A classification scheme for the inverse problem in arbitrary dimension

THoan Do & Geoff Prince

Ostrava

July 7, 2016.

A classification scheme for the inverse problem in arbitrary dimension

1. The Problem

2. Timeline: what's known, what's not

3. Geometric formulation of the IP

4. EDS

5. The classification for arbitrary n

6. An example of the progress

1. The inverse problem, related problems

"When are the solutions of

$$\ddot{x}^a = F^a(t, x^b, \dot{x}^b) \quad a, b = 1, \ldots, n$$

the solutions of

$$\frac{\partial^2 L}{\partial \dot{x}^a \partial \dot{x}^b}\ddot{x}^b + \frac{\partial^2 L}{\partial x^b \partial \dot{x}^a}\dot{x}^b + \frac{\partial^2 L}{\partial t \partial \dot{x}^a} = \frac{\partial L}{\partial x^a}$$

for some $L(t, x^a, \dot{x}^a)$?"

So, find regular g_{ab} (and L) so that

$$g_{ab}(\ddot{x}^b - F^b) \equiv \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}^a}\right) - \frac{\partial L}{\partial \dot{x}^a}$$

- The multiplier problem.

Applications in calculus of variations, geometry, classical mechanics, quantum mechanics

Helmholtz conditions (Douglas 1941, Sarlet 1982) necessary and sufficient conditions on g_{ab}:

$$g_{ab} = g_{ba}, \quad \Gamma(g_{ab}) = g_{ac}\Gamma^c_b + g_{bc}\Gamma^c_a$$
$$g_{ac}\Phi^c_b = g_{bc}\Phi^c_a, \quad \frac{\partial g_{ab}}{\partial \dot{x}^c} = \frac{\partial g_{ac}}{\partial \dot{x}^b}$$

where

$$\Gamma^a_b := -\frac{1}{2}\frac{\partial F^a}{\partial \dot{x}^b},$$
$$\Phi^a_b := -\frac{\partial F^a}{\partial x^b} - \Gamma^c_b\Gamma^a_c - \Gamma(\Gamma^a_b)$$
$$\Gamma := \frac{\partial}{\partial t} + u^a\frac{\partial}{\partial x^a} + F^a\frac{\partial}{\partial \dot{x}^a}$$

Significance

- Non-existence.

  No Lagrangian
  $\rightarrow$ No action integral or cost function
  $\rightarrow$ dissipation and/or coupling to blame?

- Non-uniqueness.

  Too many Lagrangians
  $\rightarrow$ Which one is optimal? Are constraints required?

Question:

Can we decide the U & E question without heavy calculation?

Answer:

Yes, by using a Douglas-type classification

3. Timeline: What's known, what's not

1886 Sonin solves IP for one equation ($n = 1$)

1887 Helmholtz states problem

1898 Hirsch states problem

1941 Douglas solves IP for $n = 2$

1982 Henneaux & Shepley algorithm for solving general IP, identify QM difficulties

1982 Sarlet reformulates HH conditions

1984 Crampin, Prince, Thompson geometrise problem

1986 Marmo gives seminal talk at Ghent workshop

1992 Anderson & Thompson apply EDS technique and solve first arbitrary n subcase

1994 Crampin et al reframe Douglas in geometric terms

1999 Crampin, Prince, Sarlet & Thompson solve more arbitrary n cases

2003 Aldridge applies EDS to Douglas $n = 2$ and some arbitrary n

2016 Do and Prince deliver the classification structure for arbitrary n and apply it to $n = 3$

4. Geometric formulation of the IP

When $\ddot{x}^a = F^a(t, x^b, \dot{x}^b)$ are (normalized) Euler-Lagrange equations, then Γ is the unique vectors field on E s.t.

