)\}$$

What is the spectrum?



¿ How to correctly understand $\begin{cases} -\Delta\psi = \lambda\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$?

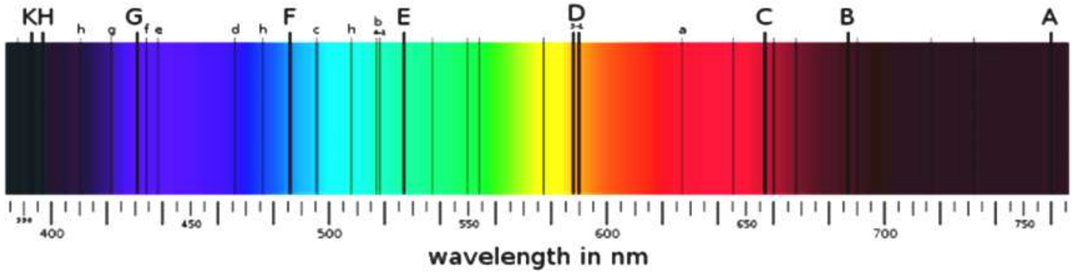
$\rightarrow$ Spectral problem for an unbounded (self-adjoint) operator in a Hilbert space:

$$-\Delta_D^\Omega : L^2(\Omega) \to L^2(\Omega) : \{\psi \mapsto -\Delta\psi\}$$
$$\text{Dom}(-\Delta_D^\Omega) := \{\psi \in W_0^{1,2}(\Omega) \mid \Delta\psi \in L^2(\Omega)\}$$

NB (spectrum of an unbounded operator H)

$$\sigma(H) := \{\lambda \in \mathbb{C} \mid H - \lambda I \quad \text{is not bijective}\}$$

What is the spectrum?



¿ How to correctly understand $\begin{cases} -\Delta\psi = \lambda\psi & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases}$?

$\rightarrow$ Spectral problem for an unbounded (self-adjoint) operator in a Hilbert space:

$$-\Delta_D^\Omega : L^2(\Omega) \to L^2(\Omega) : \{\psi \mapsto -\Delta\psi\}$$
$$\mathrm{Dom}(-\Delta_D^\Omega) := \left\{\psi \in W_0^{1,2}(\Omega) \;\middle|\; \Delta\psi \in L^2(\Omega)\right\}$$

NB (spectrum of an unbounded operator H)

$$\sigma(H) := \{\lambda \in \mathbb{C} \mid H - \lambda I \quad \text{is not bijective}\}$$
$$= \underbrace{\sigma_{\text{disc}}(H)}_{||} \,\dot{\cup}\, \sigma_{\text{ess}}(H)$$

isolated eigenvalues of finite multiplicity

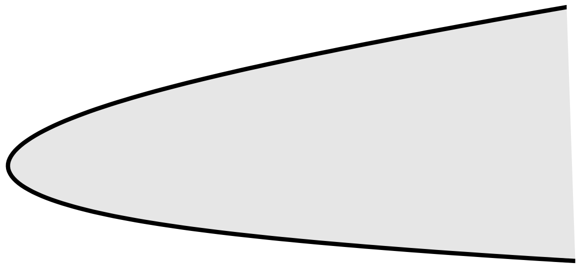
Classification of Euclidean domains

[Glazman 1963]

Classification of Euclidean domains

[Glazman 1963]

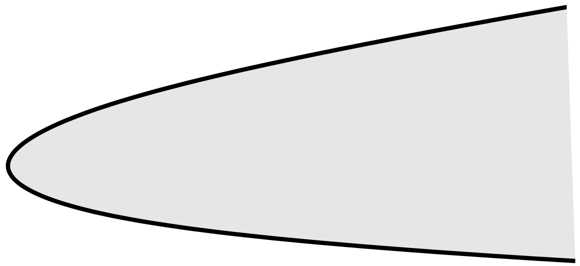
- *quasi-conical* $:\Longleftrightarrow \ \Omega \supset \{\text{arbitrarily large balls}\}$



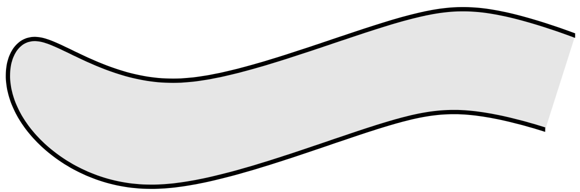
Classification of Euclidean domains

[Glazman 1963]

- *quasi-conical* $:\Longleftrightarrow$ $\Omega \supset \{\text{arbitrarily large balls}\}$



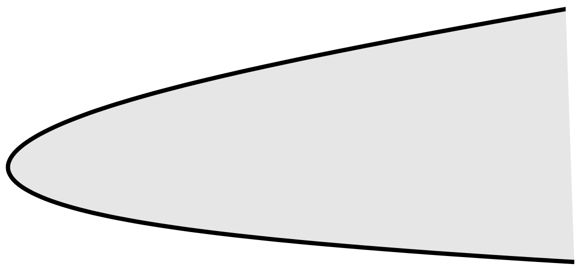
- *quasi-cylindrical* $:\Longleftrightarrow$ not q-conical but $\Omega \supseteq \{\text{sequence of identical disjoint balls}\}$



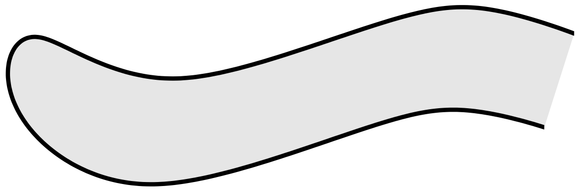
Classification of Euclidean domains

[Glazman 1963]

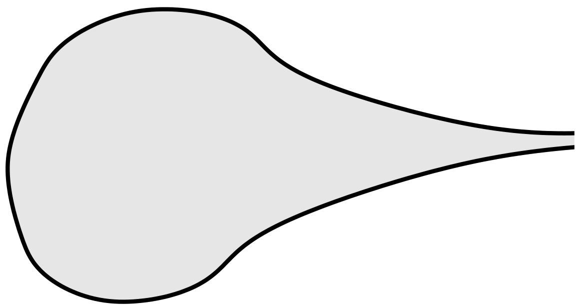
- *quasi-conical* $:\Longleftrightarrow$ $\Omega \supset$ {arbitrarily large balls}



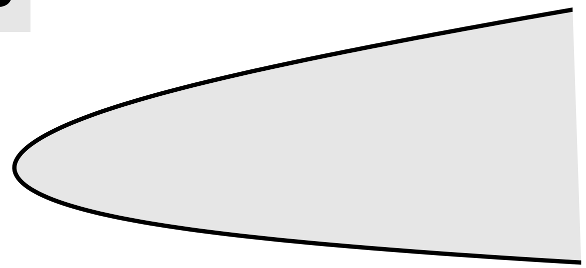
- *quasi-cylindrical* $:\Longleftrightarrow$ not q-conical but $\Omega \supseteq$ {sequence of identical disjoint balls}



- *quasi-bounded* $:\Longleftrightarrow$ neither q-conical nor q-cylindrical

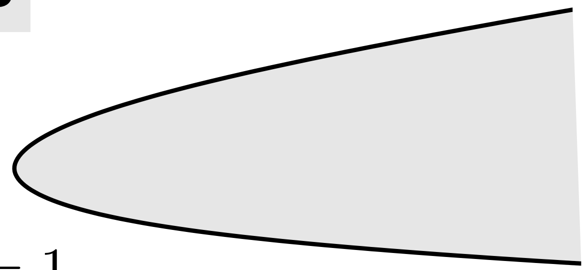


Quasi-conical domains



Theorem. $\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [0, \infty)}$

Quasi-conical domains

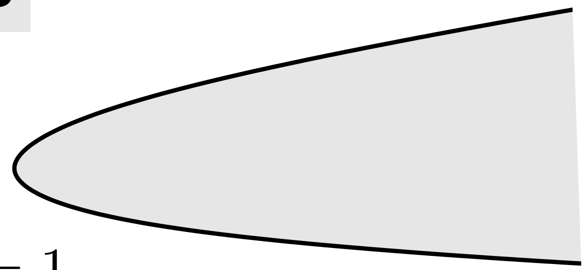


Theorem. $\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [0,\infty)}$

Proof. Weyl's criterion:

$$\lambda \in \sigma(H) \iff \exists\{\psi_n\} \subset \mathsf{Dom}(H) : \begin{cases} \|\psi_n\| = 1 \\ \|(H-\lambda)\psi_n\| \to 0 \end{cases}$$

Quasi-conical domains



Theorem. $\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\text{ess}}(-\Delta_D^\Omega) = [0, \infty)}$

Proof. Weyl's criterion:

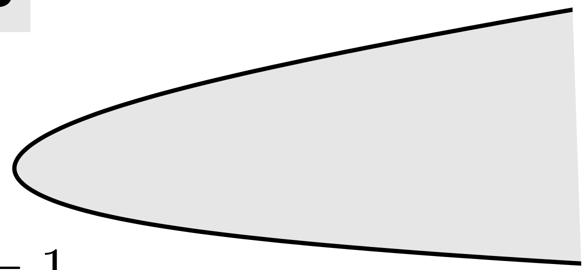
$$\lambda \in \sigma(H) \iff \exists\{\psi_n\} \subset \mathsf{Dom}(H) : \begin{cases} \|\psi_n\| = 1 \\ \|(H-\lambda)\psi_n\| \to 0 \end{cases}$$

Given $k \in \mathbb{R}^d$ such that $|k|^2 = \lambda$, one takes

$$\psi_n(x) = e^{ik\cdot x}\,\tilde{\chi}_{B_n}(x) \ \text{ where: } \begin{cases} \{B_n\} = \text{a sequence of the enlarging balls} \\ \tilde{\chi}_B = \text{mollified and normalised } \chi_B \end{cases}$$

q.e.d.

Quasi-conical domains



Theorem. $\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\text{ess}}(-\Delta_D^\Omega) = [0,\infty)}$

Proof. Weyl's criterion:

$$\lambda \in \sigma(H) \iff \exists\{\psi_n\} \subset \mathsf{Dom}(H) : \begin{cases} \|\psi_n\| = 1 \\ \|(H-\lambda)\psi_n\| \to 0 \end{cases}$$

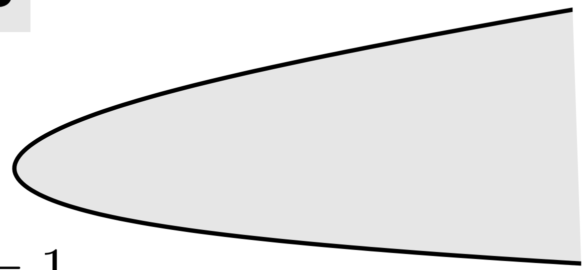
Given $k \in \mathbb{R}^d$ such that $|k|^2 = \lambda$, one takes

$$\psi_n(x) = e^{ik\cdot x}\,\tilde{\chi}_{B_n}(x) \ \text{ where: } \begin{cases} \{B_n\} = \text{a sequence of the enlarging balls} \\ \tilde{\chi}_B = \text{mollified and normalised } \chi_B \end{cases}$$

q.e.d.

¿ *Is there anything interesting to study* ?

Quasi-conical domains



Theorem. $\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [0,\infty)}$

Proof. Weyl's criterion:

$$\lambda \in \sigma(H) \iff \exists\{\psi_n\} \subset \mathsf{Dom}(H) : \begin{cases} \|\psi_n\| = 1 \\ \|(H-\lambda)\psi_n\| \to 0 \end{cases}$$

Given $k \in \mathbb{R}^d$ such that $|k|^2 = \lambda$, one takes

$$\psi_n(x) = e^{ik\cdot x}\,\tilde{\chi}_{B_n}(x) \ \text{ where: } \begin{cases} \{B_n\} = \text{a sequence of the enlarging balls} \\ \tilde{\chi}_B = \text{mollified and normalised } \chi_B \end{cases}$$

q.e.d.

¿ *Is there anything interesting to study* ?

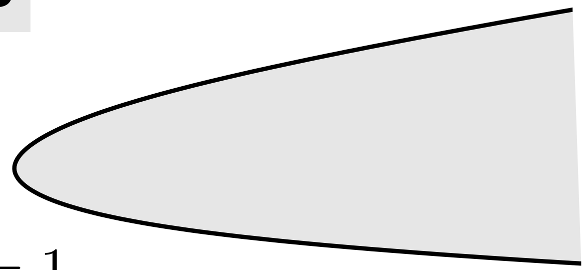
Godfrey Harold Hardy

1877–1947

Theorem (Hardy inequality). Let $d \geq 3$. Then

$$\forall \psi \in W_0^{1,2}(\Omega), \quad \boxed{\int_\Omega |\nabla\psi(x)|^2\,\mathrm{d}x \ \geq \ \left(\frac{d-2}{2}\right)^2 \int_\Omega \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x}$$

Quasi-conical domains



Theorem. $\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [0,\infty)}$

Proof. Weyl's criterion:

$$\lambda \in \sigma(H) \iff \exists \{\psi_n\} \subset \mathrm{Dom}(H) : \begin{cases} \|\psi_n\| = 1 \\ \|(H-\lambda)\psi_n\| \to 0 \end{cases}$$

Given $k \in \mathbb{R}^d$ such that $|k|^2 = \lambda$, one takes

$$\psi_n(x) = e^{ik\cdot x}\,\tilde{\chi}_{B_n}(x) \ \text{ where: } \begin{cases} \{B_n\} = \text{a sequence of the enlarging balls} \\ \tilde{\chi}_B = \text{mollified and normalised } \chi_B \end{cases}$$

q.e.d.

¿ *Is there anything interesting to study* ?

Godfrey Harold Hardy

1877–1947

Theorem (Hardy inequality). Let $d \geq 3$. Then

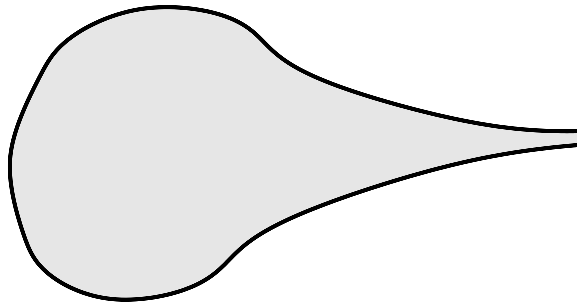
$$\forall \psi \in W_0^{1,2}(\Omega), \quad \boxed{\int_\Omega |\nabla\psi(x)|^2\,\mathrm{d}x \ \geq \ \left(\frac{d-2}{2}\right)^2 \int_\Omega \frac{|\psi(x)|^2}{|x|^2}\,\mathrm{d}x}$$

Theorem (criticality of $\mathbb{R}^1$ and $\mathbb{R}^2$**).** Let $d = 1, 2$. For *any non-positive* measurable V,

$$\boxed{\inf \sigma(-\Delta_D^{\mathbb{R}^d} + V) < 0}$$

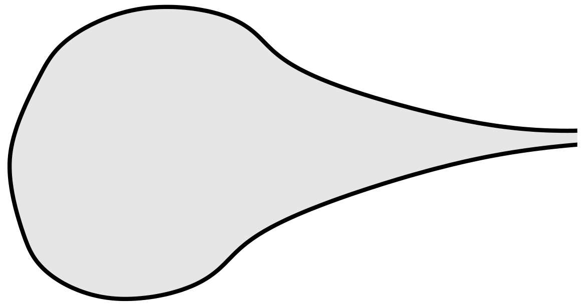
Quasi-bounded domains

Ω is quasi-bounded $\iff$ $\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x, \partial\Omega) = 0$



Quasi-bounded domains

Ω is quasi-bounded $\iff$ $\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0$

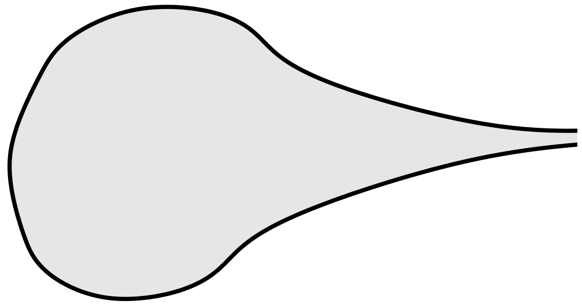


$\Uparrow \not\Downarrow$

$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact $\iff$ $\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

Quasi-bounded domains

$\boxed{\Omega \text{ is quasi-bounded} \quad \Longleftrightarrow \quad \limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0}$



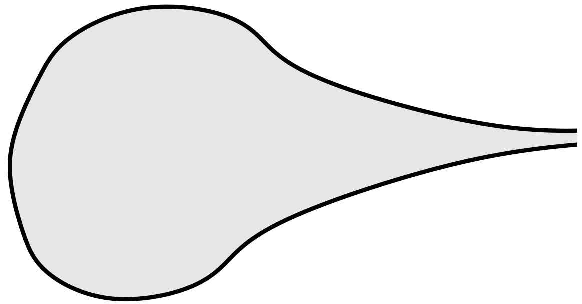
$\Uparrow \ \not\Downarrow$

$\boxed{H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact}} \quad \Longleftrightarrow \quad \boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)}$

$\Uparrow$

$\boxed{\limsup_{|x|\to\infty,\ x\in\Omega} \left|\Omega \cap B_1(x)\right| = 0}$ sufficient condition due to [Berger, Schechter 1972]

Quasi-bounded domains



$$\boxed{\Omega \text{ is quasi-bounded} \quad \Longleftrightarrow \quad \limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x, \partial\Omega) = 0}$$

$$\Uparrow \ \not\Downarrow$$

$$\boxed{H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact}} \quad \Longleftrightarrow \quad \boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)}$$

$$\Uparrow$$

$$\boxed{\limsup_{|x|\to\infty,\ x\in\Omega} \big|\Omega \cap B_1(x)\big| = 0}$$

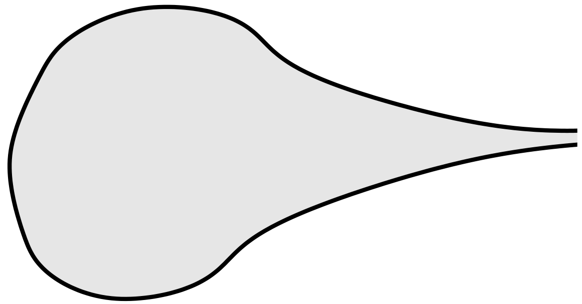
sufficient condition due to [Berger, Schechter 1972]

not satisfied for *spiny urchin*

$$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$$

Quasi-bounded domains

Ω is quasi-bounded $\iff$ $\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0$



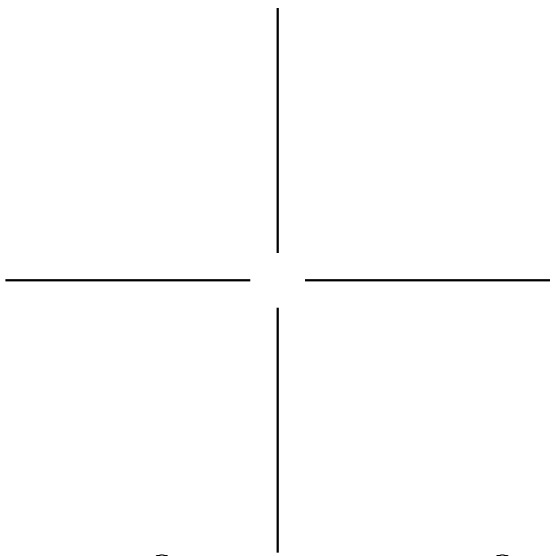
$\Uparrow \ \not\Downarrow$

$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact $\iff$ $\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

$\Uparrow$

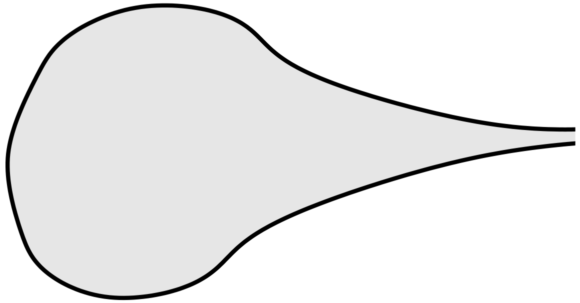
$\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \big|\Omega \cap B_1(x)\big| = 0$ sufficient condition due to [Berger, Schechter 1972]

not satisfied for *spiny urchin*



$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

Quasi-bounded domains



$\boxed{\Omega \text{ is quasi-bounded} \iff \limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0}$

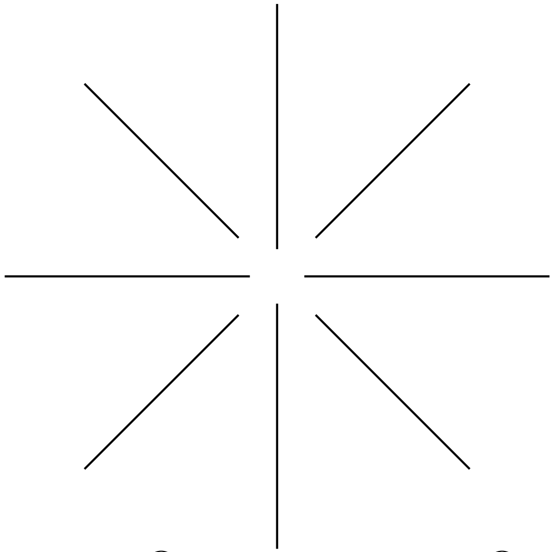
$\Uparrow \ \not\Downarrow$

$\boxed{H_0^1(\Omega) \hookrightarrow L^2(\Omega) \text{ is compact}} \iff \boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)}$

