Self–organization and multiscale approach: a mathematical theory

Mirosław LACHOWICZ

∗

Institute of Applied Mathematics and Mechanics,
University of Warsaw, Poland

∗

KwaZulu–Natal University, Durban, South Africa

Ostrava: Prosinec 2016

Collaborators

- ★ Jacek Banasiak, University of Pretoria, South Africa;
- ★ Henryk Leszczyński, University of Gdańsk, Poland;
- ★ Martin Parisot, INRIA, France;

Examples of self–organization

- **Swarm of insects**, bacteria, people
- **School (or shoal) of fish**
- **Flocks of birds**
- **Herds of animals**
- **Pack of carnivores**

Alignment of animals or cells: adaptation of orientation to that of neighborhood

Turning: change of orientation

Swarming: an example of complex behavior in biological systems

Collective motion: interplay of individual behavior

Local interaction of a large number of individuals

Interactive cooperation behavior is advantageous in **avoiding predators** or in **capturing prey**

Swarming characterized as **coherent motion** of groups of individuals into the **same direction**

The entities comprising the group are organized, i.e. **oriented into similar directions**.

We are interested in the formation of swarms that do not possess **a leader** but in which organization arises as result of local **alignment interactions**

with J. Banasiak

★ J. Banasiak, **M.L.**, On a macroscopic limit of a kinetic model of alignment, **Math. Models Methods Appl. Sci.** 23, 2013

★ J. Banasiak, **M.L.**, Methods of small parameter in mathematical biology, **Birkhäuser**, Boston 2014

A model based on a microscopic **majority-choice** interaction.

The parameter of the model: a **sensitivity** $\gamma > 0$.

$f = f(t, j, x)$ — **the probability** that an entity is
at **time** t with **orientation** j and at **point** x

$t > 0$, $j \in \{-1, 1\}$, $x \in \mathbb{R}^1$

General expression for every **migration – interaction** case

$$f(t+\Delta t, j, x + j\Delta x) =$$
$$= f(t,-j,x)\,\mathcal{P}[f](t,-j,x) + f(t,j,x)\,\mathcal{P}'[f](t,j,x)$$

$$\mathcal{P}[f](t,j,x) = \textbf{Prob}\Big(\text{the change of orientation in } \Delta t \,\Big|\, (t,j,x)\Big)\,,$$

$$\mathcal{P}'[f](t,j,x) = \textbf{Prob}\Big(\text{no changes of orientation in } \Delta t \,\Big|\, (t,j,x)\Big)$$

Arlotti–Deutsch–Lachowicz Model, 2005

$$\mathcal{P}[f](k,x) = \frac{\chi\Big(\sum_{k,l=\pm 1} f(k,x+al) > 0\Big)\Big(\sum_{l=\pm 1} f(-j,x+al)\Big)^{\gamma}}{\Big(\sum_{l=\pm 1} f(-j,x+al)\Big)^{\gamma} + \Big(\sum_{l=\pm} f(j,x+al)\Big)^{\gamma}} \Delta t$$

$\sum_{l=\pm 1} f(t,j,x+al)$ is the neighborhood density in direction j,

χ (true) $= 1$, χ (false) $= 0$, γ is the sensitivity:

- γ – small: the prob–ity of a change of orientation weakly depends on the actual orientations
- γ – large: the prob–ity of a change of orientation strongly depends on the actual orientations.

Assuming that $a = \Delta x$ **in the limit** $\Delta t = \frac{\Delta x}{j} \to 0$

$$\partial_t f_j + j\partial_x f_j = \frac{\chi(f_1 + f_{-1} > 0)}{(f_{-j})^\gamma + (f_j)^\gamma}\Big(f_{-j}(f_j)^\gamma - f_j(f_{-j})^\gamma\Big)\,.$$