$$\Gamma \lrcorner\ d\theta_L = 0, dt(\Gamma) = 1$$

where

$$\theta_L := Ldt + dL \circ S = Ldt + \frac{\partial L}{\partial u^a}\theta^a$$

$$d\theta_L = \frac{\partial^2 L}{\partial u^a \partial u^b}\psi^a \wedge \theta^b \quad \text{Cartan 2-form}$$

Theorem (CPT 1984) Given a semispray Γ, necessary and sufficient conditions for the existence of a Lagrangian whose $E-L$ field is Γ are that there exists $\Omega \in \Lambda^2(E)$:

1. Ω has max'l rank

2. $\Omega(V_1, V_2) = 0 \ \forall V_1, V_2 \in V(E)$

3. $\Gamma \lrcorner \Omega = 0$

4. $d\Omega = 0$

Usually begin the search for Ω by assuming 1, 2, 3 i.e.

$$\Omega = g_{ab}\psi^a \wedge \theta^b, |g_{ab}| \neq 0$$

and requiring

$$d\Omega = 0.$$

$d\Omega(X, Y, Z) = 0$ give the Helmholtz conditions e.g.

$$d\Omega(\Gamma, V_a, H_b) = 0 \Leftrightarrow \Gamma(g_{ab}) - g_{bc}\Gamma_a^c - g_{ac}\Gamma_b^c = 0$$
$$d\Omega(\Gamma, V_a, V_b) = 0 \Leftrightarrow g_{ab} = g_{ba}$$
$$d\Omega(\Gamma, H_a, H_b) = 0 \Leftrightarrow g_{ac}\Phi_b^c = g_{bc}\Phi_a^c$$

etc.

5. EDS and the Inverse Problem

EDS reference Bryant, Chern et al 1991.

IP reference Anderson and Thompson 1992.

In EDS terms, the I.P. is

"Find all closed, maximal rank 2-forms in $\Sigma := Sp\{\psi^a \wedge \theta^b\} \subset \Lambda^2(E)$"

3 steps

1. Find the largest differential ideal generated by Σ. An algebraic and iterative process.

2. Create a Pfaffian system from the closure condition on this ideal. A differential process.

3. Apply Cartan-Kähler to determine the generality of the solution of this Pfaffian system. An art form!

The differential ideal step.

Q. Set $\Sigma^0 := \Sigma = Sp\{\psi^a \wedge \theta^b\}$.

Is $\langle \Sigma^0 \rangle$ closed?

A. Yes - done!

No - define $\Sigma^1 := \{\omega \in \Sigma^0 : d\omega \in \langle \Sigma^0 \rangle\}$

Q. Is $\langle \Sigma^1 \rangle$ closed?

etc.

This process terminates for some (possibly trivial) Σ^{final}.

If Σ^{final} non-trivial go to step 2.

Notes

1. The differential ideal steps

$$\Sigma^0 \to \Sigma^1 \to \cdots \to \Sigma^{\text{final}}$$

   generate hierarchies of **algebraic conditions** on the multiplier, eg if $\omega \in \Sigma^k$ then

$$\omega(X^V, Y^H) = \omega(Y^V, X^H)$$
$$\omega(\Phi(X)^V, Y^H) = \omega(\Phi(Y)^V, X^H)$$
$$\cdots$$
$$\omega((\nabla^k \Phi(X))^V, Y^H) = \omega((\nabla^k \Phi(Y))^V, X^H)$$

   There is a similar hierarchy of curvature conditions.

2. If Σ^k **is** a differential ideal then we get conditions on Φ, eg, $\Phi = \lambda I$.

3. If Σ^k is a differential ideal **and** contains closed 2 forms then we get **differential conditions** on the multiplier (EDS step 2).