$\Uparrow$

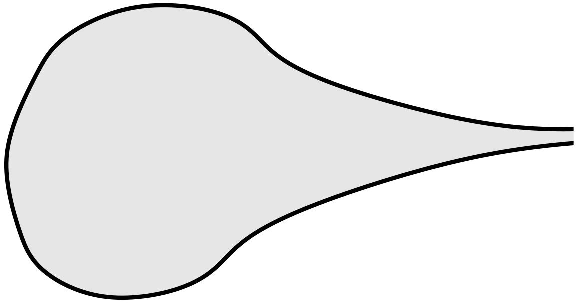
$\boxed{\limsup_{|x|\to\infty,\ x\in\Omega} \big|\Omega \cap B_1(x)\big| = 0}$ sufficient condition due to [Berger, Schechter 1972]

not satisfied for *spiny urchin*



$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

Quasi-bounded domains



Ω is quasi-bounded $\iff$ $\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x, \partial\Omega) = 0$

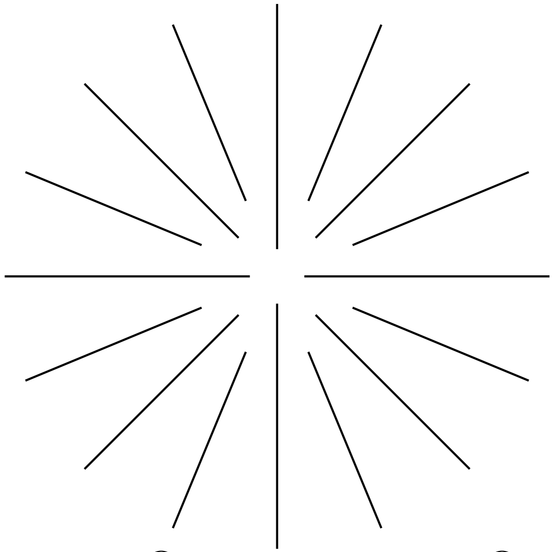
$\Uparrow \;\not\Downarrow$

$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact $\iff$ $\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

$\Uparrow$

$\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \big|\Omega \cap B_1(x)\big| = 0$ sufficient condition due to [Berger, Schechter 1972]

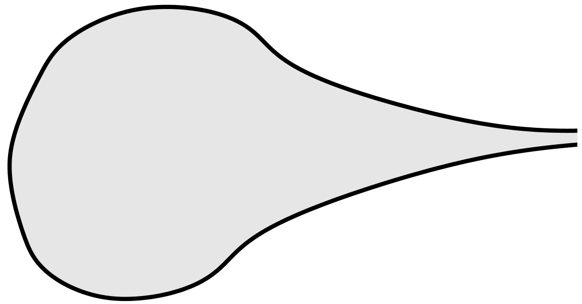
not satisfied for *spiny urchin*



$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

Quasi-bounded domains

Ω is quasi-bounded $\iff \limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0$



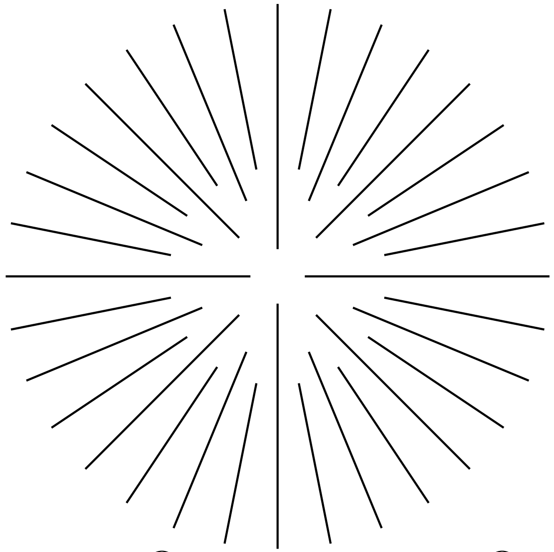
$\Uparrow \not\Downarrow$

$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact $\iff \sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

$\Uparrow$

$\limsup_{|x|\to\infty,\ x\in\Omega} \left|\Omega \cap B_1(x)\right| = 0$ sufficient condition due to [Berger, Schechter 1972]

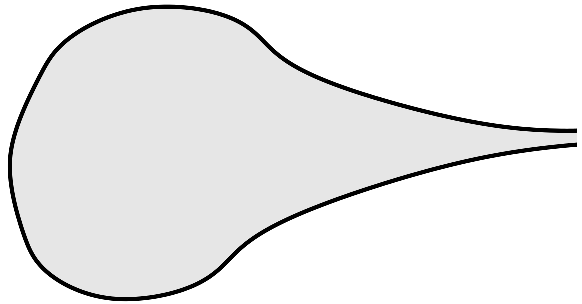
not satisfied for *spiny urchin*



$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

Quasi-bounded domains

Ω is quasi-bounded $\quad \Longleftrightarrow \quad \limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0$



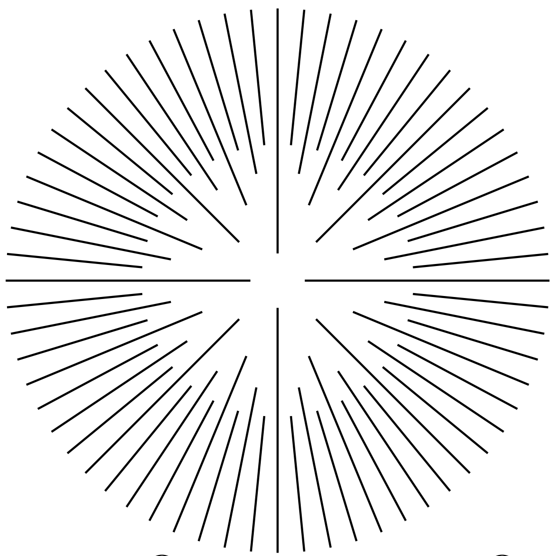
$\Uparrow \ \not\Downarrow$

$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact $\quad \Longleftrightarrow \quad \sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

$\Uparrow$

$\limsup_{|x|\to\infty,\ x\in\Omega} \left|\Omega \cap B_1(x)\right| = 0$ sufficient condition due to [Berger, Schechter 1972]

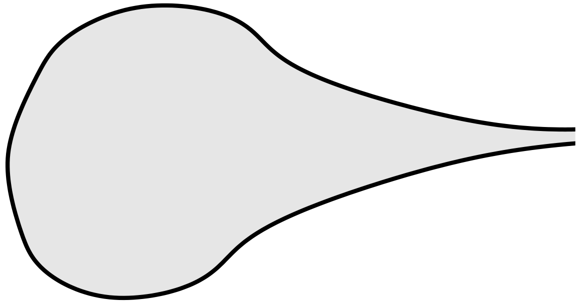
not satisfied for *spiny urchin*



$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

Quasi-bounded domains

Ω is quasi-bounded $\iff$ $\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0$



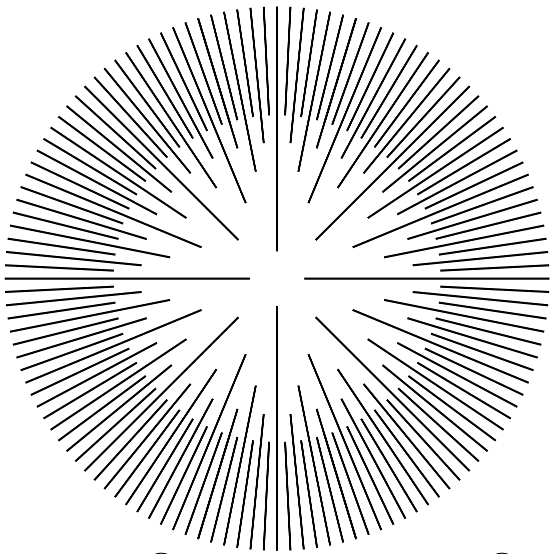
$\Uparrow \ \not\Downarrow$

$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact $\iff$ $\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

$\Uparrow$

$\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \left|\Omega \cap B_1(x)\right| = 0$ sufficient condition due to [Berger, Schechter 1972]

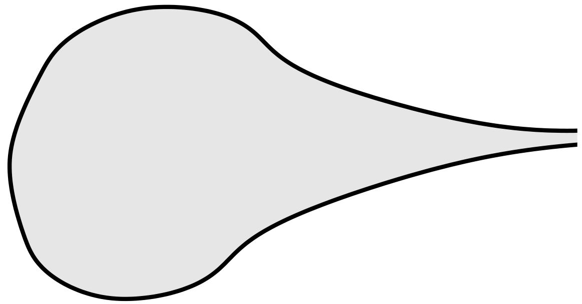
not satisfied for *spiny urchin*



$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

Quasi-bounded domains

Ω is quasi-bounded $\iff$ $\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \operatorname{dist}(x,\partial\Omega) = 0$



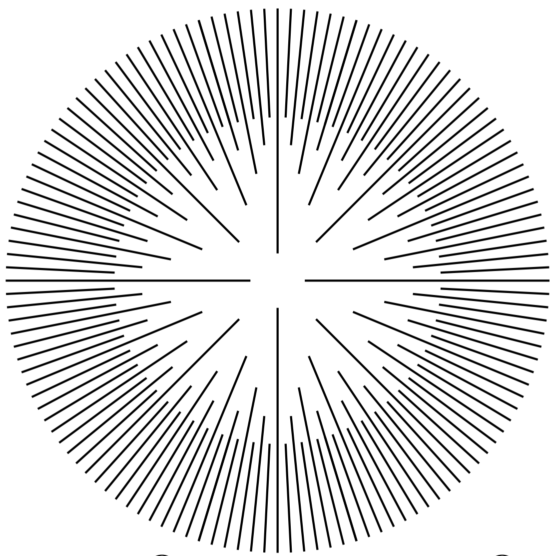
$\Uparrow \ \not\Downarrow$

$H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact $\iff$ $\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$

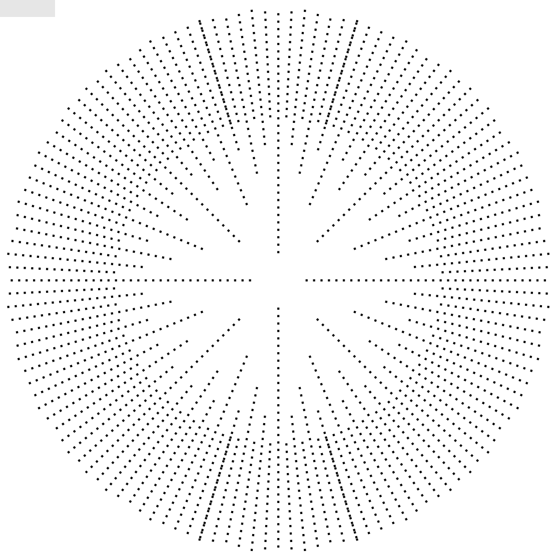
$\Uparrow$

$\displaystyle\limsup_{|x|\to\infty,\ x\in\Omega} \big|\Omega \cap B_1(x)\big| = 0$ sufficient condition due to [Berger, Schechter 1972]

not satisfied for *spiny urchin*



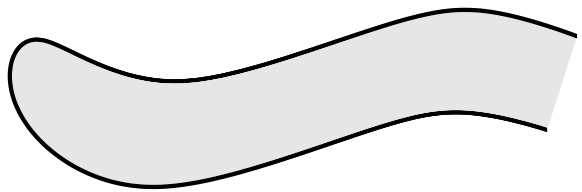
$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega)$



$\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [0,\infty)$

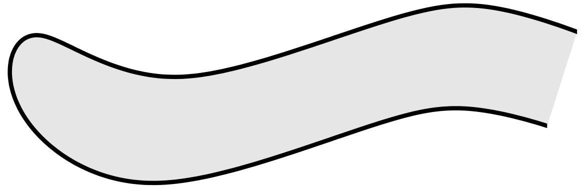
Quasi-cylindrical domains

$$\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega) \cup \underbrace{\sigma_{\mathrm{ess}}(-\Delta_D^\Omega)}_{\neq\varnothing}} \Rightarrow \text{very hard to study}$$



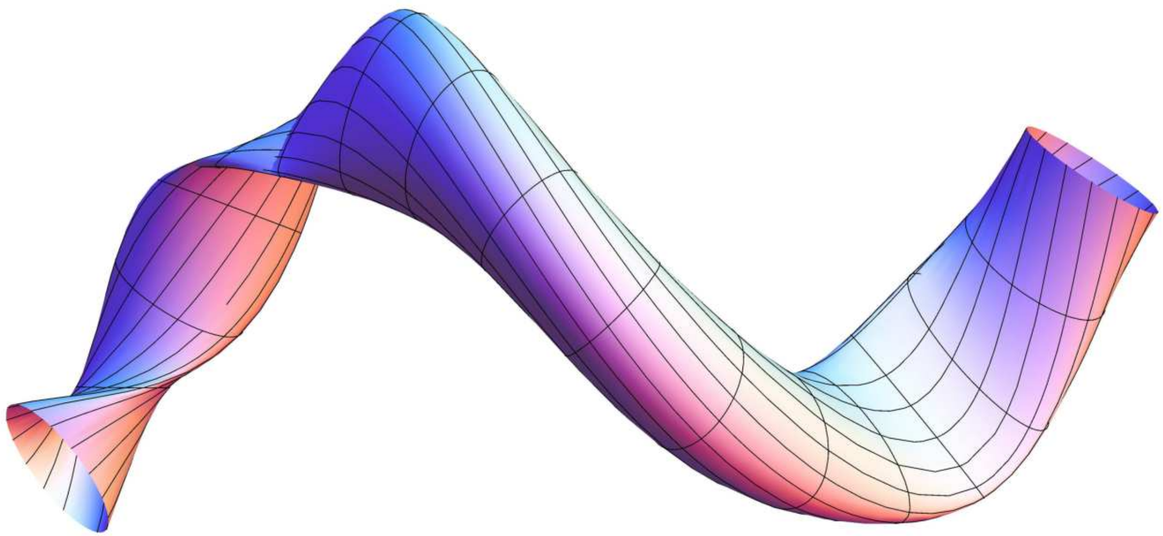
Quasi-cylindrical domains

$$\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\mathrm{disc}}(-\Delta_D^\Omega) \cup \underbrace{\sigma_{\mathrm{ess}}(-\Delta_D^\Omega)}_{\neq\varnothing}} \Rightarrow \text{very hard to study}$$



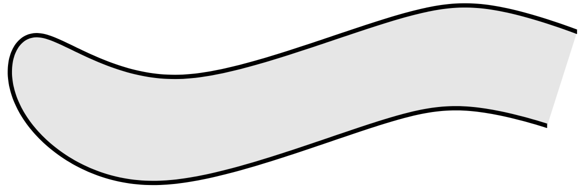
distinguished subclass: TUBES

tubular neighbourhoods of submanifolds



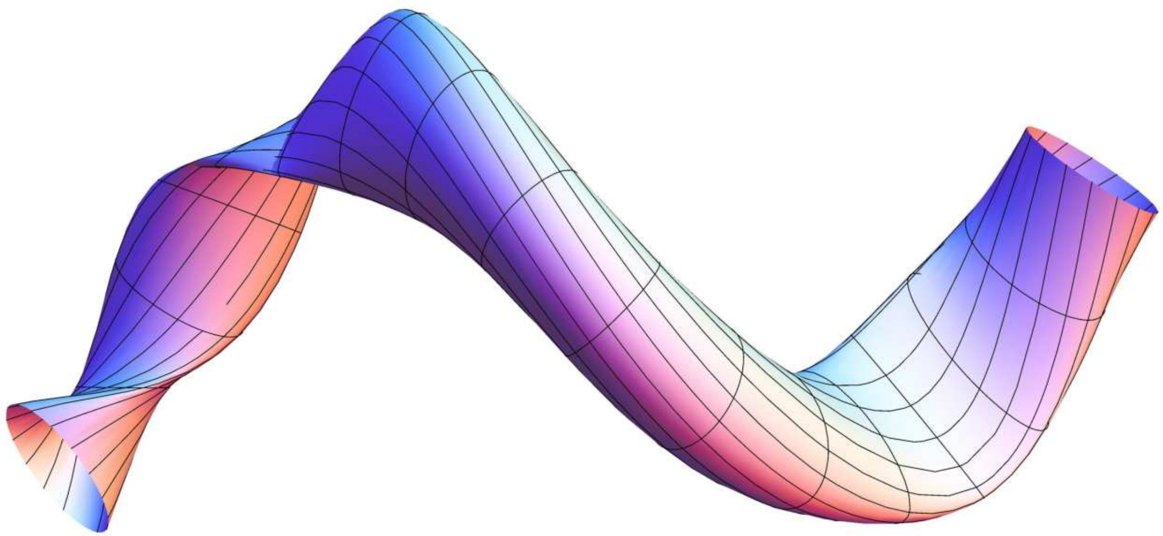
Quasi-cylindrical domains

$$\boxed{\sigma(-\Delta_D^\Omega) = \sigma_{\text{disc}}(-\Delta_D^\Omega) \cup \underbrace{\sigma_{\text{ess}}(-\Delta_D^\Omega)}_{\neq\varnothing}} \Rightarrow \text{very hard to study}$$