"Hydrodynamic" (i.e. macroscopic) limits

Two dimensionless forms

$$\left(\partial_t + j\partial_x\right) f_j = \frac{1}{\varepsilon}\left(Q[f]\right)_j, \qquad \textbf{(Hyp)}$$

$$\left(\varepsilon\partial_t + j\partial_x\right) f_j = \frac{1}{\varepsilon}\left(Q[f]\right)_j, \qquad \textbf{(Par)}$$

where the nonlinear operator Q is given by the RHS
$\varepsilon > 0$ is a **small parameter**

$$f\Big|_{t=0} = f_0$$

Singularly perturbed problems in the limit $\varepsilon \to 0$

The 0–th order term in **the Hilbert expansion** is the function $\mathfrak{M}$ (a "**Maxwellian**")

$$Q(\mathfrak{M}) = 0 \qquad \Longleftrightarrow$$

$$\mathfrak{M}(t, j, x) = \eta_j\, \varrho_0(t, x)$$

where **either**

$$\eta_1 = \eta_{-1} = \frac{1}{2}\,, \qquad \textbf{or}$$

$$\eta_k = 1\,, \qquad \eta_{-k} = 0\,, \qquad \text{for some} \quad k \in \{-1, 1\}$$

ϱ_0 is the macroscopic density.

Diffusive picture: the ("**Navier–Stokes**") order macroscopic approximation:

Hyp:

$$\partial_t \varrho = \frac{\varepsilon}{1-\gamma} \partial_x^2 \varrho$$

Par:

$$\partial_t \varrho = \frac{1}{1-\gamma} \partial_x^2 \varrho$$

These are the **diffusion equations** if $0 < \gamma < 1$

If $\gamma > 1$ they are the **backward diffusion equations**

Aligned picture: "**Navier–Stokes**"–order macroscopic approximation:

Hyp:

$$\partial_t \varrho + k\,\partial_x \varrho = 0\,.$$

Par:

$$\partial_t \varrho + k\,\partial_x \varrho = 0\,.$$

The equations can be explicitly solved. If f_0 is a smooth bounded initial datum than the unique solution, which for any $t > 0$ is bounded with respect to $x \in \mathbb{R}$, is given by

$$\varrho(t,x) = f_0(k, x - kt)\,, \qquad t \geq 0\,, \quad x \in \mathbb{R}$$

Aligned picture

We consider the ”**aligned picture**” in case $\gamma > 1$.
$k \in \{-1, 1\}$ is fixed.

Let $x \in \mathbb{T}$, where $\mathbb{T}$ is the 1–dimensional torus.
Let X_∞ be the Banach space of continuous functions on $\mathbb{T}$ with the norm

$$\|f\|_\infty = \sum_{j\in\{-1,1\}} \sup_{x\in\mathbb{T}} |f(j,x)| \, .$$

B.–L. Theorem

THEOREM: Let the initial data $f_0 \in X_\infty$ be nonnegative functions with continuous second derivatives and such that

$$f_0(-k, x) \geq \mu \,, \qquad \forall\, x \in \mathbb{T}$$

for some $\mu > 0$, is satisfied. For any $T > 0$ there exists $\varepsilon_0 > 0$ and $c_\gamma > 0$ such that **if**

$$\min_{x \in \mathbb{T}} f_0(k, x) \geq c_\gamma \, \max_{x \in \mathbb{T}} f_0(-k, x)$$

is satisfied for given $k \in \{-1, 1\}$, **then** for $\varepsilon \in \ [0, \varepsilon_0[\ $ the Cauchy Problem with the initial datum f_0, has a mild solution $f = f(t)$ in X_∞ on $[0, T]$.

Moreover,

$$\sup_{t\in[0,T]} \left\| f(t) - \varrho(t) - \mathbf{h}\left(\frac{t}{\varepsilon}\right) \right\|_\infty \leq c_T \varepsilon \,,$$

where

$$\begin{array}{rcl} \varrho(t,k,x) & = & f_0(k, x - kt)\,, \\ \varrho(t,-k,x) & = & 0\,, \end{array}$$

$$\mathbf{h}(0,x) = f_0(-k,x) \qquad \forall\, x \in \mathbb{T}\,.$$

with Martin Parisot

★ M. Parisot, **M.L.**, A kinetic model for the formation of swarms with nonlinear interactions, **Kinetic Related Models** 9, 2016

$f(t, x, v)$ — the probability density of individuals at time $t \geq 0$ and position $x \in \mathbb{R}^d$ with velocity $v \in \mathbb{V} \subset \mathbb{R}^d$.

$$\partial_t f(t,x,v) + v \cdot \partial_x f(t,x,v) = \frac{1}{\varepsilon} Q[f](t,x,v) =$$

$$= \frac{1}{\varepsilon} \int\limits_{\mathbb{V}} \Big(T[f(t,x,\,.\,)](w,v) f(t,x,w) -$$

$$- T[f(t,x,\,.\,)](v,w) f(t,x,v) \Big) \, \mathrm{d}w$$

ε — **Knudsen number**

The turning rate $T[f](v,w)$ measures the probability for an individual with the velocity v to switch the velocity to w.

