Jets
or, how to geometrize the differential calculus

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Abstract

The idea of a "jet" was introduced by Charles Ehresmann in 1951 as an equivalence class of maps, all defined in some neighbourhood of a given point, and with the same value and derivatives (up to a given order) at that point. Conceptually, therefore, a jet may be considered as an abstract Taylor polynomial.

In this talk I shall expand on this definition, and explain the structure of various spaces of jets (of which the simplest is the tangent bundle of a differentiable manifold).

I shall also explain how the use of jets can give a precise meaning to certain aspects of the Euler-Lagrange equations of the calculus of variations, such as the ideas of "differentiating with respect to derivatives" and of "total derivative".

The Euler–Lagrange equations

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$$\frac{\partial L}{\partial q^i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}^i} = 0$$

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A formal approach uses the concept of
tangent vectors on a manifold

Tangent vectors: the classical approach

Take a manifold M: a locally Euclidean topological space whose charts $x : U \to \mathbb{R}^n$, $U \subset M$ are differentiably related (we'll assume that maps $\hat{x} \circ x^{-1} : x(U \cap \hat{U}) \to \hat{x}(U \cap \hat{U})$ are C^∞)

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Formally, $\xi = \big[\big((\xi^i), x\big)\big] \in T_p M$ where $(\xi^i) \in \mathbb{R}^n$, $p \in U \subset M$, $x : U \to \mathbb{R}^n$ and

$$\big((\xi^i), x\big) \sim \big((\hat{\xi}^i), \hat{x}\big) \quad \text{if} \quad \hat{\xi}^i = \left.\frac{\partial \hat{x}^i}{\partial x^j}\right|_p \xi^j$$

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$T_p M$ is the tangent space to M at p □

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Take $\mathbb{R}(\epsilon)$, $\epsilon^2 = 0$, as the algebra of 'dual numbers'.

An algebraic tangent vector ξ at $p \in M$ is an algebra homomorphism $C^\infty(M) \to \mathbb{R}(\epsilon)$ satisfying

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Algebraic and classical tangent vectors may be identified ...
in the C^∞ case! $\square$

Tangent vectors: the jet approach

Let $\gamma : \mathbb{R} \to M$ be a curve with $\gamma(0) = p$.

A geometric tangent vector is an equivalence class $[\gamma]$ where $\tilde{\gamma} \sim \gamma$ if

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and

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An equivalence class defined in this way is called a jet $\square$

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The second condition may be written in fibred coordinates (x^i, u^α) on E around $\phi(p)$ as

$$\left.\frac{\partial\tilde{\phi}^\alpha}{\partial x^i}\right|_p = \left.\frac{\partial\phi^\alpha}{\partial x^i}\right|_p$$

The 1-jet manifold

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The source map π_1 defines a fibred manifold
The target map $\pi_{1,0}$ defines an affine bundle (always!) □

The affine structure

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So regard $J^1\pi$ as a sub-bundle of $\pi^*T^*M \otimes_E TE$
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In coordinates

$$j_p^1\phi \sim dx^i|_p \otimes \left(\left.\frac{\partial}{\partial x^i}\right|_{\phi(p)} + \left.\frac{\partial \phi^\alpha}{\partial x^i}\right|_p \left.\frac{\partial}{\partial u^\alpha}\right|_{\phi(p)} \right)$$

Prolonging local sections

Suppose $U \subset M$ is open, and $\phi : U \to E$ is a local section

For each $p \in U$ there is a 1-jet $j^1_p\phi \in J^1\pi$

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NOT EVERY LOCAL SECTION OF $\pi_1 : J^1\pi \to M$ IS A PROLONGATION!

□

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Thinking of a jet $j_p^1\phi$ as a tensor at $\phi(p) \in E$,
a connection is a tensor field (horizontal projector)

$$P_\Gamma = dx^i \otimes \left(\frac{\partial}{\partial x^i} + \Gamma_i^\alpha \frac{\partial}{\partial u^\alpha} \right)$$

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the *covariant differential* ∇^Γ is defined by

$$\nabla^\Gamma\phi = j^1\phi - \Gamma \circ \phi$$

k-jets of sections

Continue with a fibred manifold $\pi : E \to M$

Take $p \in M$. Two local sections ϕ, $\tilde{\phi}$ defined near p are *k-equivalent at p* if

- $\tilde{\phi}(p) = \phi(p)$ and
- $(f \circ \tilde{\phi} \circ \gamma)^{(r)}(0) = (f \circ \phi \circ \gamma)^{(r)}(0)$ for every $f \in C^\infty(E)$ and every curve γ in M with $\gamma(0) = p \quad (1 \leq r \leq k)$