distinguished subclass: TUBES

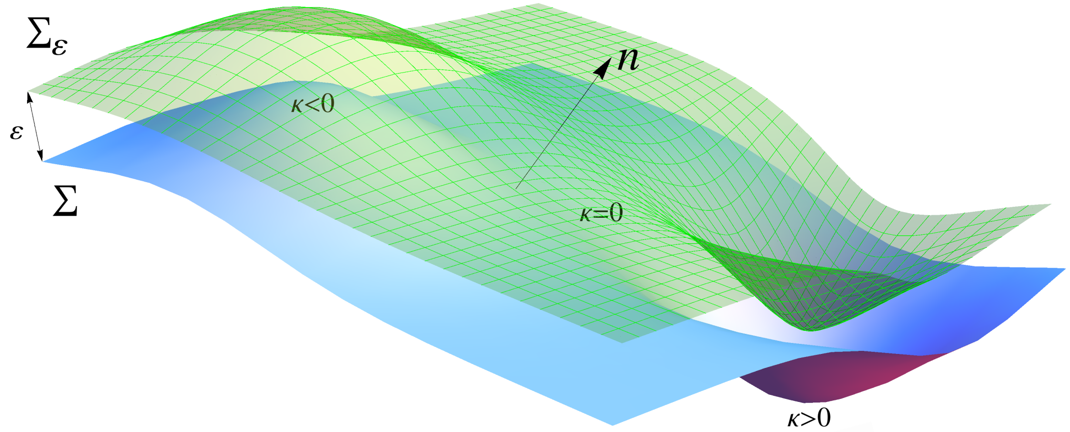
tubular neighbourhoods of submanifolds

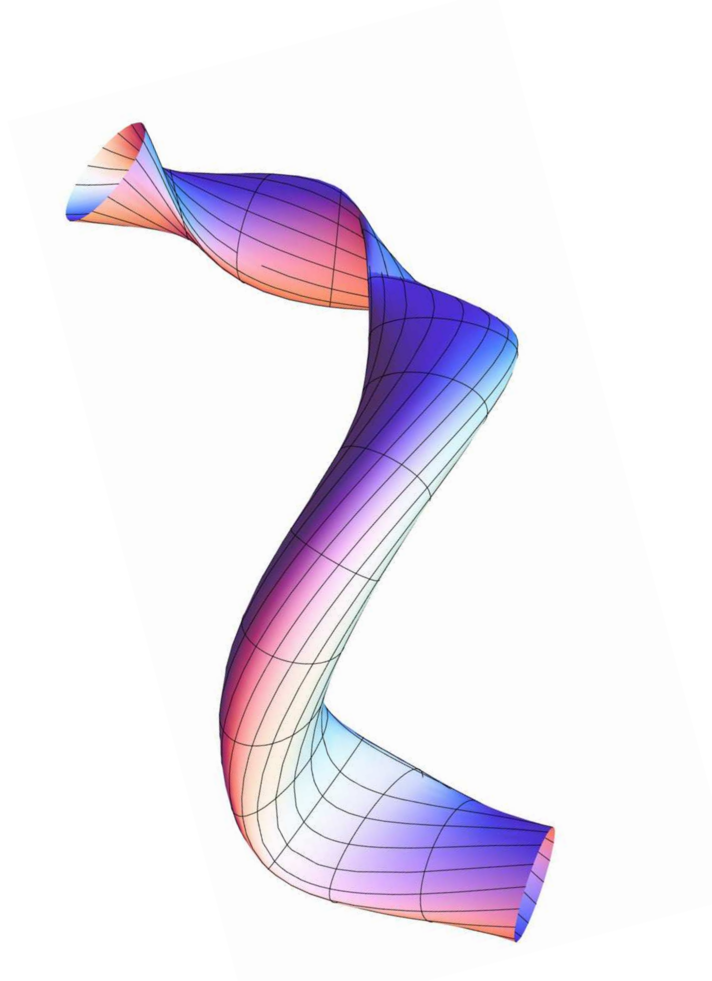


$\longrightarrow$ location of the essential spectrum

$\longrightarrow$ existence of the discrete spectrum

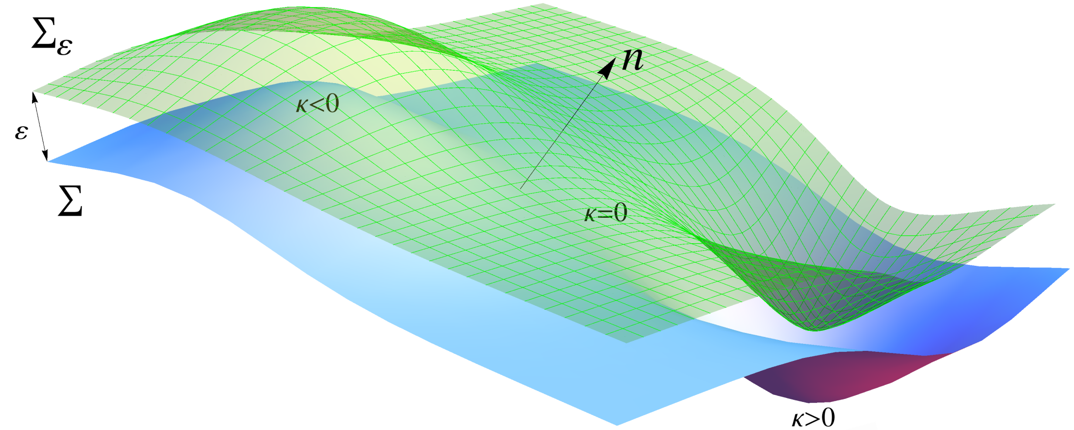
The geometry of tubes

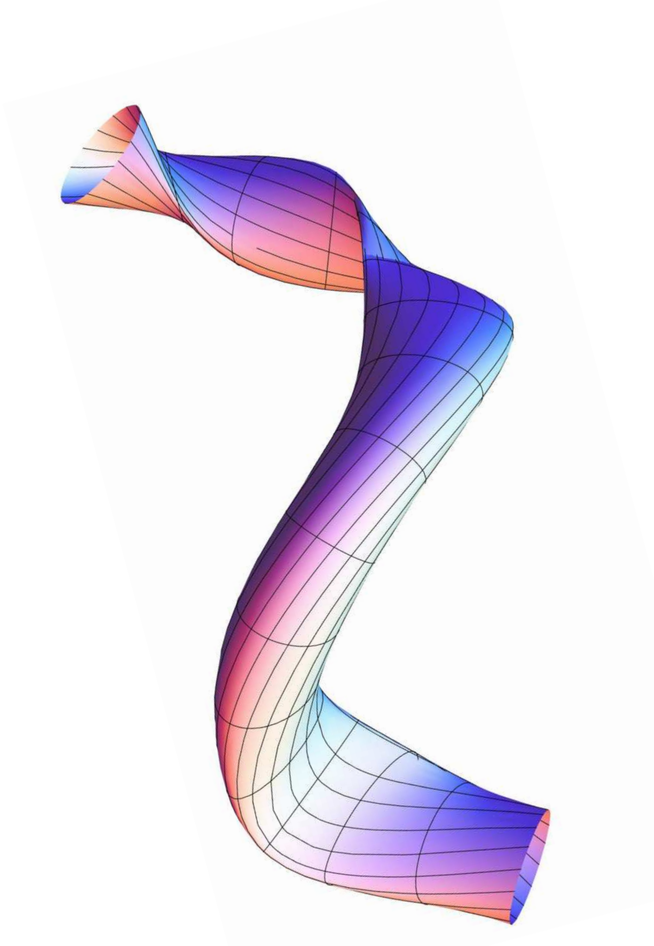




The geometry of tubes

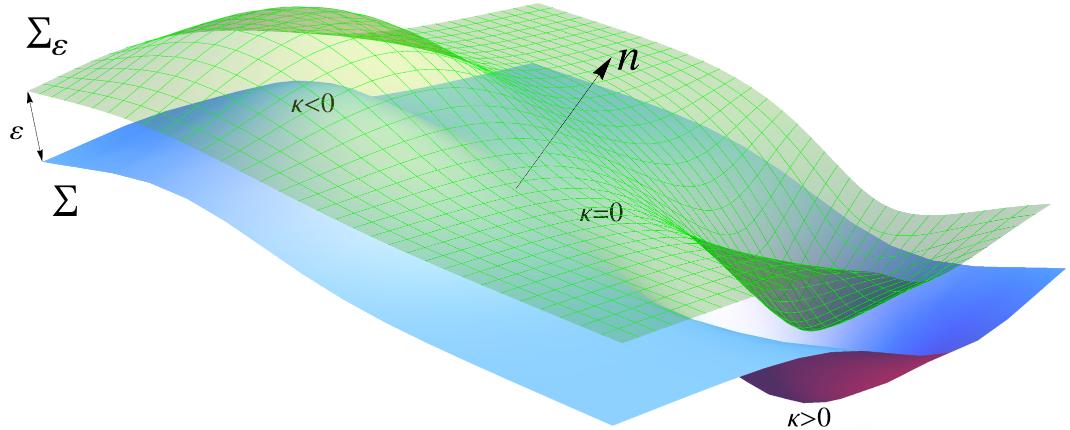
- ambient Riemannian manifold $\mathbb{R}^d$, $d \geq 2$

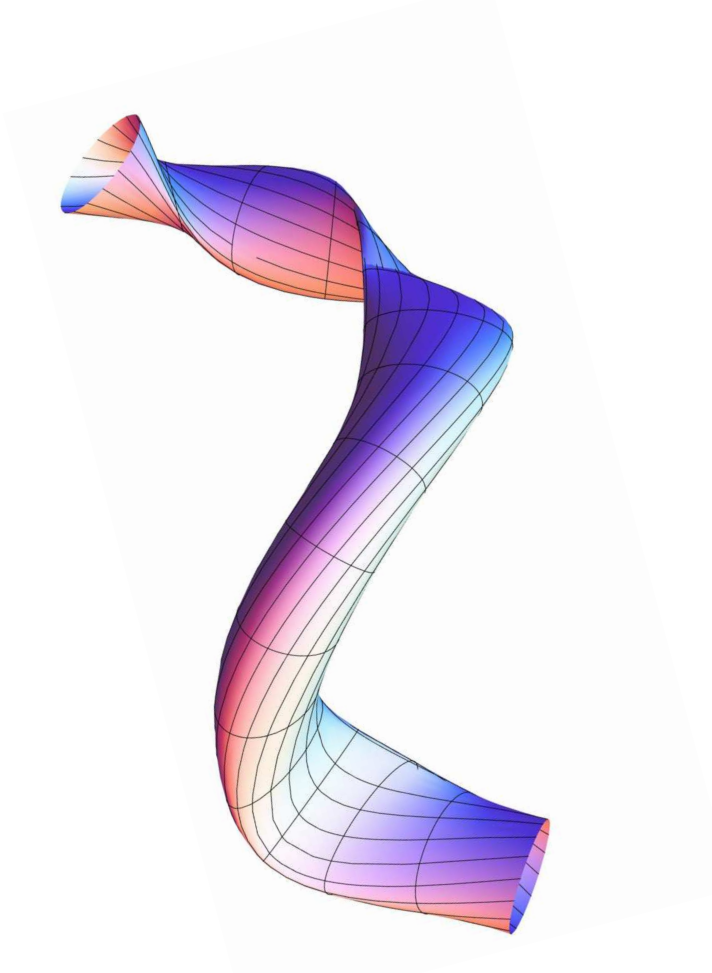




The geometry of tubes

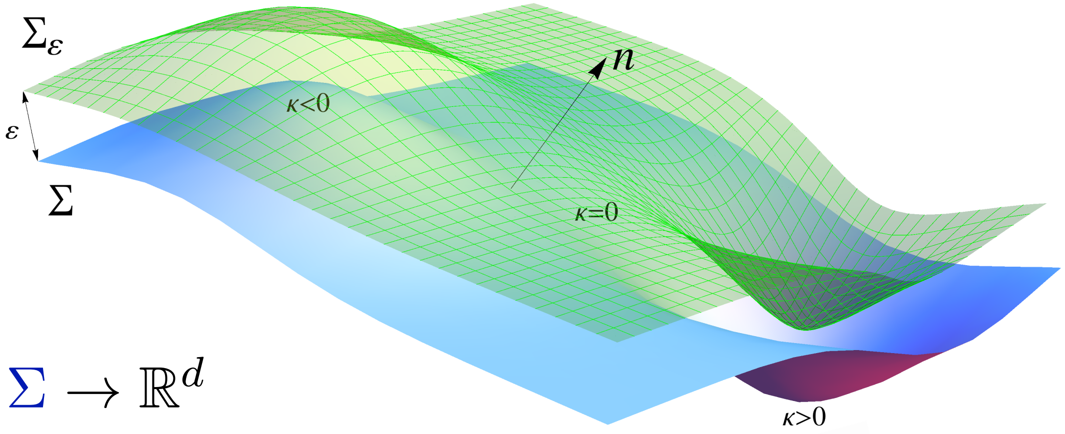
- ambient Riemannian manifold $\mathbb{R}^d$, $d \geq 2$
- oriented submanifold $\Sigma \subset \mathbb{R}^d$ complete and non-compact, $\dim \Sigma < d$

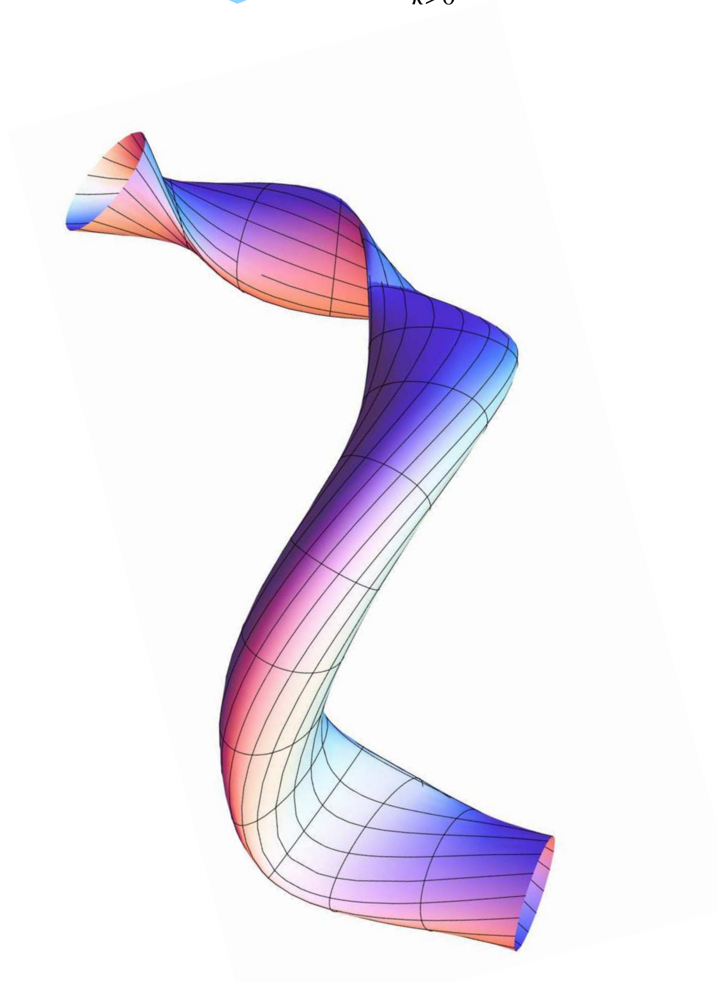




The geometry of tubes

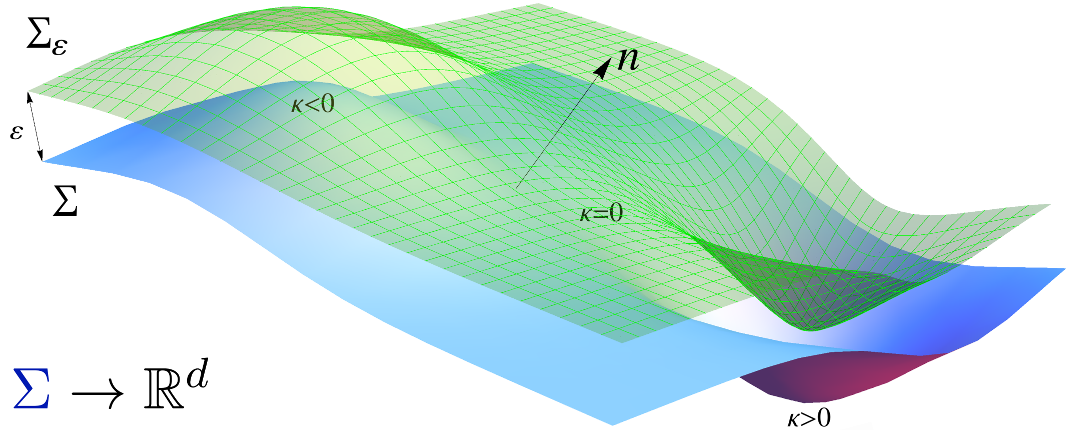
- ambient Riemannian manifold $\mathbb{R}^d$, $d \geq 2$
- oriented submanifold $\Sigma \subset \mathbb{R}^d$ complete and non-compact, $\dim \Sigma < d$
- orthonormal vector fields $n_1, \ldots, n_{\operatorname{codim} \Sigma} : \Sigma \to \mathbb{R}^d$

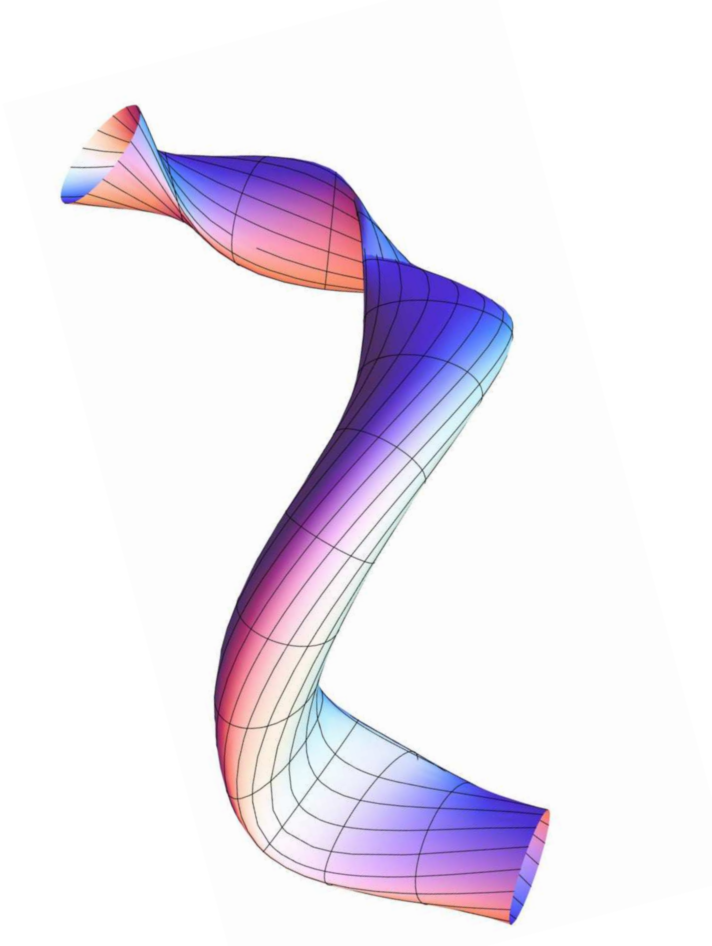




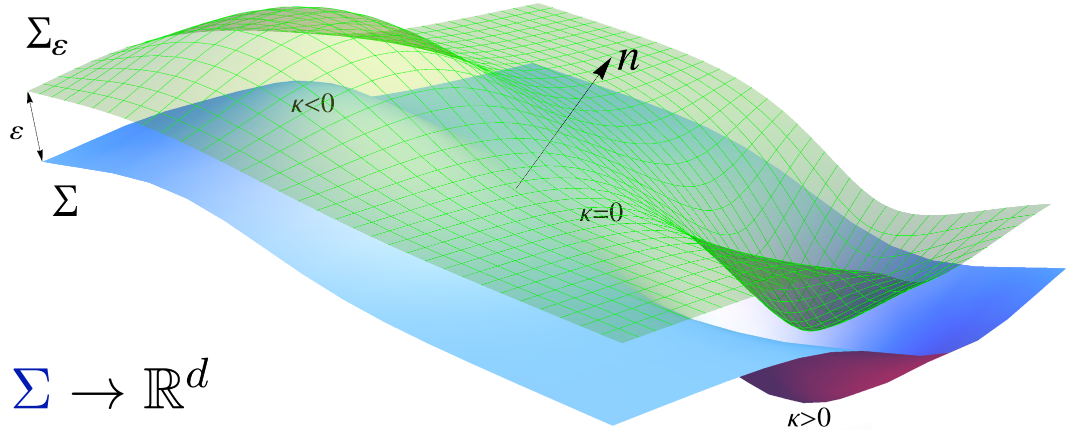
The geometry of tubes

- ambient Riemannian manifold $\mathbb{R}^d$, $d \geq 2$
- oriented submanifold $\Sigma \subset \mathbb{R}^d$
  complete and non-compact, $\dim \Sigma < d$
- orthonormal vector fields $n_1, \dots, n_{\operatorname{codim}\Sigma} : \Sigma \to \mathbb{R}^d$
- cross-section $\omega \subset \mathbb{R}^{\operatorname{codim}\Sigma}$
  bounded domain