$$T[f(t,x,\,.\,)](v,w) = \beta(v,w)\Big(\frac{f(t,x,w)}{\rho}\Big)^{\gamma(\rho)}$$

$\rho(t,x) = \int_{\mathbb{V}} f(t,x,v)\,\mathrm{d}v$ — the **total density of individuals**
$\beta \,:\, \mathbb{V}^2 \to \mathbb{R}_+$ — the **interaction rate** (a symmetric, positive and bounded function)
$\gamma \,:\, \mathbb{R}_+ \to \mathbb{R}_+$ — the **attractiveness**

P.–L. theorem

THEOREM. Let $0 \leq \gamma < 1$, $\beta \in L^\infty_+(\mathbb{V}^2)$ and $f_0(v) \in L^\infty_+(\mathbb{V})$ be such that there exist constants $0 < m \leq M < +\infty$ and a.a.

$$m \leq f_0(v) \leq M\,.$$

Then **there exists an unique global solution** $f \in C^1\Big(\mathbb{R}_+; L^\infty(\mathbb{V})\Big)$ with $f(0,v) = f_0(v)$. In addition, the solution satisfy the maximum principle, i.e. for any $t > 0$ and a.a. $v \in \mathbb{V}$

$$m \leq f(t,v) \leq M.$$

Results	
$0 < \gamma < 1$	$\gamma > 1$
solitarious	gregarious
entropy dissipation law	self–organization (swarms)
maximum principle	blow–up in a finite time
diffusion equation	**transport equation**

see

★ M. Parisot, **M.L.**, A kinetic model for the formation of swarms with nonlinear interactions, **Kinetic Related Models** 9, 2016

Hyperbolic scaling — diffusive picture

$$\partial_t \varrho - \varepsilon \partial_x \cdot \left(\frac{\kappa}{\varrho^\gamma} \partial_x \varrho \right) = 0$$

where

$$\kappa = \kappa(\gamma, \mathbb{V}) > 0\,.$$

Aligned picture

There is **NO** steady states that are stable in L^∞.

Space–homogeneous case

$$\partial_t f = (\beta * f) f^\gamma - (\beta * f^\gamma) f\,, \qquad f\Big|_{t=0} = f_0$$

$$\beta * f(v) = \int_{\mathbb{V}} \beta(v,w) f(w)\,\mathrm{d}w$$

THEOREM. Let $\gamma > 1$, $h \in L^\infty_+(\mathbb{V}^2)$ and $f_0 \in L^\infty_+(\mathbb{V})$. Then, there exists an unique **local solution** and it is non–negative.

with H. Leszyński and M. Parisot

★ **M.L.**, H. Leszczyński, M. Parisot, A simple kinetic equation of swarm formation: Blow–up and global existence, **Appl. Math. Letters** 57, 2016

Space homogeneous case: all functions are independent of x. Moreover: $\beta = 1$, and $\gamma > 1$.

$$\partial_t f = \overline{f}\, f^{\gamma} - \overline{f^{\gamma}}\, f\,, \qquad\qquad f\Big|_{t=0} = f_0$$

$$\overline{f}(t) = \int\limits_{\mathbb{V}} f(t,v)\,\mathrm{d}v\,.$$

We may rewrite as

$$\partial_t f(t,v) = f^\gamma(t,v) - f(t,v)\,\|f(t,\,.\,)\|_\gamma^\gamma\ ,$$

with

$$f(0,v) = f_0\,(v)\ , \qquad t \geq 0\,,$$

where $v \in \mathbb{V}$.

Let $z\,(t) = \int\limits_0^t \|f(s,\,.\,)\|_\gamma^\gamma \,\mathrm{d}s$. It satisfies

$$\mathrm{d}_t z(t) = \frac{1}{\exp\left\{\gamma z(t)\right\}} \int\limits_{\mathbb{V}} \frac{1}{\left(\frac{1}{f_0^{\gamma-1}} - (\gamma - 1)u(t)\right)^{\frac{\gamma}{\gamma-1}}} \mathrm{d}v$$

$$u\,(t) = \int\limits_0^t \frac{1}{\exp\left\{(\gamma - 1)\,z(s)\right\}}\,\mathrm{d}s$$

This equation determines **global existence** or **blow–up**.
Function $u = u(t)$ is **increasing** and **concave**.