Example 1. (n=3)

$$\Sigma^1 = Sp\{\omega^{11}, \omega^{22}, \omega^{33}\}$$

$$(\omega^{aa} := \psi^a \wedge \theta^a)$$

$$\Sigma^2 = Sp\{\omega^1 := \omega^{11} + r_3^1 \omega^{33},\ \omega^2 := \omega^{22} + r_3^2 \omega^{33}\}$$

For $\omega = \omega^1 + p\omega^2 \in \Sigma^2$, $\exists\ \lambda_1, \lambda_2 :$

$$d\omega \in \langle \Sigma^2 \rangle \iff$$

$$d\omega^1 + pd\omega^2 = \lambda_1 \wedge \omega^1 + \lambda_2 \wedge \omega^2 \qquad (1)$$

Now we use $d\omega^1, d\omega^2 \in \langle \Sigma^1 \rangle$ and

$$\omega^1 := \omega^{11} + r_3^1 \omega^{33},\ \omega^2 := \omega^{22} + r_3^2 \omega^{33}$$

to get (1) in terms of ω^{11}, ω^{22}.

We get a linear system of 4 equations in 5 unknowns (p and 4 components of λ_1, λ_2) whose rank depends on the known coefficients r_3^1, r_3^2.

If there is no condition on p then $\langle \Sigma^2 \rangle$ is a differential ideal and we are finished.

The differential ideal step generates all the necessary and sufficient conditions in a basis calculation.

2. Pfaffian system step.

Let $\Sigma^{\text{final}} = Sp\{\omega^k\},\ k = 1, \ldots, d$

then $d\omega^k = \xi^k_h \wedge \omega^h$

$\xi^k_h \in \Lambda^1(E)$ are known.

In order to find $\omega := r_k \omega^k$ with $d\omega = 0$ we will need

$$\rho_k \in \Lambda^1(E) : \rho_k \wedge \omega^k = 0$$

Suppose the solutions are

$$(\rho^A_k) := (\rho^A_1, \ldots, \rho^A_d), A = 1, \ldots, e$$

with

$$\rho^A_k \wedge \omega^k = 0$$

- an e-dim'l module of d-tuples of 1-forms on E.

Then $\omega = r_k \omega^k$ with $d\omega = 0, \; r_k \in \mathcal{F}(E)$

becomes

$$
\begin{aligned}
& d(r_k \omega^k) = 0 \\
\Rightarrow \quad & dr_k \wedge \omega^k + r_k d\omega^k = 0 \\
\Rightarrow \quad & dr_k \wedge \omega^k + r_k \xi^k_h \wedge \omega^h = 0 \\
\Rightarrow \quad & (dr_k + r_k \xi^k_h) \wedge \omega^h = 0 \\
\Rightarrow \quad & dr_k + r_k \xi^k_h = -p_A \rho^A_k \\
\Rightarrow \quad & \boxed{\sigma_k := dr_k + r_k \xi^k_h + p_A \rho^A_k = 0}
\end{aligned}
$$

Define

$$N := E \bigotimes \mathbb{R}^d \bigotimes \mathbb{R}^e$$

co-ords: $(\underbrace{t, x^a, u^a}_{y^\mu}; r_k; p_A)$

which are sections of N over E.

Basis for $\Lambda^1(N)$:

$\{\alpha_\mu, \sigma_k, \pi_A := dp_A\}$
$\{\alpha_\mu\}$ a basis for $\Lambda^1(E)$ pulled back by the section.

So

- $\alpha_1 \wedge \ldots \wedge \alpha_{2n+1} \neq 0$ on the image of the section.

INDEPENDENCE CONDITION

Frobenius integrability of $D^1_\sigma = Sp\{\sigma_k\}$:

Want $d\sigma_k \equiv 0$ mod σ_k

But

$$d\sigma_k \;\equiv\; \underset{\text{part}}{\pi\Lambda\alpha} \;+\; \underbrace{\underset{\text{part}}{\alpha\Lambda\ \alpha}}_{\substack{\text{never zero}\\ \text{on section}}}$$

$\alpha\Lambda\alpha$ part is the "torsion" - an obstruction to integrability.