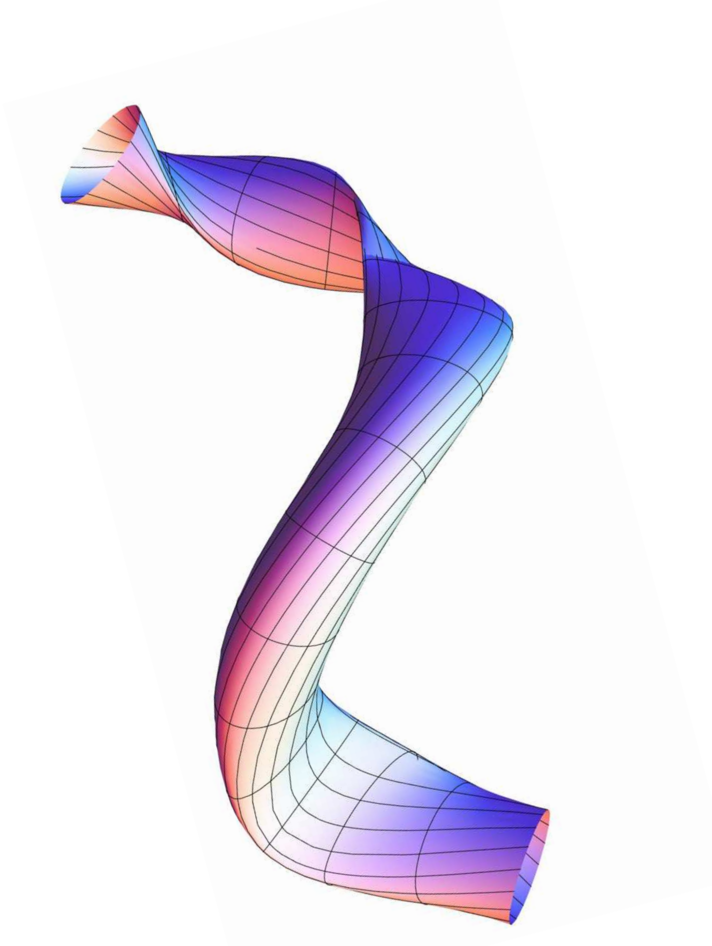




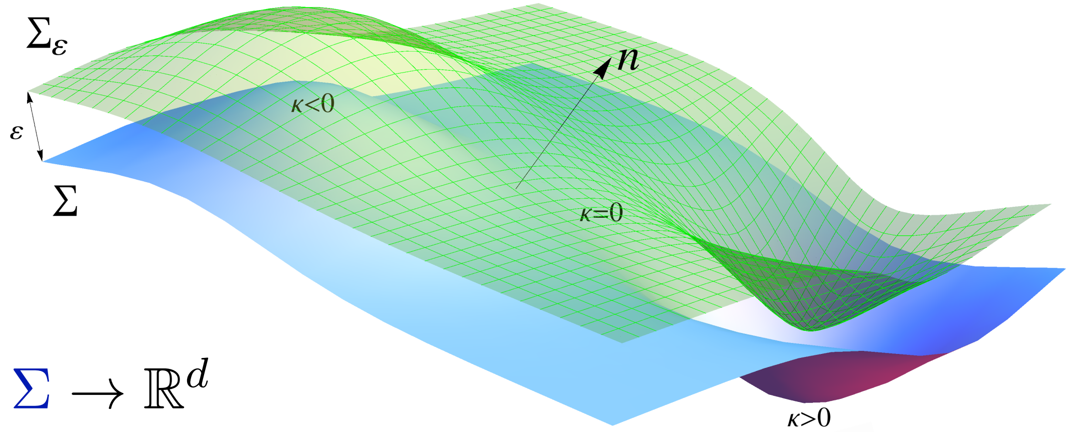
The geometry of tubes



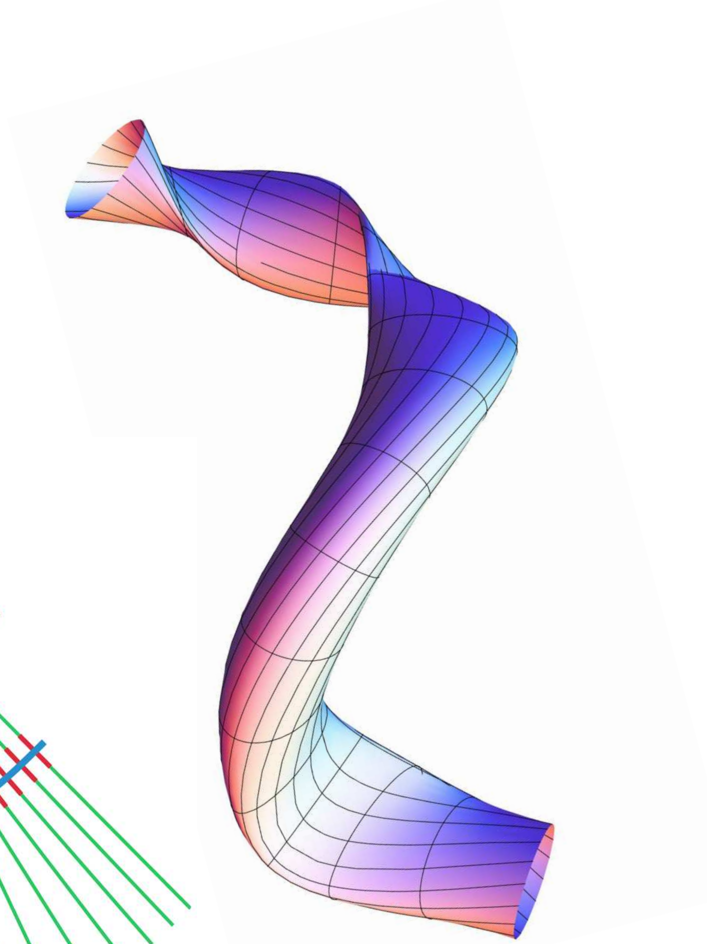
- ambient Riemannian manifold $\mathbb{R}^d$, $d \geq 2$
- oriented submanifold $\Sigma \subset \mathbb{R}^d$
  complete and non-compact, $\dim \Sigma < d$
- orthonormal vector fields $n_1, \dots, n_{\operatorname{codim} \Sigma} : \Sigma \to \mathbb{R}^d$
- cross-section $\omega \subset \mathbb{R}^{\operatorname{codim} \Sigma}$
  bounded domain
- tube $\Omega := \left\{ x + \sum_{j=1}^{\operatorname{codim} \Sigma} t_j\, n_j(x) \ : \ (x,t) \in \Sigma \times \omega \right\}$



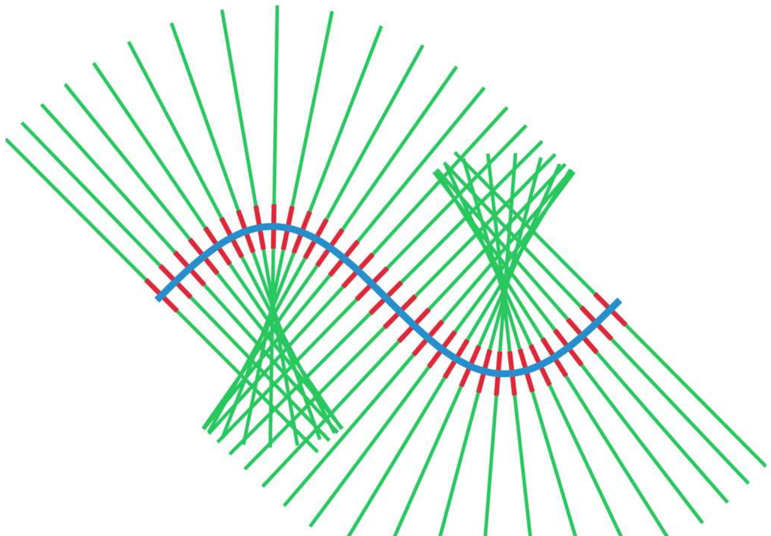
The geometry of tubes



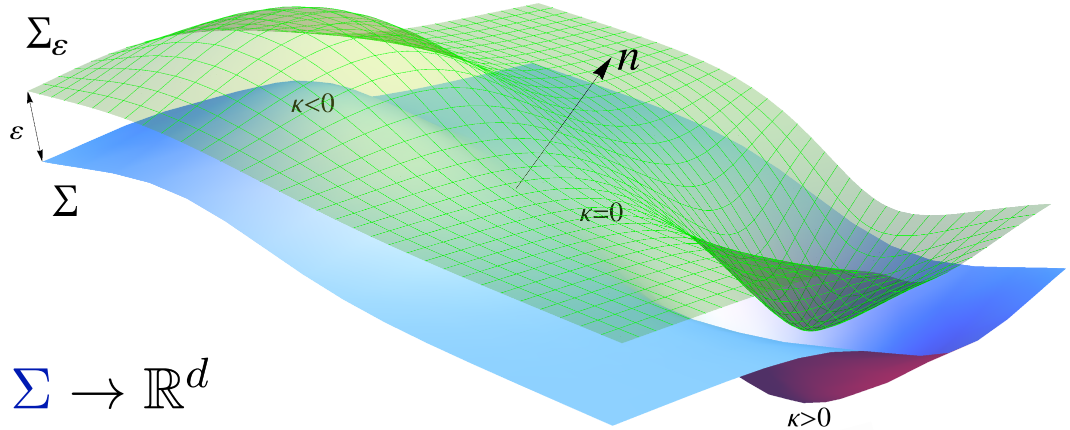
- ambient Riemannian manifold $\mathbb{R}^d$, $d \geq 2$
- oriented submanifold $\Sigma \subset \mathbb{R}^d$
  complete and non-compact, $\dim \Sigma < d$
- orthonormal vector fields $n_1, \ldots, n_{\operatorname{codim}\Sigma} : \Sigma \to \mathbb{R}^d$
- cross-section $\omega \subset \mathbb{R}^{\operatorname{codim}\Sigma}$
  bounded domain
- tube $\Omega := \left\{ x + \sum_{j=1}^{\operatorname{codim}\Sigma} t_j\, n_j(x) \;:\; (x,t) \in \Sigma \times \omega \right\}$



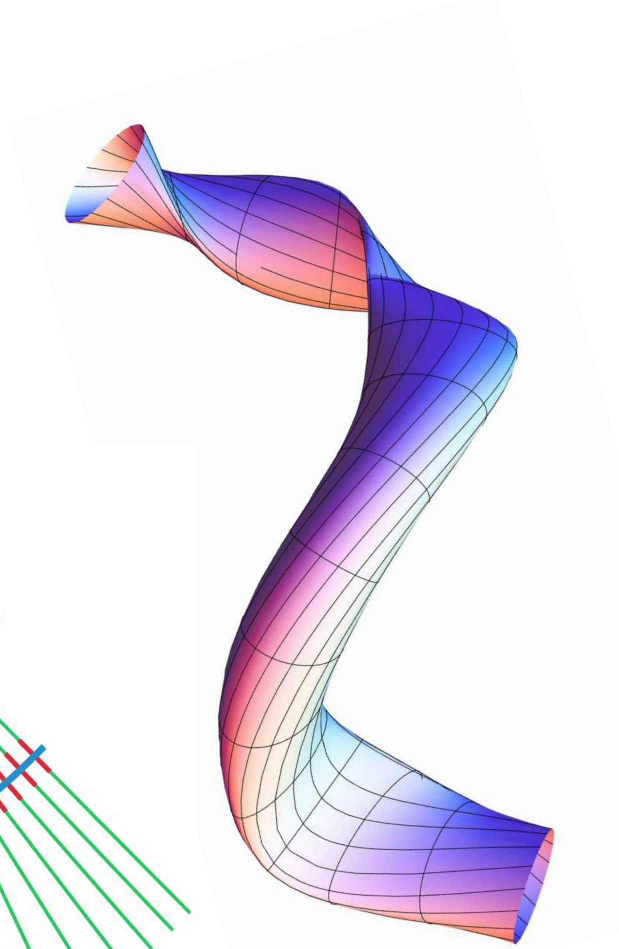
Assumption. No self-intersections.



The geometry of tubes



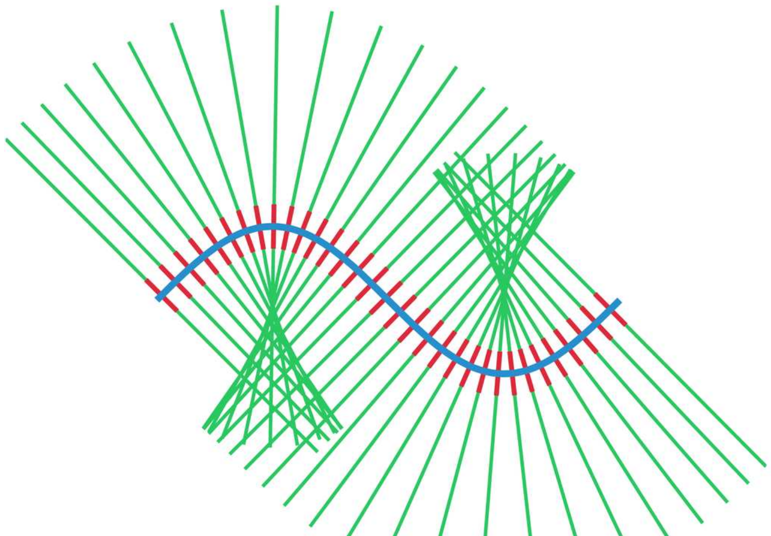
- ambient Riemannian manifold $\mathbb{R}^d$, $d \geq 2$
- oriented submanifold $\Sigma \subset \mathbb{R}^d$
  complete and non-compact, $\dim \Sigma < d$
- orthonormal vector fields $n_1, \dots, n_{\operatorname{codim}\Sigma} : \Sigma \to \mathbb{R}^d$
- cross-section $\omega \subset \mathbb{R}^{\operatorname{codim}\Sigma}$
  bounded domain
- tube $\Omega := \left\{ x + \sum_{j=1}^{\operatorname{codim}\Sigma} t_j\, n_j(x) \ : \ (x,t) \in \Sigma \times \omega \right\}$



Assumption. No self-intersections.

$\longrightarrow$ unbounded geometry

$\longrightarrow$ uniform cross-section



Premillennial history

Many results for
- complete manifolds (both compact and non-compact),
- compact submanifolds (with boundary),

but no systematic spectral-theoretic study of non-compact non-complete manifolds.

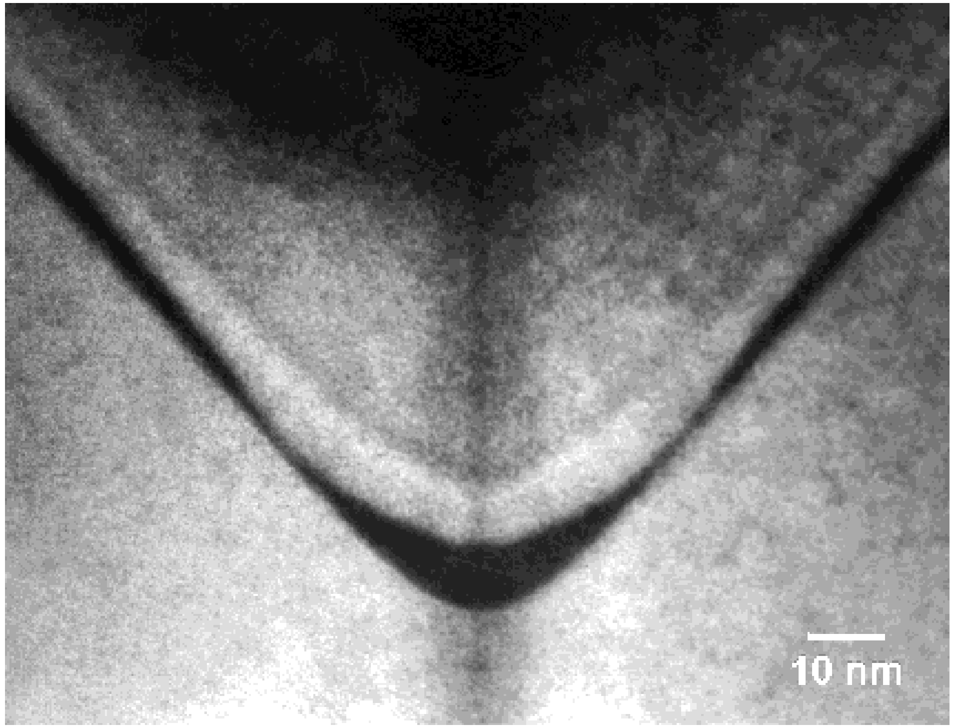
Premillennial history

Many results for
- complete manifolds (both compact and non-compact),
- compact submanifolds (with boundary),

but no systematic spectral-theoretic study of non-compact non-complete manifolds.

New motivations from physics:

nanostructures



GaAs/AlGaAs crescent shaped quantum wire

Premillennial history

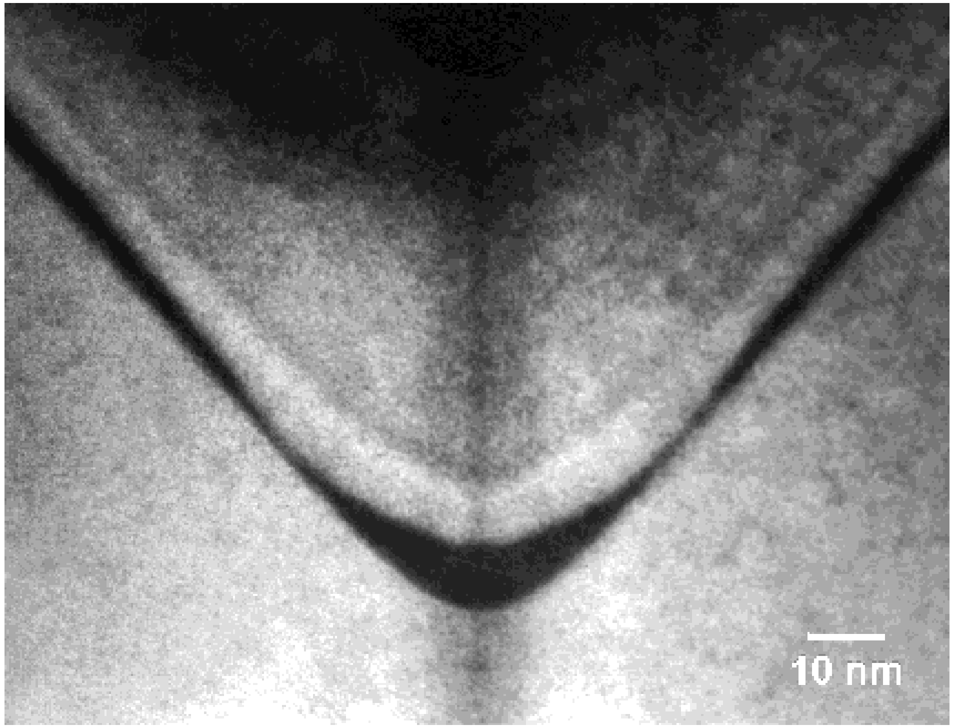
Many results for
- ◦ complete manifolds (both compact and non-compact),
- ◦ compact submanifolds (with boundary),

but no systematic spectral-theoretic study of non-compact non-complete manifolds.

New motivations from physics:

nanostructures

[Exner, Šeba 1989] $\dim \Sigma = 1$, $\operatorname{codim} \Sigma = 1$
$\rightarrow$ existence of quantum bound states (σ_{disc})



GaAs/AlGaAs crescent shaped quantum wire

Premillennial history

Many results for
- ∘ complete manifolds (both compact and non-compact),
- ∘ compact submanifolds (with boundary),

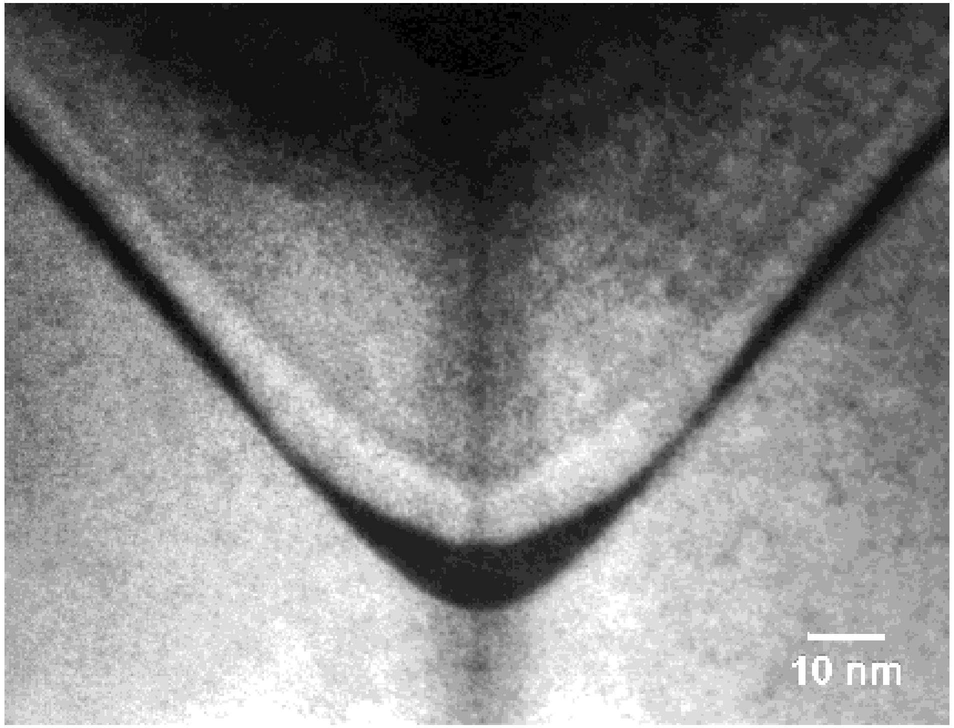
but no systematic spectral-theoretic study of non-compact non-complete manifolds.

New motivations from physics:

nanostructures

[Exner, Šeba 1989] $\dim\Sigma = 1$, $\operatorname{codim}\Sigma = 1$
$\rightarrow$ existence of quantum bound states (σ_{disc})

[Goldstone, Jaffe 1992] $\dim\Sigma = 1$, "$\operatorname{codim}\Sigma = 2$"
$\rightarrow$ variational proof



GaAs/AlGaAs crescent shaped quantum wire

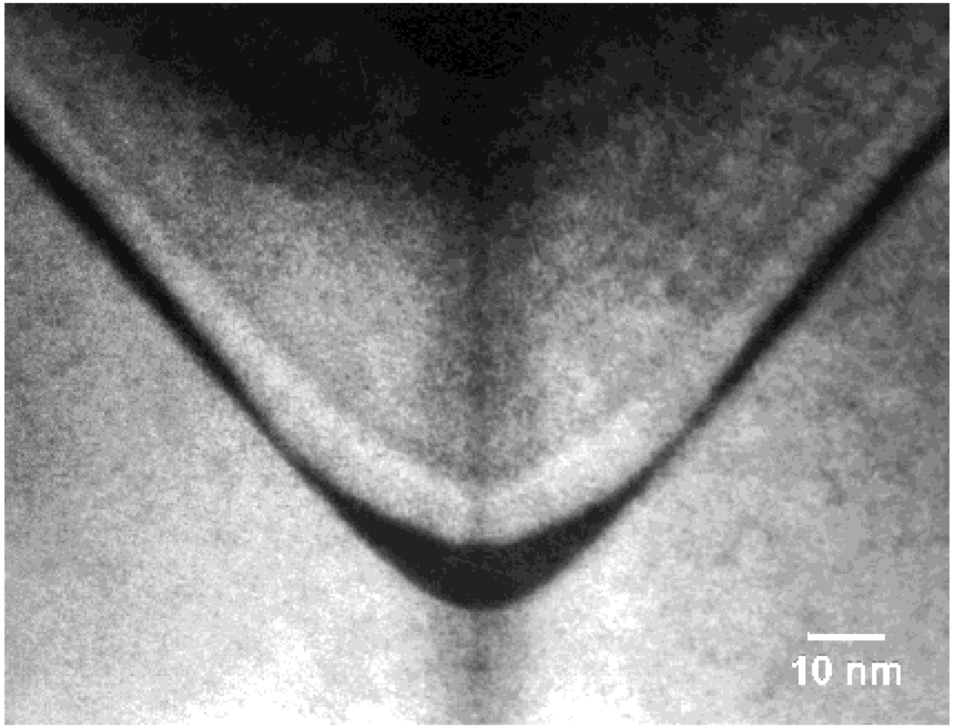
Premillennial history

Many results for
- ∘ complete manifolds (both compact and non-compact),
- ∘ compact submanifolds (with boundary),

but no systematic spectral-theoretic study of non-compact non-complete manifolds.