We obtain

$$\mathrm{d}_t\, u = \frac{1}{\left(\displaystyle\int_{\mathbb{V}} \frac{f_0(v)}{\left(1-(\gamma-1) f_0^{\gamma-1}(v)\, u\right)^{\frac{1}{\gamma-1}}}\, \mathrm{d}v\right)^{\gamma-1}}$$

A **blow–up** occurs for $T > 0$ such that

$$(\gamma - 1)\,\|f_0\|_\infty^{\gamma-1}\; u\,(T) = 1$$

Theorem L.–L.–P.

First: $\mathbb{W} = \{v \in \mathbb{V} \ : \ f_0(v) = \|f_0\|_\infty\}\,, \ \ |\mathbb{W}| > 0$

THEOREM.
Let the probability density f_0 be in $L^\infty\,(\mathbb{V})$ and $|\mathbb{W}| > 0$. Then, for any $T > 0$, there exists a unique solution in $C^1\left(\left[0, T\right) ; L^\infty(\mathbb{V})\right)$.
Moreover, the solution $f = f(t)$, $t > 0$, is a probability density.

Case $|\mathbb{W}| = 0$ **is more complex**.
We denote the RHS of Equation for u by Φ

$$\Phi(u) = \frac{1}{\left(\int\limits_{\mathbb{V}} \frac{f_0(v)}{\left(1-(\gamma-1) f_0^{\gamma-1}(v)\, u\right)^{\frac{1}{\gamma-1}}} \, \mathrm{d}v \right)^{\gamma-1}}$$

Φ is defined on $[\,0, u_0\,)$, where $u_0 = \frac{1}{(\gamma-1)\|f_0\|_\infty^{\gamma-1}}$

$$\Phi(u_0) := \lim_{u \uparrow u_0} \Phi(u)$$

THEOREM. Let the probability density f_0 be in $L^\infty(\mathbb{V})$ and $|\mathbb{W}| = 0$.

1. If $\Phi(u_0) > 0$, then there is a blow–up in a finite time;

2. If $\Phi(u_0) = 0$, then

a. if $\int_0^{u_0} \frac{1}{\Phi(u)}\, \mathrm{d}u < \infty$

then there is a blow–up in a finite time;

b. if $\int_0^{u_0} \frac{1}{\Phi(u)}\, \mathrm{d}u = \infty$

then for each $T > 0$ there exists a unique solution on $[0, T]$.

Two conditions

$$\Phi(u_0) > 0 \qquad \text{and} \qquad \Phi(u_0) = 0$$

are equivalent to

$$\widetilde{f_0}\left(1 - \widetilde{f_0}^{\gamma-1}\right)^{-\frac{1}{\gamma-1}} \in L^1(\mathbb{V}) \qquad \text{and} \qquad \widetilde{f_0}\left(1 - \widetilde{f_0}^{\gamma-1}\right)^{-\frac{1}{\gamma-1}} \notin L^1(\mathbb{V})$$

respectively, where $\widetilde{f_0}\,(v) = \frac{f_0(v)}{\|f_0\|_\infty}$.

We rephrase Item 2 of Theorem as follows

Corollary. Let p.d. f_0 be in $L^\infty(\mathbb{V})$, $|\mathbb{W}| = 0$, and

$$\widetilde{f}_0 \left(1 - \widetilde{f}_0^{\gamma-1}\right)^{-\frac{1}{\gamma-1}} \notin L^1(\mathbb{V})$$

1. **If** $\displaystyle\int_0^1 \left(\int_{\mathbb{V}} \frac{\widetilde{f}_0(v)}{\left(1-\widetilde{f}_0^{\gamma-1}(v)y\right)^{\frac{1}{\gamma-1}}} \, \mathrm{d}v \right)^{\gamma-1} \mathrm{d}y < \infty$ **then** there is a **blow–up** in a finite time.
2. **If** $\displaystyle\int_0^1 \left(\int_{\mathbb{V}} \frac{\widetilde{f}_0(v)}{\left(1-\widetilde{f}_0^{\gamma-1}(v)y\right)^{\frac{1}{\gamma-1}}} \, \mathrm{d}v \right)^{\gamma-1} \mathrm{d}y = \infty$ **then**, **for any time $T > 0$**, there exists a unique solution on $[0, T]$.