If $\alpha\Lambda\alpha$ part can't be absorbed into $\pi\Lambda\alpha$ part by $\pi \to \overline{\pi}$ **there is no solution.**

So absorb torsion and go to Cartan-Kähler theorem.

3. Cartan-Kähler thm to determine generality of solutions.

$$d\sigma_k \equiv \pi_k^\mu \wedge \alpha_\mu (\text{mod } \sigma_k.)$$

$$\begin{array}{c|ccccccc}
 & \alpha_1 & \alpha_2 & & & & & \alpha_{2n+1} \\
\hline
d\sigma_1 & \pi_1^1 & \pi_1^2 & \cdot & \cdot & \cdot & \cdot & \pi_1^{2n+1} \\
 & \cdot & \cdot & & & & & \cdot \\
 & \cdot & \cdot & & & & & \cdot \\
 & \cdot & \cdot & & & & & \cdot \\
 & \cdot & \cdot & & & & & \cdot \\
d\sigma_d & \pi_d^1 & \pi_d^2 & \cdot & \cdot & \cdot & \cdot & \pi_d^{2n+1}
\end{array}$$

(a) "Optimise" this tableau and calculate Cartan characters

$$\underbrace{s_1, \ldots, s_\ell, 0, \ldots 0}_{2n+1} \quad s_1 \geq s_2 \geq \cdots \geq s > 0$$

(b) Apply Cartan involution test.

Involutive? No - prolong and start again.

? Involutive

Yes!!

- general solution g_{ab} specified by s_ℓ arbitrary functions of ℓ variables.

Calculate L!

Notes

- $\omega = r_k \omega^k$ are not explicitly calculated.

- computations are easy for a given F^a (in low dim'n).

- classification à la Douglas proceeds by examining diagonalisability of Φ.

- usual $\Phi, \mathbb{R}$ hierachies appear at diff'l ideal step.

5. The classification for arbitrary n

Do and Prince 2015.

A. $\Phi = \lambda I_n \Leftrightarrow \langle \Sigma^0 \rangle$is a differential ideal

B. Φ diagonalisable with distinct eigenvalues

- Subcases according to the integrability of eigenspaces of Φ i.e. p eigenspaces are non-integrable and $n-p$ are integrable

- Non-existence: no differential ideal before step $p+1 \Rightarrow$ no non-degenerate multiplier.

- Existence subcases: a differential ideal is generated at step 1, step 2,..., up to step p.

- Further subcases: integrable subspaces of non-integrable eigenspaces.

C. Φ is diagonalisable with repeated eigenvalues

Subcases: similar to case B above.

D. Φ is not diagonalisable

Subcases: integrability of normal forms of Φ.

Douglas treated the differential ideal conditions first and the eigenspace integrability second!

Examples

Theorem (Do, 2016)

Assume that $\mathbf{\Phi}$ is diagonalisable with distinct (real) eigenvalues and with $p \geq 2$ non-integrable eigen co-distributions. Suppose that eigen co-distributions are ordered such that $Sp\{\phi^{AV}, \phi^{AH}\}$, $A = 1, \ldots, p$ are non-integrable. Suppose further that $rank(\mathbf{A}_1) = p-1$. Then the necessary and sufficient conditions for the existence of a solution for the associated inverse problem are that the given conditions hold. Moreover, **the solution (if it exists) depends on $n-p$ arbitrary functions of 2 variables each**.

Example $n = 4$ (Do, 2016)

$$\ddot{x} = x, \; \ddot{y} = 0, \; \ddot{z} = \frac{\dot{y}}{\dot{z}}, \; \ddot{w} = \dot{w}$$

has no solution (condition checking alone, no PDEs!)

Question

What can we do we couldn't do before?

Answers

- Decide and elaborate variationality by checking, not integrating
- Explicitly find all Lagrangians for particular examples at any dimension
- Explicitly elaborate all $n = 3$ cases
- Retire the problem?