New motivations from physics:

nanostructures



GaAs/AlGaAs crescent shaped quantum wire

[Exner, Šeba 1989] $\dim\Sigma = 1,\ \mathrm{codim}\,\Sigma = 1$
$\to$ existence of quantum bound states (σ_{disc})

[Goldstone, Jaffe 1992] $\dim\Sigma = 1,$ "$\mathrm{codim}\,\Sigma = 2$"
$\to$ variational proof

[Duclos, Exner 1995] *idem*

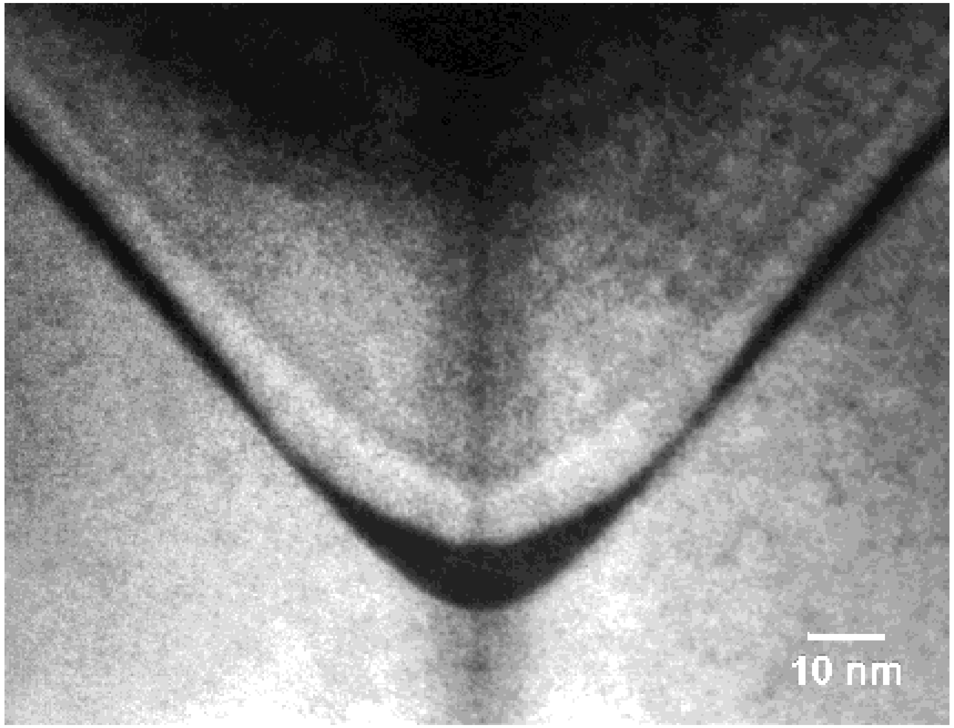
Premillennial history

Many results for
- ◦ complete manifolds (both compact and non-compact),
- ◦ compact submanifolds (with boundary),

but no systematic spectral-theoretic study of non-compact non-complete manifolds.

New motivations from physics:

nanostructures



GaAs/AlGaAs crescent shaped quantum wire

[Exner, Šeba 1989] $\dim\Sigma = 1$, $\operatorname{codim}\Sigma = 1$
$\rightarrow$ existence of quantum bound states (σ_{disc})

[Goldstone, Jaffe 1992] $\dim\Sigma = 1$, "$\operatorname{codim}\Sigma = 2$"
$\rightarrow$ variational proof

[Duclos, Exner 1995] *idem*

[Duclos, Exner, D.K. 2001] $\dim\Sigma = 2$, $\operatorname{codim}\Sigma = 1$

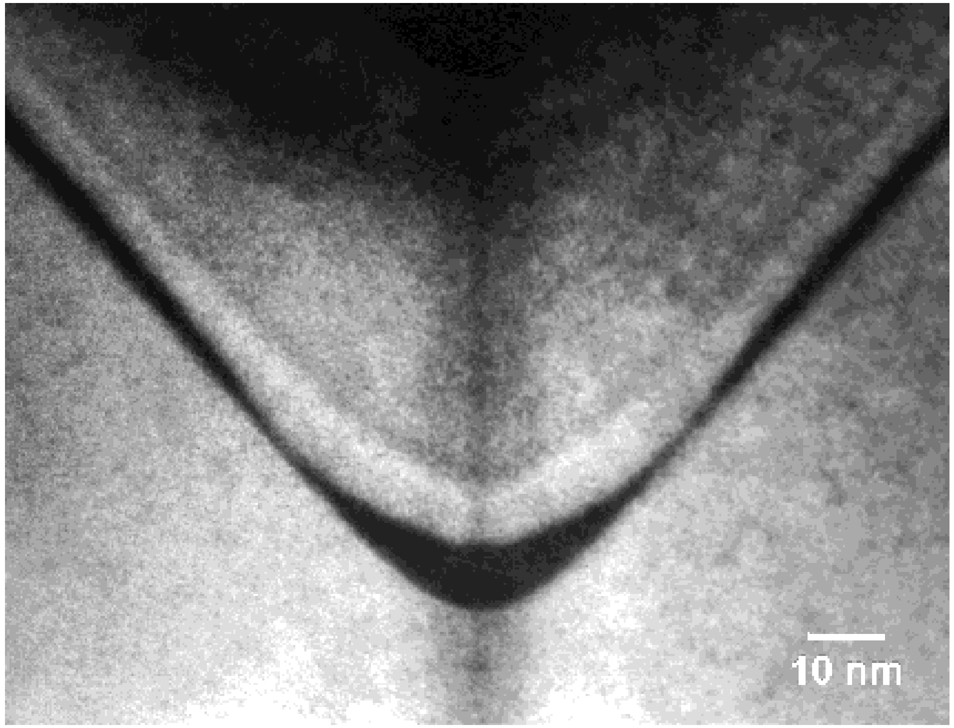
Premillennial history

Many results for
- ∘ complete manifolds (both compact and non-compact),
- ∘ compact submanifolds (with boundary),

but no systematic spectral-theoretic study of non-compact non-complete manifolds.

New motivations from physics:

nanostructures



GaAs/AlGaAs crescent shaped quantum wire

[Exner, Šeba 1989] $\dim \Sigma = 1, \operatorname{codim} \Sigma = 1$
→ existence of quantum bound states (σ_{disc})

[Goldstone, Jaffe 1992] $\dim \Sigma = 1$, "$\operatorname{codim} \Sigma = 2$"
→ variational proof

[Duclos, Exner 1995] *idem*

[Duclos, Exner, D.K. 2001] $\dim \Sigma = 2, \operatorname{codim} \Sigma = 1$

⋮

[Exner, Kovařík 2015]
→ book

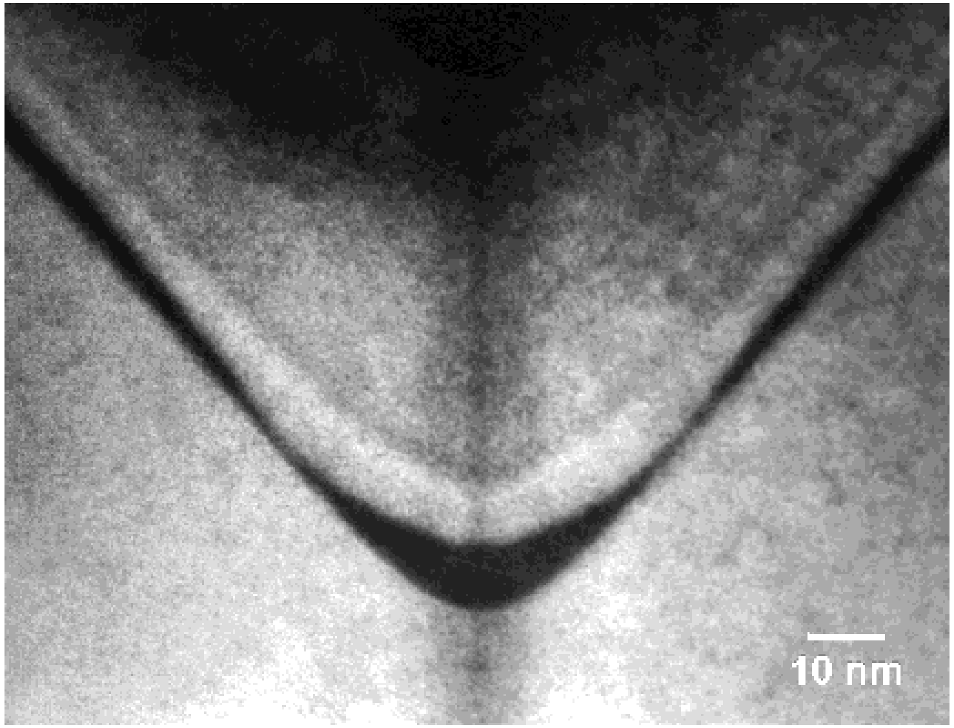
Premillennial history

Many results for
- ◦ complete manifolds (both compact and non-compact),
- ◦ compact submanifolds (with boundary),

but no systematic spectral-theoretic study of non-compact non-complete manifolds.

New motivations from physics:

nanostructures



GaAs/AlGaAs crescent shaped quantum wire

[Exner, Šeba 1989] $\dim \Sigma = 1$, $\operatorname{codim} \Sigma = 1$
$\rightarrow$ existence of quantum bound states (σ_{disc})

[Goldstone, Jaffe 1992] $\dim \Sigma = 1$, "$\operatorname{codim} \Sigma = 2$"
$\rightarrow$ variational proof

[Duclos, Exner 1995] *idem*

[Duclos, Exner, D.K. 2001] $\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

⋮

[Exner, Kovařík 2015] . . . [Exner, Barseghyan 2014] . . .
$\rightarrow$ book

The essential spectrum

Theorem ([D.K., Lu 2014 (*J. Math. Phys.*)]). If

- Σ is asymptotically flat (2^{nd} fundamental form goes to 0 at infinity),

and

- the transport of ω along Σ is asymptotically parallel (relevant only if $\operatorname{codim}\Sigma \geq 2$),

then

$$\boxed{\sigma_{\text{ess}}(-\Delta_D^\Omega) = [E_1, \infty)\,.}$$

$$E_1 := \min \sigma(-\Delta_D^\omega)$$

The essential spectrum

Theorem ([D.K., Lu 2014 (*J. Math. Phys.*)]). If

- Σ is asymptotically flat (2$^{\text{nd}}$ fundamental form goes to 0 at infinity),

and

- the transport of ω along Σ is asymptotically parallel (relevant only if $\operatorname{codim} \Sigma \geq 2$),

then

$$\boxed{\sigma_{\text{ess}}(-\Delta_D^\Omega) = [E_1, \infty)\,.}$$

$$E_1 := \min \sigma(-\Delta_D^\omega)$$

Proof.

$$\begin{array}{ccc} -\Delta_D^\Omega & \simeq & -G^{\frac{1}{2}} \partial_i G^{\frac{1}{2}} G^{ij} \partial_j \\ L^2(\Omega) & & L^2(\Sigma \times \omega, \mathrm{d}x\, \mathrm{d}t) \end{array}$$

The essential spectrum

Theorem ([D.K., Lu 2014 (*J. Math. Phys.*)]). If

- Σ is asymptotically flat (2$^{\mathrm{nd}}$ fundamental form goes to 0 at infinity),

and

- the transport of ω along Σ is asymptotically parallel (relevant only if $\operatorname{codim}\Sigma \geq 2$),

then

$$\boxed{\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = [E_1, \infty)\,.}$$

$$E_1 := \min \sigma(-\Delta_D^\omega)$$

Proof.

$$\begin{array}{ccc} -\Delta_D^\Omega & \simeq & -G^{\frac{1}{2}}\partial_i G^{\frac{1}{2}} G^{ij}\partial_j \\ L^2(\Omega) & & L^2(\Sigma \times \omega, \mathrm{d}x\,\mathrm{d}t) \end{array}$$

Weyl's criterion adapted to quadratic forms:

$$\lambda \in \sigma(H) \iff \exists\{\psi_n\} \subset \underbrace{\mathsf{Dom}\big(H^{\frac{1}{2}}\big)}_{\mathcal{H}_1} : \begin{cases} \|\psi_n\|_{\mathcal{H}} & = 1 \\ \|(H-\lambda)\psi_n\|_{\mathcal{H}_1^*} & \to 0 \end{cases}$$

$$\mathsf{Dom}(H) \subset \mathcal{H}_1 \subset \mathcal{H} = \mathcal{H}^* \subset \mathcal{H}_1^*$$

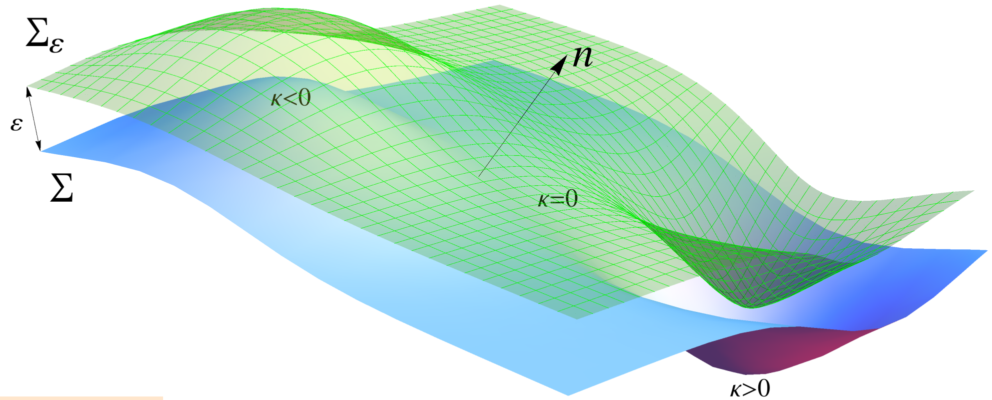
q.e.d.

Quantum layers: the discrete spectrum

$$\dim \Sigma = 2, \ \operatorname{codim} \Sigma = 1$$

Theorem ([Duclos, Exner, D.K. 2001 (*Comm. Math. Phys.*)], [Carron, Exner, D.K. 2004 (*J. Math. Phys.*)]).

Let $K \in L^2(\Sigma)$ and $\Sigma \neq \mathbb{R}^2$. If



- $\int_\Sigma K \leq 0$,

or

- ω is thin enough,

or

- $\int_\Sigma M^2 = \infty$ but $\nabla M \in L^2(\Sigma)$,

or

- $\Sigma \supset$ cylindrically symmetric end E with $\int_E K > 0$,

then

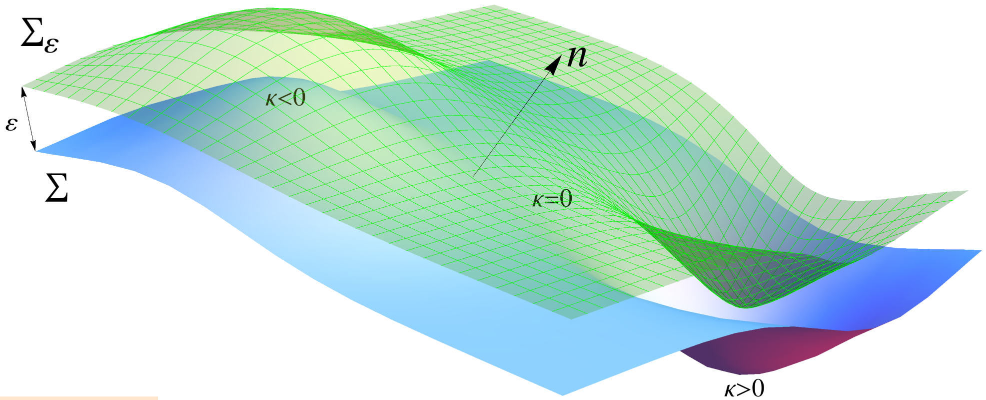
$$\boxed{\inf \sigma(-\Delta_D^\Omega) < E_1 \,.}$$

Quantum layers: the discrete spectrum

$\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

Theorem ([Duclos, Exner, D.K. 2001 (*Comm. Math. Phys.*)], [Carron, Exner, D.K. 2004 (*J. Math. Phys.*)]).

Let $K \in L^2(\Sigma)$ and $\Sigma \neq \mathbb{R}^2$. If



- $\int_\Sigma K \leq 0$,

or

- ω is thin enough,

or

- $\int_\Sigma M^2 = \infty$ but $\nabla M \in L^2(\Sigma)$,

or

- $\Sigma \supset$ cylindrically symmetric end E with $\int_E K > 0$,

then

$$\boxed{\inf \sigma(-\Delta_D^\Omega) < E_1 \,.}$$

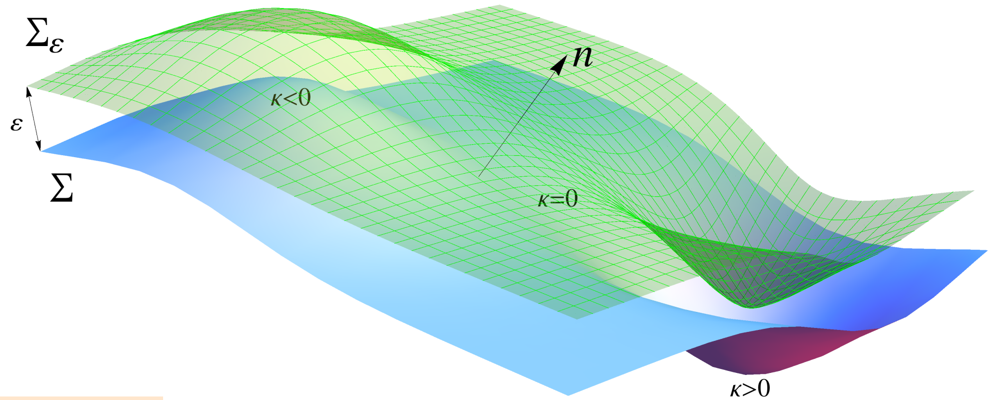
Corollary. If Σ is asymptotically flat and any of the conditions above hold, then

$$\boxed{\sigma_{\rm disc}(-\Delta_D^\Omega) \neq \varnothing \,.}$$

Quantum layers: the discrete spectrum

$\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

Theorem ([Duclos, Exner, D.K. 2001 (*Comm. Math. Phys.*)],
[Carron, Exner, D.K. 2004 (*J. Math. Phys.*)]**)**.