Corollary. Let p.d. $f_0 \in L^\infty(\mathbb{V})$, $|\mathbb{W}| = 0$,

$$\widetilde{f}_0 \left(1 - \widetilde{f}_0^{\gamma-1}\right)^{-\frac{1}{\gamma-1}} \notin L^1(\mathbb{V})$$

$$g(v) := \widetilde{f}_0^{2-\gamma}(v)\left(1 - \widetilde{f}_0^{\gamma-1}(v)\right)^{\frac{\gamma-2}{\gamma-1}}$$

$$h(v) := -\log\left(1 - \widetilde{f}_0^{\gamma-1}(v)\right)$$

Then

1. If $1 < \gamma < 2$ and $g \in L^1(\mathbb{V})$ then the solution **blows–up** in a finite time;
2. If $1 < \gamma \leq 2$ and $h \notin L^1(\mathbb{V})$ then the solution **is global**;
3. If $\gamma \geq 2$ and $h \in L^1(\mathbb{V})$ then the solution **blows–up** in a finite time.

If $\gamma = 2$ then Items 2 and 3 of Corollary give a sufficient and necessary condition for blow–up in terms of h.
We have

$$\int_0^1 \left(\int_{\mathbb{V}} \frac{\widetilde{f}_0(v)}{1 - \widetilde{f}_0(v)\, y} \, \mathrm{d}v \right) \mathrm{d}y = - \int_{\mathbb{V}} \log\left(1 - \widetilde{f}_0(v)\right) \mathrm{d}v \,.$$

Conditions reduce to the question **whether** $\log(1 - \widetilde{f}_0)$ **is in** $L^1(\mathbb{V})$

Let $\alpha > 0$ and $\widetilde{f}_0(v) = 1 - \exp\left(-\frac{1}{|v|^\alpha}\right)$ on $\mathbb{V} = (-1, 1) \setminus \{0\}$.

Then $\quad h \in L^1(\mathbb{V}) \quad \Leftrightarrow \quad \alpha < 1$.

Examples of initial data such that the solution

- **blows up** in a finite time: $\quad \widetilde{f}_0(v) = 1 - \exp\left(-\frac{1}{|v|^{\frac{1}{2}}}\right)$,
- **is global**: $\quad \widetilde{f}_0(v) = 1 - \exp\left(-\frac{1}{|v|^2}\right)$.

Asymptotic behavior

★ **M.L.**, H. Leszczyński, M. Parisot, Blow–up and global existence for a kinetic equation of swarm formation, **Math. Models Methods Appl. Sci.**, to appear

THEOREM. Let f_0 be a p.d. on $\mathbb{V}$, $f_0 \in L^\infty(\mathbb{V})$ and $|\mathbb{W}| > 0$. Then the unique solution f, corresponding to the initial datum f_0, satisfies

$$\lim_{t\to\infty} f(t,v) = \frac{1}{|\mathbb{W}|}\chi(v \in \mathbb{W}),$$

where $\chi(\mathrm{true}) = 1$ and $\chi(\mathrm{false}) = 0$.

THEOREM. Let $f_0 \in L^\infty(\mathbb{V})$ be a p.d., $|\mathbb{W}| = 0$ and f be the unique solution in $C^1\big(0, t_*; L^\infty(\mathbb{V})\big)$, corresponding to the initial datum f_0, where t_* is either $t_* = \infty$ or $t_* = T$ (a **blow–up time**).

1. If $\Phi(u_0) = 0$ then

$$\lim_{t\to t_*} f(t,v) = 0 \qquad \text{for} \quad v \notin \mathbb{W}\,,$$

2. If $\Phi(u_0) > 0$ then

$$\lim_{t\to t_*} f(t,v) = \frac{\tilde{f}_0(v)}{\left(1 - \tilde{f}_0^{\gamma-1}(v)\right)^{\frac{1}{\gamma-1}} \int\limits_{\mathbb{V}} \frac{\tilde{f}_0(w)}{\left(1-\tilde{f}_0^{\gamma-1}(w)\right)^{\frac{1}{\gamma-1}}}\, \mathrm{d}w}\,,$$

where $\tilde{f}_0 = \frac{f_0}{\|f_0\|_\infty}$.