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The bundle structure of k-jets

The *source map* π_k, the *target map* $\pi_{k,0}$
and the *order-reduction map* $\pi_{k,l} : J^k\pi \to J^l\pi$ $(1 \leq l \leq k)$ are

$$\begin{array}{ccc} \pi_k : J^k\pi \to M & \pi_{k,0} : J^k\pi \to E & \pi_{k,l} : J^k\pi \to J^l\pi \\ \pi_k(j^k_p\phi) = p & \pi_{k,0}(j^k_p\phi) = \phi(p) & \pi_{k,l}(j^k_p\phi) = j^l_p\phi \end{array}$$

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If $\phi : U \to E$ is a local section,
Define a local section $j^k\phi : U \to J^k\pi$ by $j^k\phi(p) = j^k_p\phi$ called
the *prolongation* of ϕ $\square$

Symmetric coordinates and multi-indices

$J^2\pi$ is a manifold with coordinates $(x^i, u^\alpha, u^\alpha_i, u^\alpha_{ij})$ where

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In general $u^\alpha_i \neq u^\alpha_{\cdot,i}$ and $u^\alpha_{i;j} \neq u^\alpha_{j;i}$ $\square$

Holonomic and semiholonomic jets

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There is, in general, no canonical projection $J^1\pi_1 \to J^2\pi$.

But there is a submanifold $\hat{J}^2\pi \subset J^1\pi_1$ of *semiholonomic jets* where $u^\alpha_{\cdot,j} = u^\alpha_j$, but $u^\alpha_{i,j}$ need not be symmetric

Holonomic and semiholonomic jets

$$u_i^\alpha(j_p^1\psi) = \psi_i^\alpha(p)\,, \qquad u_{\cdot,j}^\alpha(j_p^1\psi) = \left.\frac{\partial\psi^\alpha}{\partial x^j}\right|_p\,, \qquad u_{i,j}^\alpha(j_p^1\psi) = \left.\frac{\partial\psi_i^\alpha}{\partial x^j}\right|_p$$

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and $\hat{J}^2\pi = J^2\pi \oplus_{J^1\pi} \left(\bigwedge^2 \pi_1^* T^*M \otimes \pi_{1,0}^* V\pi\right)$

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Can also define *q-contact r-forms* and *exactly q-contact r-forms* $(q \leq r)$

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So given a vector field X on M, the corresponding *total derivative* is

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A map $f : E \to F$ is a *fibred map* if $y, z \in E_p$ implies $f(y) = f(z)$
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$$X = X^i \frac{\partial}{\partial x^i} + X^\alpha \frac{\partial}{\partial u^\alpha}$$

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If X is not projectable, it can still be prolonged!
(Unlike diffeomorphisms) □

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An *infinitesimal contact transformation* is a vector field X on $J^k\pi$ such that $\mathcal{L}_X\omega$ is a contact form whenever ω is a contact form $\square$

The calculus of variations

A *Lagrangian form* is a horizontal m-form λ on $J^k\pi$ ($m = \dim M$)

The *variational problem* defined by λ and a compact connected m-dimensional submanifold $C \subset M$ is

$$\frac{d}{dt}\int_C (j^k\phi_t)^*\lambda = 0$$

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The 1-contact part ε_λ of $d\theta_\lambda$ is called the *Euler–Lagrange form*

In coordinates, if $\lambda = L dx^1 \wedge \cdots \wedge dx^m$ then

$$\varepsilon_\lambda = \left(\frac{\partial L}{\partial u^\alpha} - \sum_{|I|=1}^{2k-1} (-1)^{|I|-1} \frac{d^{|I|}}{dx^I}\frac{\partial L}{\partial u^\alpha_I}\right) du^\alpha \wedge dx^1 \wedge \cdots \wedge dx^m \, .$$

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The skew-symmetric part is the *curvature* of the connection □

Formal integrability of PDEs

Let $R \subset J^k\pi$ be a closed fibred submanifold with $\pi_k(R) = M$

R is a *differential equation*
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For C^∞ systems the formal series might not define a solution! □

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Consider also the submanifold $T^{\circ k}_n$ of *regular velocities* where γ is an immersion near zero $\square$

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Consider also the *oriented jet group* L_n^{k+}
where φ has positive Jacobian determinant at zero □

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Taking the quotient by the oriented jet group L^{k+}_n gives oriented contact elements $J^{k+}_n M$ □