Let $K \in L^2(\Sigma)$ and $\Sigma \neq \mathbb{R}^2$. If

- $\int_\Sigma K \leq 0$,

or

- ω is thin enough,

or

- $\int_\Sigma M^2 = \infty$ but $\nabla M \in L^2(\Sigma)$,

or

- $\Sigma \supset$ cylindrically symmetric end E with $\int_E K > 0$,

then

$$\boxed{\inf \sigma(-\Delta_D^\Omega) < E_1 \,.}$$

Corollary. If Σ is asymptotically flat and any of the conditions above hold, then

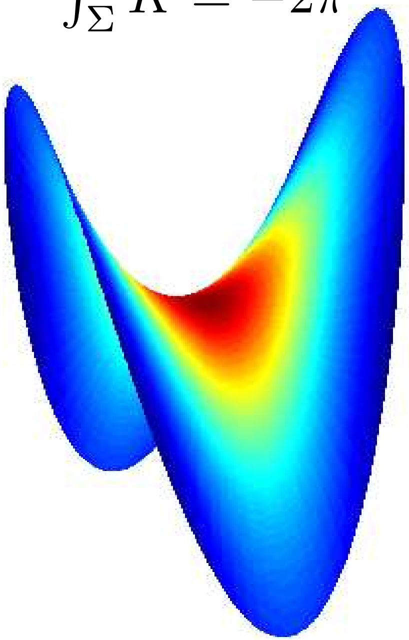
$$\boxed{\sigma_{\text{disc}}(-\Delta_D^\Omega) \neq \varnothing \,.}$$

Proof. Test function $1 \times \mathcal{J}_1$ where $\mathcal{J}_1$ is the first eigenfunction of $-\Delta_D^\omega$, *etc.* *q.e.d.*

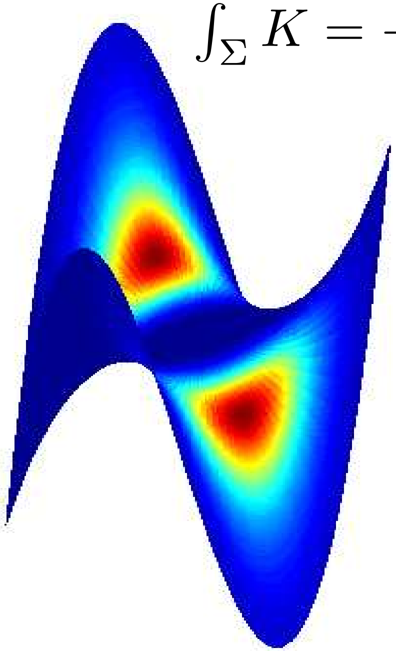
Quantum layers: examples

$\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

$\int_\Sigma K = -2\pi$



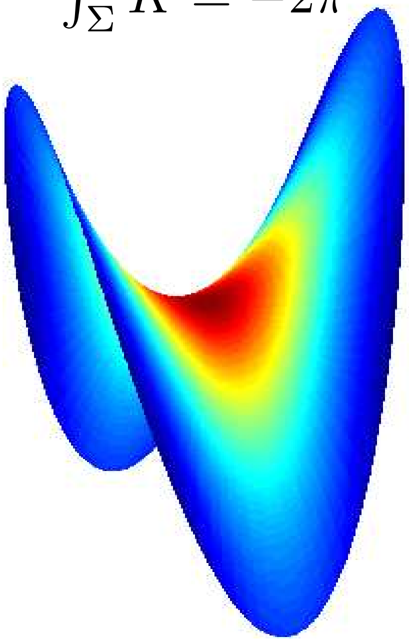
$\int_\Sigma K = -4\pi$



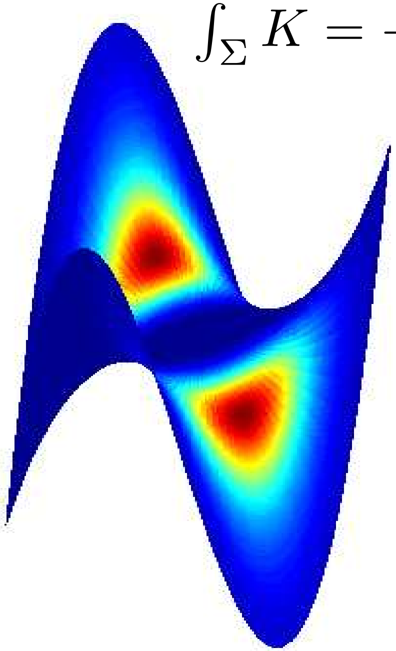
Quantum layers: examples

$\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

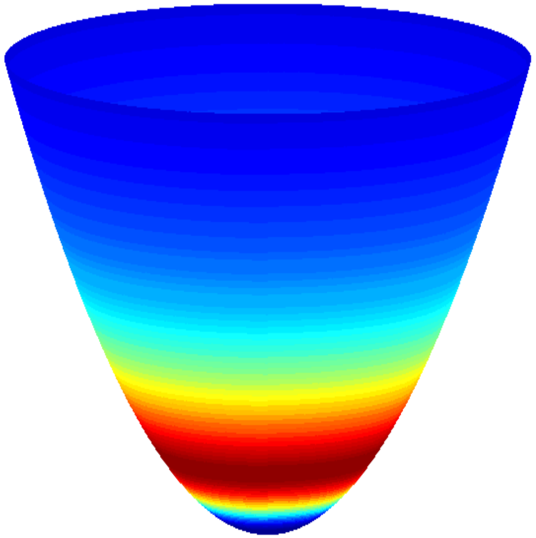
$\int_\Sigma K = -2\pi$



$\int_\Sigma K = -4\pi$



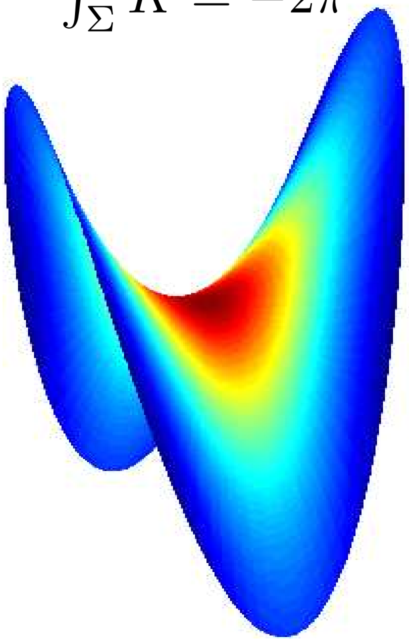
$\int_\Sigma K = 2\pi$



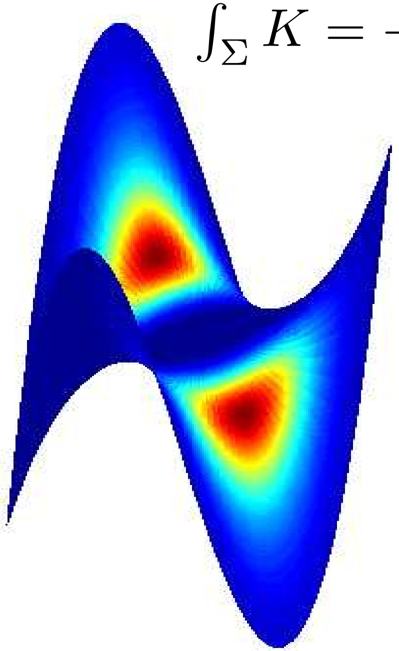
Quantum layers: examples

$\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

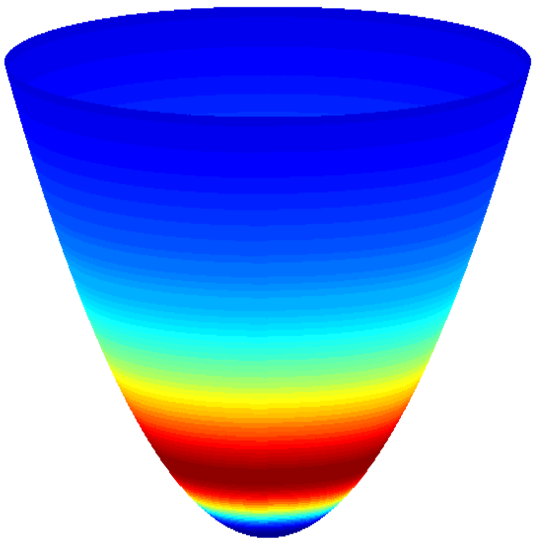
$\int_\Sigma K = -2\pi$

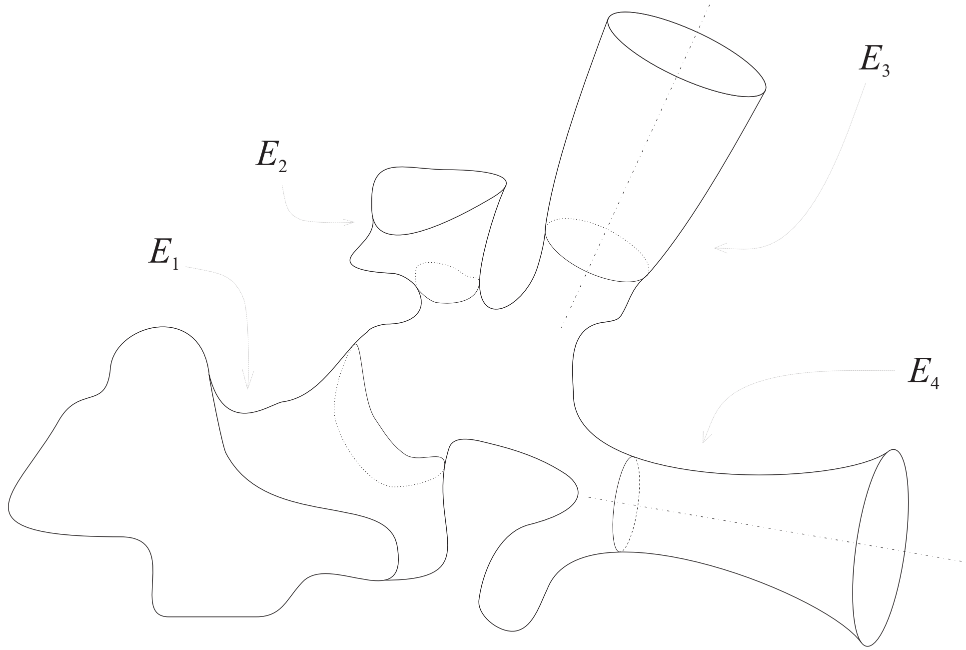


$\int_\Sigma K = -4\pi$



$\int_\Sigma K = 2\pi$





Quantum layers: further results

$\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

Quantum layers: further results

$$\dim \Sigma = 2, \ \mathrm{codim}\, \Sigma = 1$$

- extensions to higher dimensions and codimensions
  [Lin, Lu 2006], [Lin, Lu 2007]

Quantum layers: further results

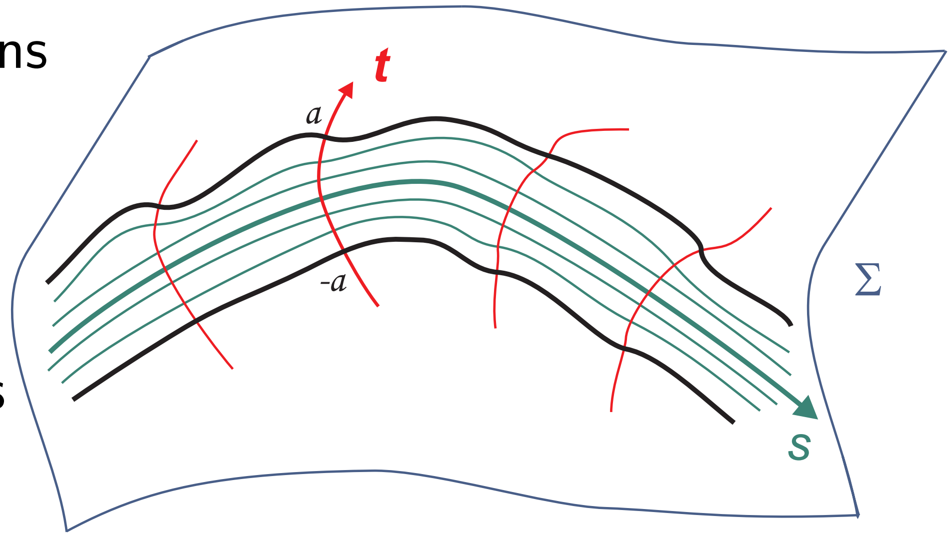
$\dim \Sigma = 2$, $\operatorname{codim} \Sigma = 1$

- extensions to higher dimensions and codimensions
  [Lin, Lu 2006], [Lin, Lu 2007]
- alternative sufficient conditions
  [Lin, Lu 2007], [Lu, Rowlett 2012]

Quantum layers: further results

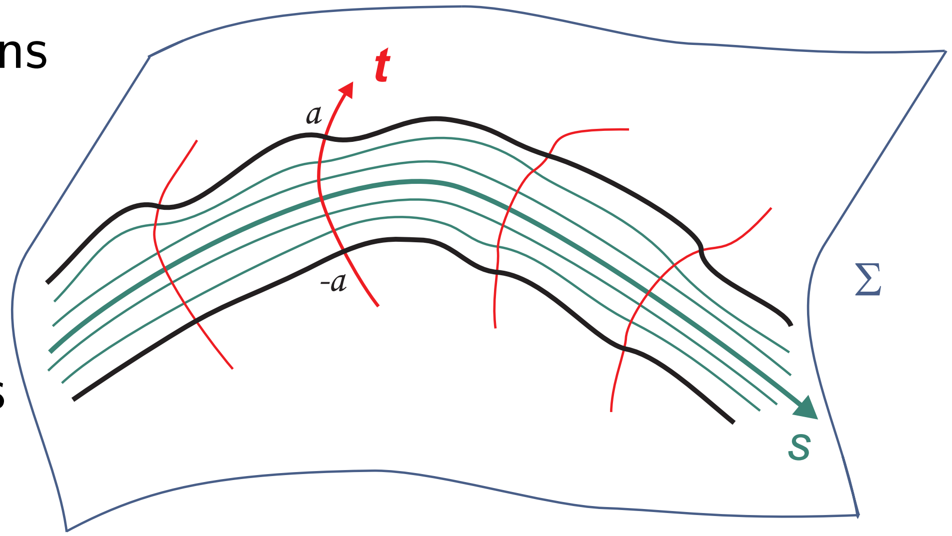
$$\dim \Sigma = 2, \ \operatorname{codim} \Sigma = 1$$

- extensions to higher dimensions and codimensions
  [Lin, Lu 2006], [Lin, Lu 2007]
- alternative sufficient conditions
  [Lin, Lu 2007], [Lu, Rowlett 2012]
- extensions to tubes in curved ambient manifolds
  [D.K. 2003], [D.K. 2006], [Kolb, D.K. 2014],
  [Wachsmuth, Teufel 2013]



Quantum layers: further results

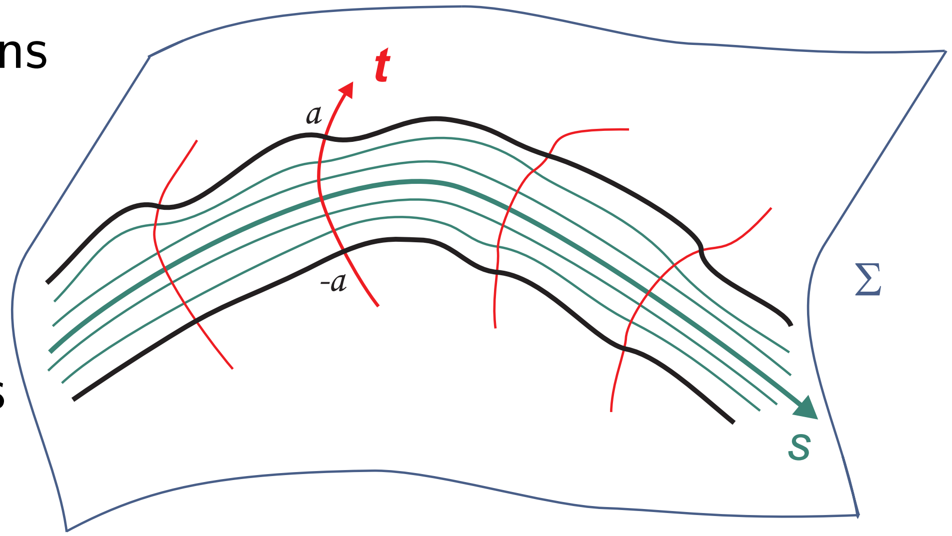
$\dim \Sigma = 2, \ \mathrm{codim}\, \Sigma = 1$



- extensions to higher dimensions and codimensions
  [Lin, Lu 2006], [Lin, Lu 2007]
- alternative sufficient conditions
  [Lin, Lu 2007], [Lu, Rowlett 2012]
- extensions to tubes in curved ambient manifolds
  [D.K. 2003], [D.K. 2006], [Kolb, D.K. 2014], [Wachsmuth, Teufel 2013]
- magnetic field $-\Delta \ \rightsquigarrow \ (-i\nabla - A)^2$
  [D.K., Raymond, Tušek 2015]

Quantum layers: further results

$$\dim \Sigma = 2, \ \mathrm{codim}\, \Sigma = 1$$



- extensions to higher dimensions and codimensions
  [Lin, Lu 2006], [Lin, Lu 2007]
- alternative sufficient conditions
  [Lin, Lu 2007], [Lu, Rowlett 2012]
- extensions to tubes in curved ambient manifolds
  [D.K. 2003], [D.K. 2006], [Kolb, D.K. 2014], [Wachsmuth, Teufel 2013]
- magnetic field $-\Delta \ \rightsquigarrow \ (-i\nabla - A)^2$
  [D.K., Raymond, Tušek 2015]

Theorem ([D.K., Raymond, Tušek 2015 (*J. Geom. Anal.*)]).
Replace $\omega \mapsto \varepsilon\omega$ with $\varepsilon > 0$. Then $\left(\dim \Sigma = 2, \ \mathrm{codim}\, \Sigma = 1, \ A = 0\right)$

$$\boxed{-\Delta_D^{\Omega_\varepsilon} - \frac{E_1}{\varepsilon^2} \quad \xrightarrow[\varepsilon \to 0]{\text{n.r.s.}} \quad -\Delta^\Sigma + K - M^2}$$

NB $K - M^2 = -\frac{1}{4}(k_1 - k_2)^2 \leq 0$

Quantum tubes: the geometry

$$\dim \Sigma = 1, \; \operatorname{codim} \Sigma = 2$$

$\Sigma := \{\Gamma(s) : s \in \mathbb{R}\}$, $\Gamma : \mathbb{R} \to \mathbb{R}^3$ unit-speed ($|\dot\Gamma| = 1$) immersion, curvature $\kappa := |\ddot\Gamma|$

Quantum tubes: the geometry

$$\dim \Sigma = 1, \ \operatorname{codim} \Sigma = 2$$

$\Sigma := \{\Gamma(s) : s \in \mathbb{R}\}$, $\Gamma : \mathbb{R} \to \mathbb{R}^3$ unit-speed ($|\dot{\Gamma}| = 1$) immersion, curvature $\kappa := |\ddot{\Gamma}|$

Frenet frame

$$\begin{pmatrix} \dot{\Gamma} \\ N \\ B \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \dot{\Gamma} \\ N \\ B \end{pmatrix}$$

$$\kappa > 0$$

Quantum tubes: the geometry

$$\dim \Sigma = 1, \ \operatorname{codim} \Sigma = 2$$

$\Sigma := \{\Gamma(s) : s \in \mathbb{R}\}$, $\Gamma : \mathbb{R} \to \mathbb{R}^3$ unit-speed ($|\dot\Gamma| = 1$) immersion, curvature $\kappa := |\ddot\Gamma|$

Frenet frame *versus* relatively parallel frame

$$\begin{pmatrix} \dot\Gamma \\ N \\ B \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \dot\Gamma \\ N \\ B \end{pmatrix} \qquad \begin{pmatrix} \dot\Gamma \\ N_1 \\ N_2 \end{pmatrix}^{\cdot} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot\Gamma \\ N_1 \\ N_2 \end{pmatrix}$$

$$\kappa > 0 \qquad \kappa^2 = k_1^2 + k_2^2$$

[Bishop 1975]
[D.K., Šediváková 2012]

Quantum tubes: the geometry

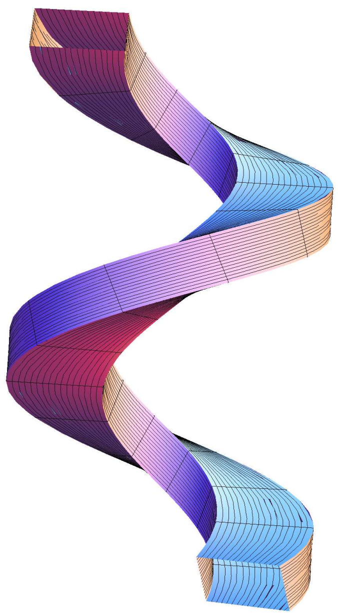
$$\dim\Sigma = 1, \ \operatorname{codim}\Sigma = 2$$

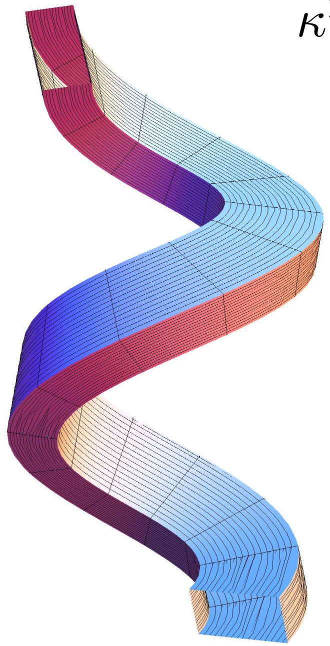
$\Sigma := \{\Gamma(s) : s \in \mathbb{R}\}$, $\Gamma : \mathbb{R} \to \mathbb{R}^3$ unit-speed ($|\dot\Gamma| = 1$) immersion, curvature $\kappa := |\ddot\Gamma|$

Frenet frame *versus* relatively parallel frame

$$\begin{pmatrix}\dot\Gamma \\ N \\ B\end{pmatrix}^{\boldsymbol{\cdot}} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix}\begin{pmatrix}\dot\Gamma \\ N \\ B\end{pmatrix} \qquad \begin{pmatrix}\dot\Gamma \\ N_1 \\ N_2\end{pmatrix}^{\boldsymbol{\cdot}} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix}\begin{pmatrix}\dot\Gamma \\ N_1 \\ N_2\end{pmatrix}$$

$\kappa > 0$ $\qquad$ $\kappa^2 = k_1^2 + k_2^2$





[Bishop 1975]
[D.K., Šediváková 2012]

Quantum tubes: the geometry

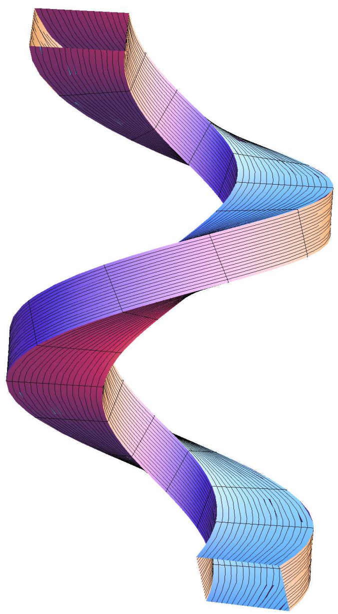
$$\dim \Sigma = 1, \ \operatorname{codim} \Sigma = 2$$