If $|\mathbb{W}| = 0$ and $\Phi\left(u_0\right) = 0$ then the limit of $f(t, v)$, as $t \to t_*$, is a kind of Dirac–delta concentrated on $\mathbb{W}$.

Challenges — open problems

h with a small support

$$\partial_t f = f^\gamma \int_{v-r}^{v+r} f \,\mathrm{d}w - f \int_{v-r}^{v+r} f^\gamma \,\mathrm{d}w$$

- J. Banasiak, **M.L.**, On a macroscopic limit of a kinetic model of alignment, **Math. Models Methods Appl. Sci.** 23, 2013
- J. Banasiak, **M.L.**, Methods of small parameter in mathematical biology, **Birkhäuser**, Boston 2014
- **M.L.**, H. Leszczyński, M. Parisot, A simple kinetic equation of swarm formation: Blow–up and global existence, **Appl. Math. Letters** 57, 2016
- M. Parisot, **M.L.**, A kinetic model for the formation of swarms with nonlinear interactions, **Kinetic Related Models** 9, 2016
- **M.L.**, H. Leszczyński, M. Parisot, Blow–up and global existence for a kinetic equation of swarm formation, **Math. Models Methods Appl. Sci.**, to appear

- J. Banasiak, **M.L.**, On a macroscopic limit of a kinetic model of alignment, **Math. Models Methods Appl. Sci.** 23, 2013
- J. Banasiak, **M.L.**, Methods of small parameter in mathematical biology, **Birkhäuser**, Boston 2014
- **M.L.**, H. Leszczyński, M. Parisot, A simple kinetic equation of swarm formation: Blow–up and global existence, **Appl. Math. Letters** 57, 2016
- M. Parisot, **M.L.**, A kinetic model for the formation of swarms with nonlinear interactions, **Kinetic Related Models** 9, 2016
- **M.L.**, H. Leszczyński, M. Parisot, Blow–up and global existence for a kinetic equation of swarm formation, **Math. Models Methods Appl. Sci.**, to appear

- J. Banasiak, **M.L.**, On a macroscopic limit of a kinetic model of alignment, **Math. Models Methods Appl. Sci.** 23, 2013
- J. Banasiak, **M.L.**, Methods of small parameter in mathematical biology, **Birkhäuser**, Boston 2014
- **M.L.**, H. Leszczyński, M. Parisot, A simple kinetic equation of swarm formation: Blow–up and global existence, **Appl. Math. Letters** 57, 2016
- M. Parisot, **M.L.**, A kinetic model for the formation of swarms with nonlinear interactions, **Kinetic Related Models** 9, 2016
- **M.L.**, H. Leszczyński, M. Parisot, Blow–up and global existence for a kinetic equation of swarm formation, **Math. Models Methods Appl. Sci.**, to appear

- J. Banasiak, **M.L.**, On a macroscopic limit of a kinetic model of alignment, **Math. Models Methods Appl. Sci.** 23, 2013
- J. Banasiak, **M.L.**, Methods of small parameter in mathematical biology, **Birkhäuser**, Boston 2014
- **M.L.**, H. Leszczyński, M. Parisot, A simple kinetic equation of swarm formation: Blow–up and global existence, **Appl. Math. Letters** 57, 2016
- M. Parisot, **M.L.**, A kinetic model for the formation of swarms with nonlinear interactions, **Kinetic Related Models** 9, 2016
- **M.L.**, H. Leszczyński, M. Parisot, Blow–up and global existence for a kinetic equation of swarm formation, **Math. Models Methods Appl. Sci.**, to appear

- J. Banasiak, **M.L.**, On a macroscopic limit of a kinetic model of alignment, **Math. Models Methods Appl. Sci.** 23, 2013
- J. Banasiak, **M.L.**, Methods of small parameter in mathematical biology, **Birkhäuser**, Boston 2014
- **M.L.**, H. Leszczyński, M. Parisot, A simple kinetic equation of swarm formation: Blow–up and global existence, **Appl. Math. Letters** 57, 2016
- M. Parisot, **M.L.**, A kinetic model for the formation of swarms with nonlinear interactions, **Kinetic Related Models** 9, 2016
- **M.L.**, H. Leszczyński, M. Parisot, Blow–up and global existence for a kinetic equation of swarm formation, **Math. Models Methods Appl. Sci.**, to appear