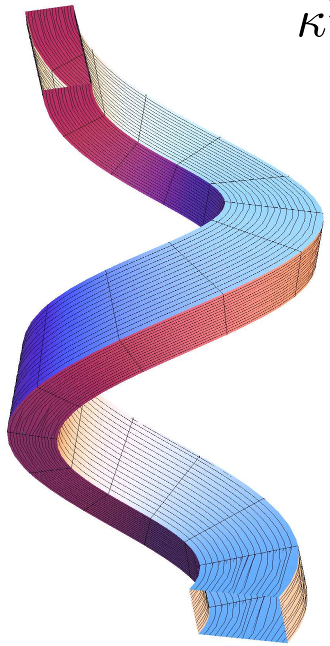
$\Sigma := \{\Gamma(s) : s \in \mathbb{R}\}$, $\Gamma : \mathbb{R} \to \mathbb{R}^3$ unit-speed ($|\dot{\Gamma}| = 1$) immersion, curvature $\kappa := |\ddot{\Gamma}|$

Frenet frame *versus* relatively parallel frame

$$\begin{pmatrix} \dot{\Gamma} \\ N \\ B \end{pmatrix}^{\boldsymbol{\cdot}} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} \dot{\Gamma} \\ N \\ B \end{pmatrix} \qquad \begin{pmatrix} \dot{\Gamma} \\ N_1 \\ N_2 \end{pmatrix}^{\boldsymbol{\cdot}} = \begin{pmatrix} 0 & k_1 & k_2 \\ -k_1 & 0 & 0 \\ -k_2 & 0 & 0 \end{pmatrix} \begin{pmatrix} \dot{\Gamma} \\ N_1 \\ N_2 \end{pmatrix}$$

$$\kappa > 0 \qquad \kappa^2 = k_1^2 + k_2^2$$



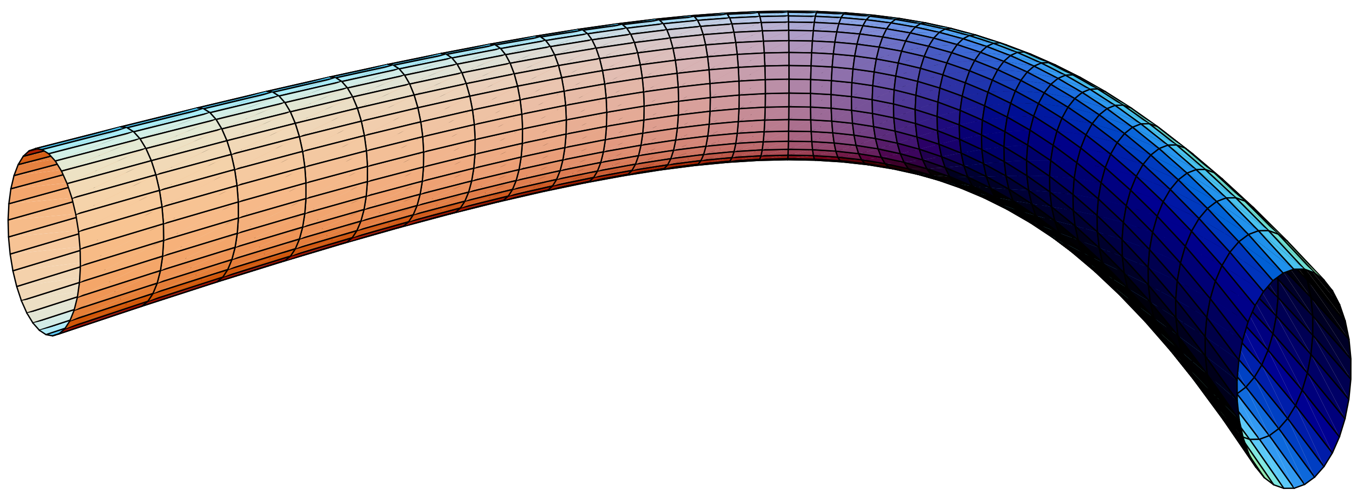


[Bishop 1975]
[D.K., Šediváková 2012]

$$\begin{pmatrix} n_1 \\ n_2 \end{pmatrix} := \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}, \qquad \theta : \mathbb{R} \to \mathbb{R} \ \text{(twisting angle)}$$

Quantum tubes: effect of bending

$\dim \Sigma = 1$, $\operatorname{codim} \Sigma \geq 1$

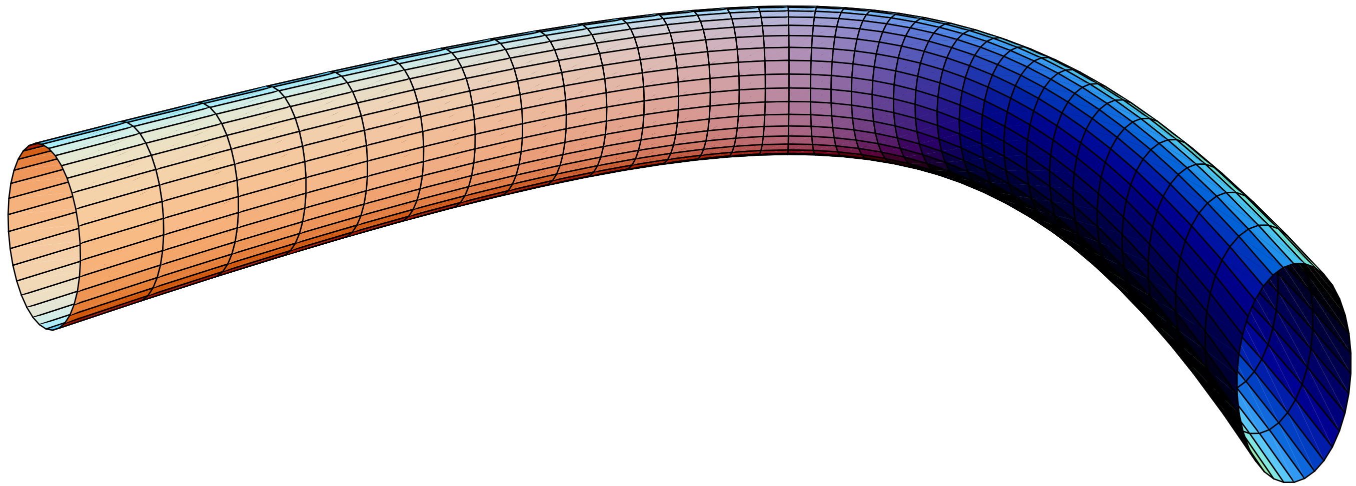


Quantum tubes: effect of bending

$\dim \Sigma = 1, \ \mathrm{codim}\, \Sigma \geq 1$

Theorem ([Chenaud, Duclos, Freitas, D.K. 2005 (*Differential Geom. Appl.*)]).

If $\kappa \neq 0$ and $\dot{\theta} = 0$, then $\boxed{\inf \sigma(-\Delta_D^\Omega) < E_1}$



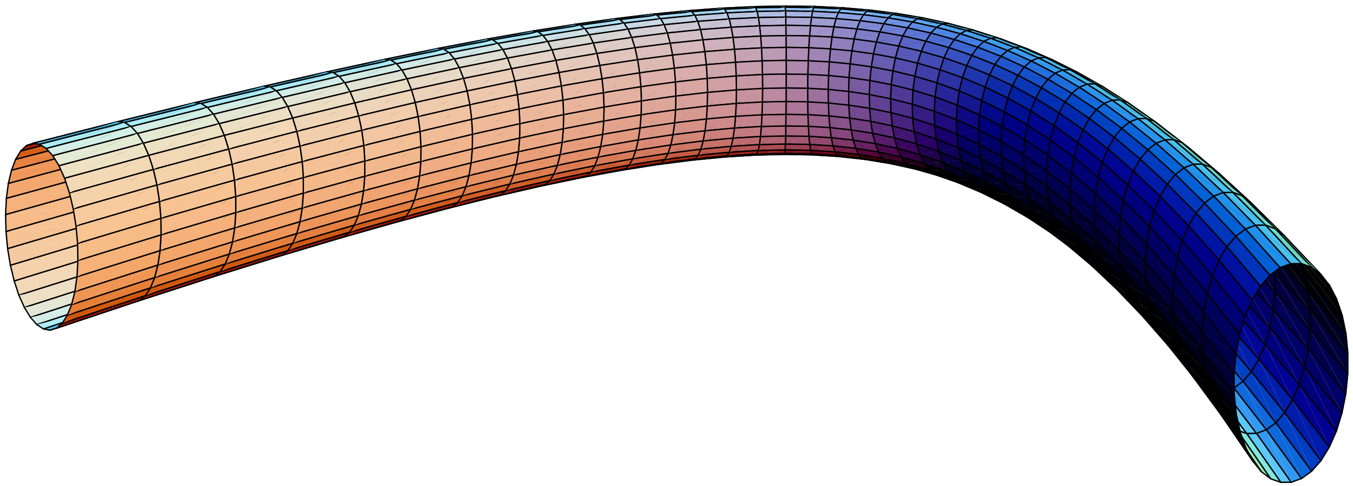
Quantum tubes: effect of bending

$\dim \Sigma = 1, \ \operatorname{codim} \Sigma \geq 1$

Theorem ([Chenaud, Duclos, Freitas, D.K. 2005 (*Differential Geom. Appl.*)]).

If $\kappa \neq 0$ and $\dot{\theta} = 0$, then $\boxed{\inf \sigma(-\Delta_D^\Omega) < E_1}$

If, in addition, $\lim_{|s| \to \infty} \kappa(s) = 0$, then $\boxed{\sigma_{\mathrm{disc}}(-\Delta_D^\Omega) \neq \varnothing}$



Quantum tubes: effect of bending

$$\dim \Sigma = 1, \ \operatorname{codim} \Sigma \geq 1$$

Theorem ([Chenaud, Duclos, Freitas, D.K. 2005 (*Differential Geom. Appl.*)]).

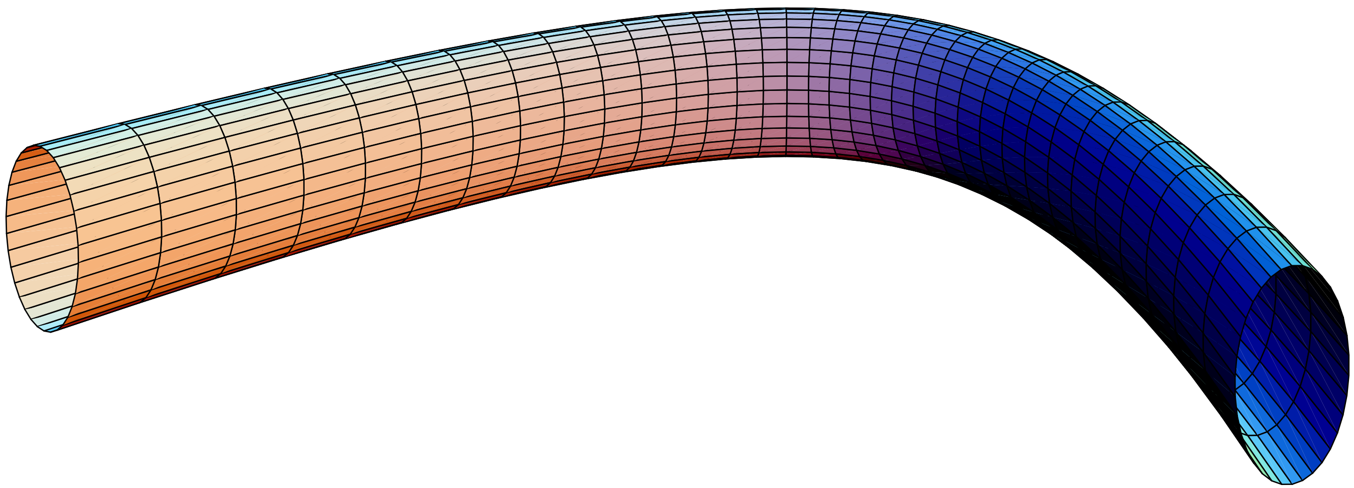
If $\kappa \neq 0$ and $\dot{\theta} = 0$, then

$$\boxed{\inf \sigma(-\Delta_D^\Omega) < E_1}$$

If, in addition, $\lim_{|s| \to \infty} \kappa(s) = 0$, then

$$\boxed{\sigma_{\mathrm{disc}}(-\Delta_D^\Omega) \neq \varnothing}$$

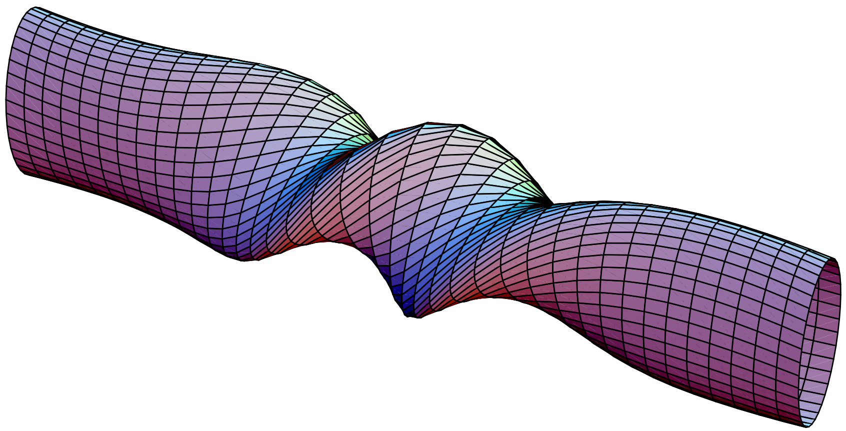
bending acts as an *attractive* interacion



Quantum tubes: effect of twisting

$$\dim \Sigma = 1, \ \operatorname{codim} \Sigma = 2$$

Proposition. If $\kappa = 0$ and $\lim_{|s|\to\infty} \dot\theta(s) = 0$, then $\boxed{\sigma(-\Delta_D^\Omega) = [E_1, \infty)}$



Quantum tubes: effect of twisting

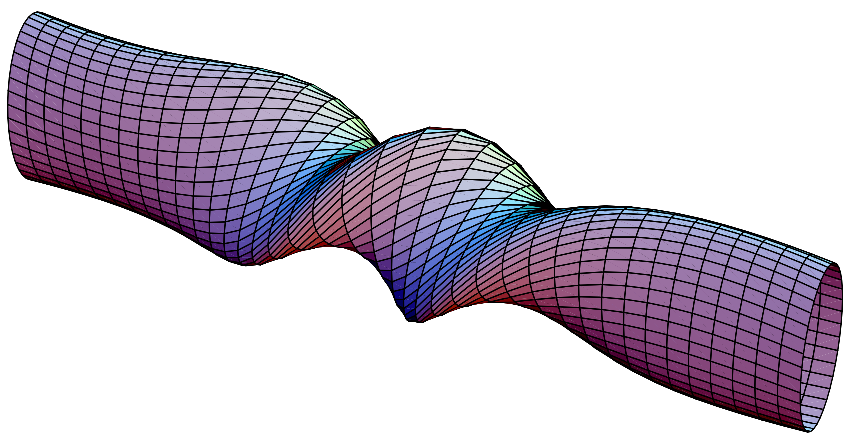
$\dim \Sigma = 1, \ \operatorname{codim} \Sigma = 2$

Proposition. If $\kappa = 0$ and $\lim\limits_{|s|\to\infty} \dot{\theta}(s) = 0$, then $\boxed{\sigma(-\Delta_D^\Omega) = [E_1, \infty)}$

Proof. $-\Delta_D^\Omega \ \simeq \ -\big(\partial_s - \dot{\theta}(s)\partial_\tau\big)^2 - \Delta_t \geq -\Delta_t \geq E_1,$

$(s,t) \in \mathbb{R} \times \omega$

$\partial_\tau := t_2 \partial_{t_1} - t_1 \partial_{t_2}$ *q.e.d.*



Quantum tubes: effect of twisting

$\dim \Sigma = 1, \ \operatorname{codim} \Sigma = 2$

Proposition. If $\kappa = 0$ and $\lim_{|s|\to\infty} \dot\theta(s) = 0$, then $\boxed{\sigma(-\Delta_D^\Omega) = [E_1, \infty)}$

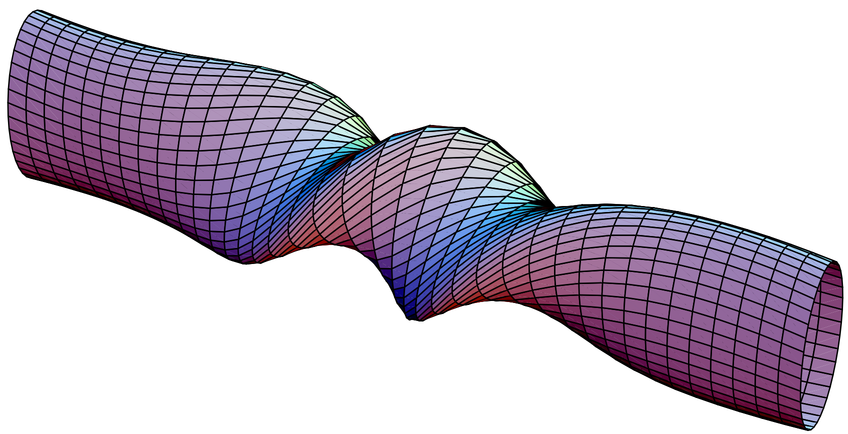
Proof. $-\Delta_D^\Omega \ \simeq \ -\big(\partial_s - \dot\theta(s)\partial_\tau\big)^2 - \Delta_t \geq -\Delta_t \geq E_1,$ $\qquad (s,t) \in \mathbb{R} \times \omega$

$\partial_\tau := t_2 \partial_{t_1} - t_1 \partial_{t_2}$ *q.e.d.*

Theorem ([Ekholm, Kovařík, D.K. 2008 (*Arch. Ration. Mech. Anal.*)]).

If $\kappa = 0$, $\dot\theta \neq 0$, $\dot\theta \in C_0(\mathbb{R})$ and ω is not circular, then there exists $c > 0$,

$$\forall \psi \in W_0^{1,2}(\Omega), \qquad \boxed{\int_\Omega |\nabla \psi(x)|^2 \, \mathrm{d}x - E_1 \int_\Omega |\psi(x)|^2 \, \mathrm{d}x \ \geq \ c \int_\Omega \frac{|\psi(x)|^2}{1 + |x|^2} \, \mathrm{d}x}$$



Quantum tubes: effect of twisting

$\dim \Sigma = 1$, $\operatorname{codim} \Sigma = 2$

Proposition. If $\kappa = 0$ and $\lim_{|s|\to\infty} \dot\theta(s) = 0$, then $\boxed{\sigma(-\Delta_D^\Omega) = [E_1, \infty)}$

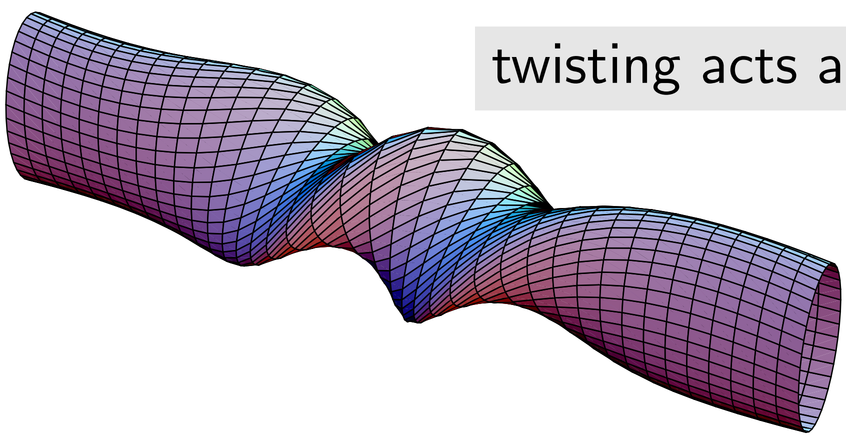
Proof. $-\Delta_D^\Omega \;\simeq\; -\big(\partial_s - \dot\theta(s)\partial_\tau\big)^2 - \Delta_t \geq -\Delta_t \geq E_1,$ $\quad (s,t) \in \mathbb{R} \times \omega$

$\partial_\tau := t_2 \partial_{t_1} - t_1 \partial_{t_2}$ *q.e.d.*

Theorem ([Ekholm, Kovařík, D.K. 2008 (*Arch. Ration. Mech. Anal.*)]).

If $\kappa = 0$, $\dot\theta \neq 0$, $\dot\theta \in C_0(\mathbb{R})$ and ω is not circular, then there exists $c > 0$,

$$\forall \psi \in W_0^{1,2}(\Omega), \qquad \boxed{\int_\Omega |\nabla \psi(x)|^2 \,\mathrm{d}x - E_1 \int_\Omega |\psi(x)|^2 \,\mathrm{d}x \;\geq\; c \int_\Omega \frac{|\psi(x)|^2}{1+|x|^2} \,\mathrm{d}x}$$

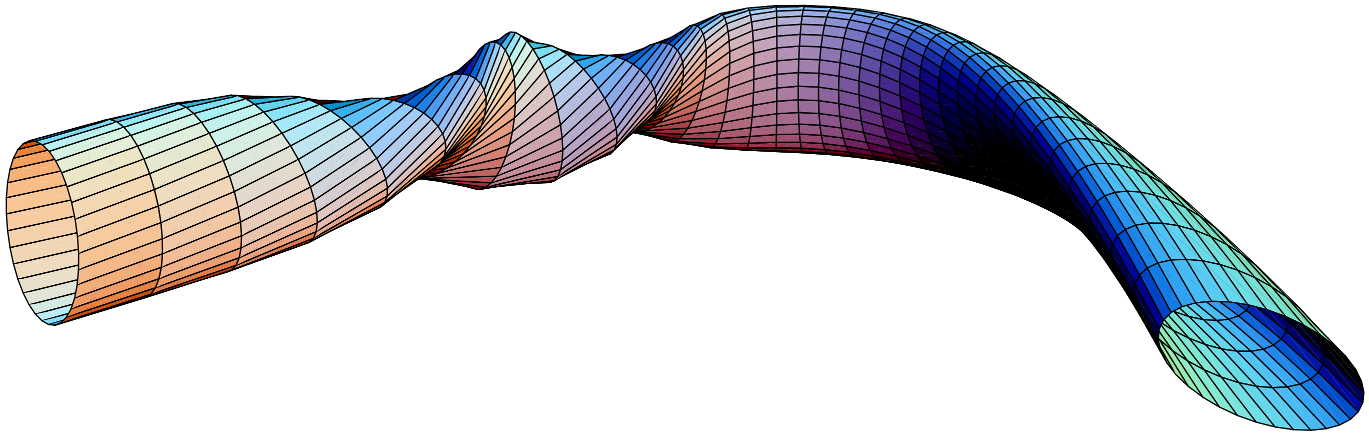


twisting acts as a *repulsive* interacion

Twisting *versus* bending

Corollary. Let $\dot\theta \neq 0$, $\dot\theta \in C_0(\mathbb{R})$ and ω is not circular. Then there exists $\epsilon > 0$,

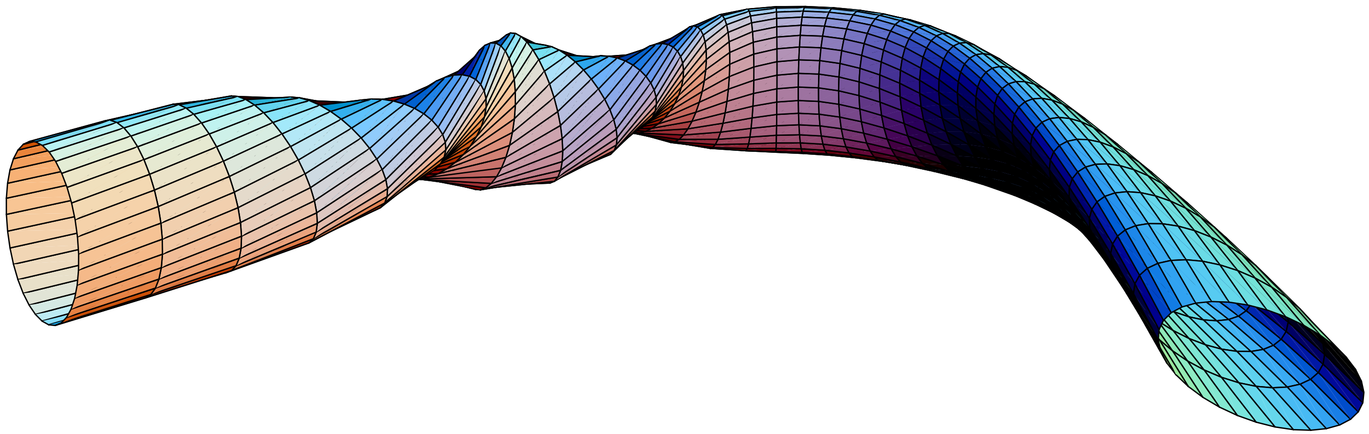
$$\boxed{\forall s \in \mathbb{R}, \quad |\kappa(s)| \leq \frac{\epsilon}{1+s^2} \qquad \Longrightarrow \qquad \sigma(-\Delta_D^\Omega) = [E_1, \infty)}$$



Twisting *versus* bending

Corollary. Let $\dot\theta \neq 0$, $\dot\theta \in C_0(\mathbb{R})$ and ω is not circular. Then there exists $\epsilon > 0$,

$$\forall s \in \mathbb{R}, \quad |\kappa(s)| \leq \frac{\epsilon}{1+s^2} \qquad \Longrightarrow \qquad \sigma(-\Delta_D^\Omega) = [E_1, \infty)$$



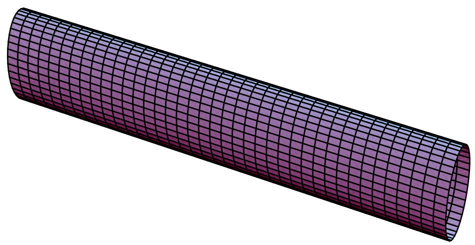
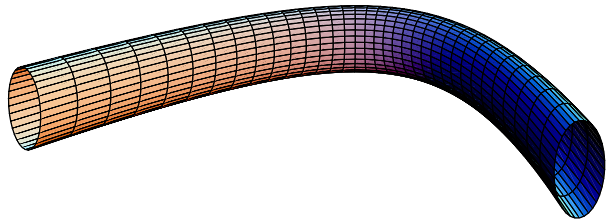
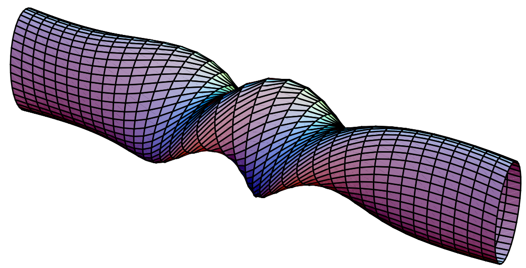
Theorem ([D.K., Šediváková 2012 (*Rev. Math. Phys.*)]).
Replace $\omega \mapsto \varepsilon\omega$ with $\varepsilon > 0$. Then

$$-\Delta_D^{\Omega_\varepsilon} - \frac{E_1}{\varepsilon^2} \quad \xrightarrow[\varepsilon\to 0]{\text{n.r.s.}} \quad -\Delta^\Sigma - \frac{\kappa^2}{4} + \|\partial_\tau \mathcal{J}_1\|^2 \, \dot\theta^2$$

Remark. Previous related results:
[Bouchitté, Mascarenhas, Trabucho 2007], [Wachsmuth, Teufel 2013], [de Oliveira 2010].

Application to the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_x u = 0\,, & (x,t) \in \Omega \times (0,\infty)\,, \\ u(x,0) = u_0(x)\,. & \end{cases}$$

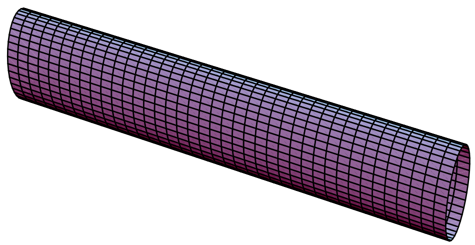
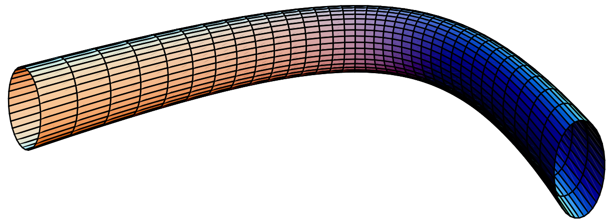
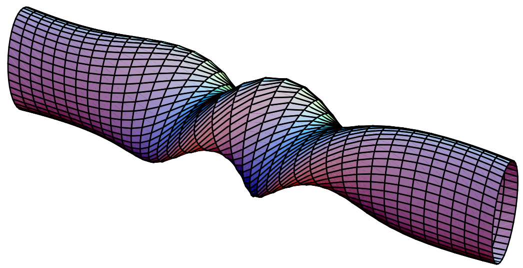
$u(t) \sim t^{-1/4}\, e^{-E_1 t}\, u_0$	$u(t) \sim e^{-\lambda_1 t}\, u_0$	$u(t) \sim t^{-3/4}\, e^{-E_1 t}\, u_0$
straight	bent $(\lambda_1 < E_1)$	twisted
		

[D.K., Zuazua 2010 (*J. Math. Pures Appl.*)] norm-wise

[Grillo, Kovařík, Pinchover 2014 (*Arch. Ration. Mech. Anal.*)] point-wise

Application to the heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - \Delta_x u = 0\,, & (x,t) \in \Omega \times (0,\infty)\,, \\ \quad u(x,0) = u_0(x)\,. \end{cases}$$

$u(t) \sim t^{-1/4}\, e^{-E_1 t}\, u_0$	$u(t) \sim e^{-\lambda_1 t}\, u_0$	$u(t) \sim t^{-3/4}\, e^{-E_1 t}\, u_0$
straight	bent $(\lambda_1 < E_1)$	twisted
		

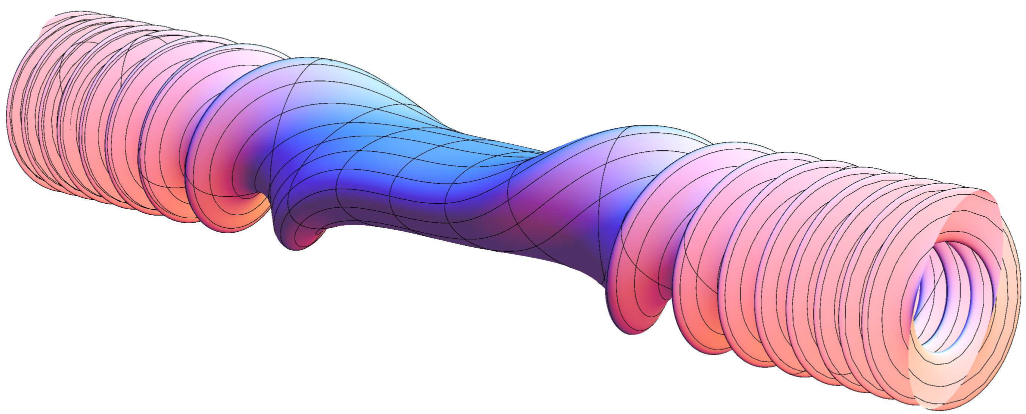
[D.K., Zuazua 2010 (*J. Math. Pures Appl.*)] norm-wise

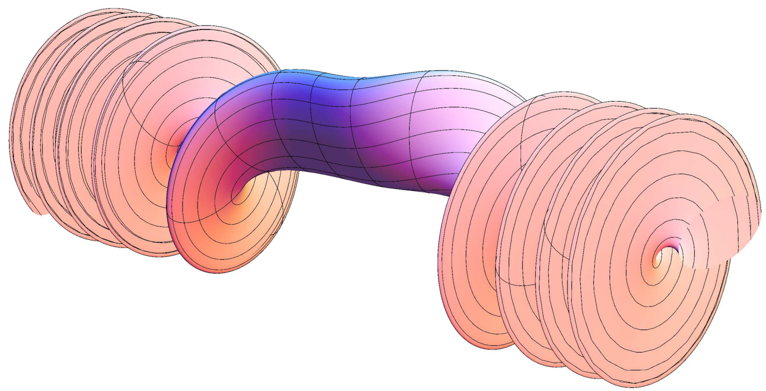
[Grillo, Kovařík, Pinchover 2014 (*Arch. Ration. Mech. Anal.*)] point-wise

twisting $\Longrightarrow$ faster cool down (death of a Brownian particle) in twisted tubes

Diverging twisting

Theorem ([D.K. 2012 (*Appl. Math. Lett.*)]). Let $\lim_{|s|\to\infty} |\dot{\theta}(s)| = \infty.$

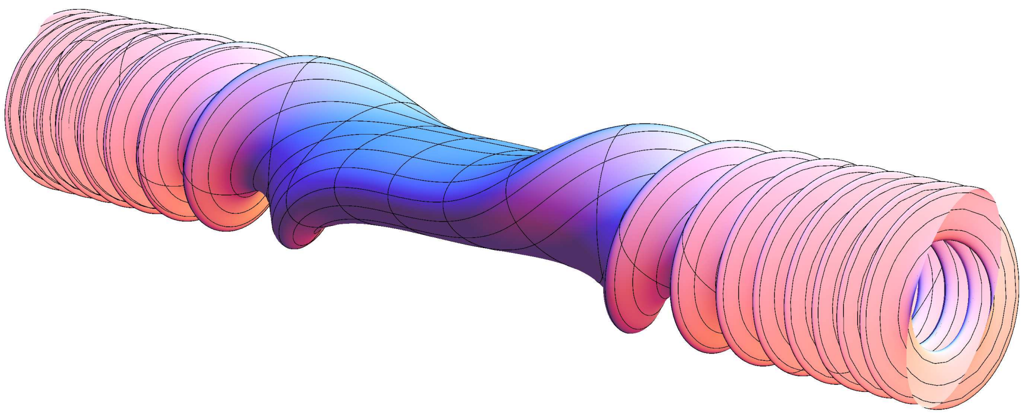


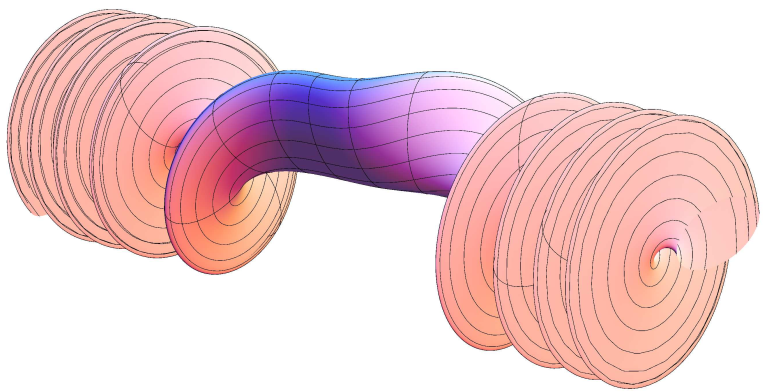


Diverging twisting

Theorem ([D.K. 2012 (*Appl. Math. Lett.*)]). Let $\boxed{\lim_{|s|\to\infty} |\dot\theta(s)| = \infty.}$

1. If $\boxed{0 \in \omega,}$ then $\boxed{\sigma(-\Delta_D^\Omega) \supset [\mu_1, \infty)}$ where $\mu_1 := \inf \sigma(-\Delta_D^{B_r})$, $r := \mathrm{dist}(0, \partial\omega)$.



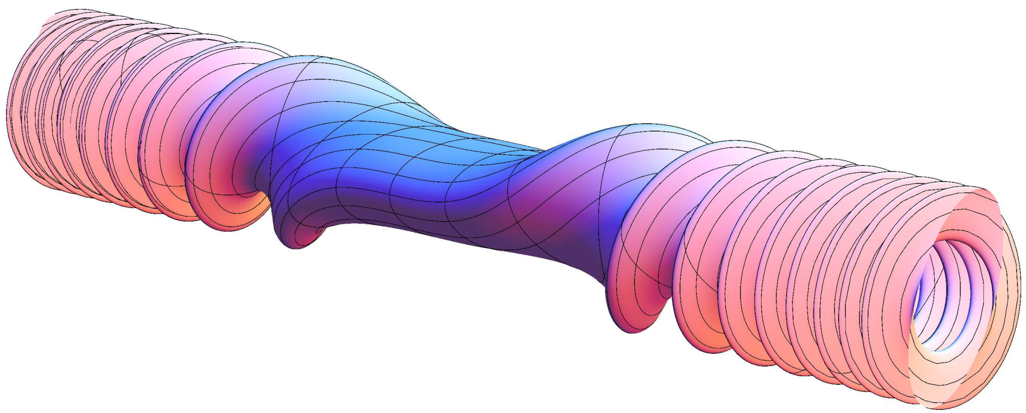


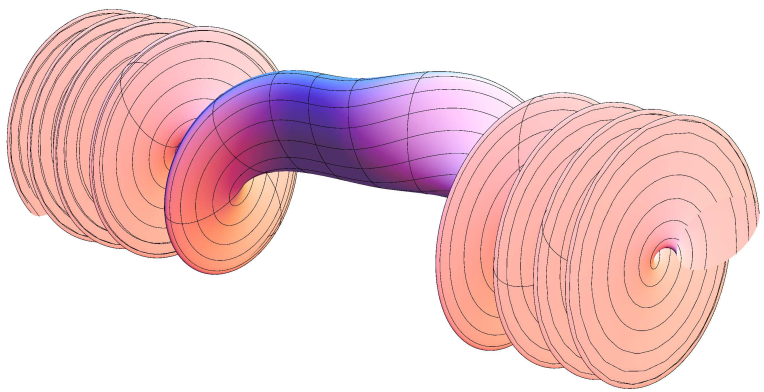
Diverging twisting

Theorem ([D.K. 2012 (*Appl. Math. Lett.*)]). Let $\lim_{|s|\to\infty} |\dot\theta(s)| = \infty.$

1. If $0 \in \omega,$ then $\boxed{\sigma(-\Delta_D^\Omega) \supset [\mu_1, \infty)}$ where $\mu_1 := \inf \sigma(-\Delta_D^{B_r})$, $r := \operatorname{dist}(0, \partial\omega)$.

2. If $\omega \subset \{t_1 > 0\},$ then $\boxed{\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = \varnothing}$ (¡ quasi-bounded realisation !)



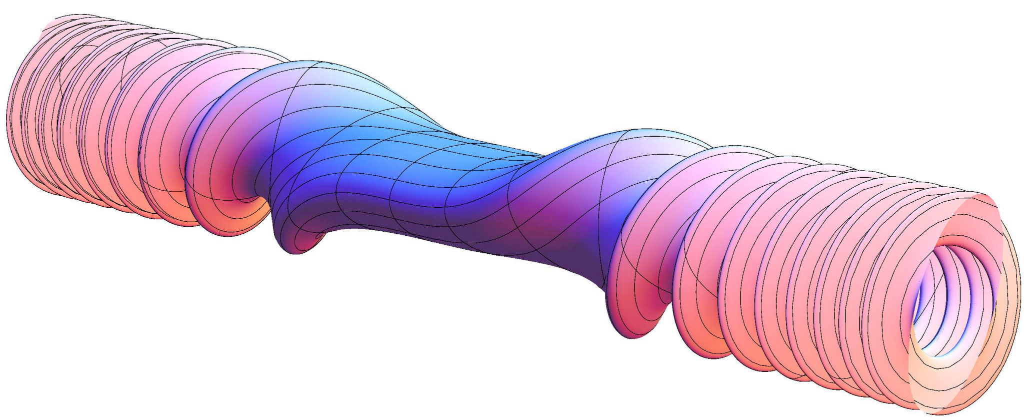


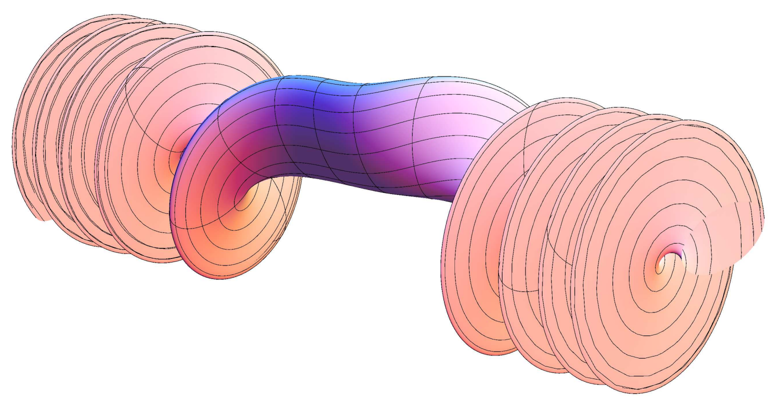
Diverging twisting

Theorem ([D.K. 2012 (*Appl. Math. Lett.*)]**).** Let $\boxed{\lim_{|s|\to\infty} |\dot\theta(s)| = \infty.}$

1. If $0 \in \omega$, then $\boxed{\sigma(-\Delta_D^\Omega) \supset [\mu_1, \infty)}$ where $\mu_1 := \inf \sigma(-\Delta_D^{B_r})$, $r := \operatorname{dist}(0, \partial\omega)$.

2. If $\omega \subset \{t_1 > 0\}$, then $\boxed{\sigma_{\mathrm{ess}}(-\Delta_D^\Omega) = \varnothing}$ (¡ quasi-bounded realisation !)





Open problem: ¿ Non-standard Weyl-type asymptotics in the case 2 ?