Geometric Structures on Lie groups and post Lie algebras

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1

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Pre-Lie algebras

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• If $(V, \cdot)$ is a pre-Lie algebra, then for $x, y \in V$ the binary operation

$$[x, y] := x \cdot y - y \cdot x$$

defines a Lie algebra.

Definition

A bilinear product $x \cdot y$ on $\mathfrak{g} \times \mathfrak{g}$ is called a pre-Lie algebra structure on $\mathfrak{g}$, if it satisfies

$$\begin{aligned} x \cdot y - y \cdot x &= [x, y], \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z), \end{aligned}$$

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for all $x, y, z \in \mathfrak{g}$.

Definition

A Lie algebra $\mathfrak{g}$ over a field K is said to admit a pre-Lie algebra structure, if there exists a pre-Lie algebra structure on $\mathfrak{g}$.

Example

The Heisenberg Lie algebra $\mathfrak{n}_3(K)$ *of dimension* 3 *with basis* $\{e_1, e_2, e_3\}$ *and Lie brackets* $[e_1, e_2] = e_3$ *admits a pre-Lie algebra structure, given by*

$$e_1 \cdot e_2 = \frac{1}{2} e_3,$$
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Example

The Lie algebra $\mathfrak{sl}_2(K)$ *over a field* K *of characteristic zero does not admit a pre-Lie algebra structure.*

The affine group

- Denote by $\mathrm{Aff}(\mathbb{R}^n) \simeq \mathbb{R}^n \rtimes GL_n(\mathbb{R})$ the group of affine transformations of $\mathbb{R}^n$.

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• We may represent the elements of $\mathrm{Aff}(\mathbb{R}^n)$ by block matrices $\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix}$ with $A \in GL_n(\mathbb{R})$, $v \in \mathbb{R}^n$ and multiplication

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} B & w \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} AB & Aw + v \\ 0 & 1 \end{pmatrix}.$$

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- $\mathrm{Aff}(\mathbb{R}^n)$ acts on $\mathbb{R}^n$ by

$$\begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ 1 \end{pmatrix} = \begin{pmatrix} Ax + v \\ 1 \end{pmatrix}.$$

- The affine group is a linear algebraic group represented by

$$\mathrm{Aff}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in GL_n(\mathbb{R}), v \in \mathbb{R}^n \right\}.$$

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- It generalizes the isometry group of $\mathbb{R}^n$,

$$\mathrm{Iso}(\mathbb{R}^n) = \left\{ \begin{pmatrix} A & v \\ 0 & 1 \end{pmatrix} \mid A \in O_n(\mathbb{R}), v \in \mathbb{R}^n \right\}.$$

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- The translations in $\mathrm{Aff}(\mathbb{R}^n)$ form a normal subgroup, given by

$$T(n) = \left\{ \begin{pmatrix} I_n & v \\ 0 & 1 \end{pmatrix} \mid v \in \mathbb{R}^n \right\}.$$

Simply transitive groups

- A group G acts simply transitively on $\mathbb{R}^n$ by affine transformations if there is a homomorphism $\rho\colon G \to \mathrm{Aff}(\mathbb{R}^n)$ letting G act on $\mathbb{R}^n$, such that for all $y_1, y_2 \in \mathbb{R}^n$ there is a unique $g \in G$ such that $\rho(g)(y_1) = y_2$.

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• L. Auslander named such groups simply transitive groups of affine motions. They are connected, simply connected n-dimensional Lie groups homeomorphic to $\mathbb{R}^n$.

• An example of a simply transitive group of affine motions is the normal subgroup $T(n)$ of translations.

Proposition (L. Auslander 1977)

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More generally the following result holds, which more or less can be found in G. Hochschild's book The Structure of Lie Groups (1965).

Proposition

Let G be a Lie group which is homeomorphic to $\mathbb{R}^n$ for some $n \geq 1$. If G admits a faithful linear representation then G is solvable.

Affinely flat manifolds

• An affinely flat structure on an n-dimensional manifold M is a collection of coordinate homeomorphisms

$$f_\alpha \colon U_\alpha \to V_\alpha \subseteq \mathbb{R}^n,$$

where the U_α are open sets covering M, and the V_α are open subsets of $\mathbb{R}^n$; whenever $U_\alpha \cap U_\beta \neq \emptyset$, it is required that the change of coordinate homeomorphism

$$f_\beta f_\alpha^{-1} \colon f_\alpha(U_\alpha \cap U_\beta) \to f_\beta(U_\alpha \cap U_\beta)$$

extends to an affine transformation in $\mathrm{Aff}(\mathbb{R}^n)$. We call M together with this structure an affinely flat manifold, or affine manifold.

• A special case of affine flat manifolds are Riemannian-flat manifolds, where the coordinate changes extend to isometries in $\mathrm{Iso}(\mathbb{R}^n)$, i.e., to affine transformations $x \mapsto Ax + b$ with $A \in O_n(\mathbb{R})$.

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Theorem (Benzecri 1959)

A closed surface admits an affine (affinely flat) structure if and only if its Euler characteristic vanishes.

• In particular, a closed surface different from the 2-torus or the Klein bottle does not admit any affine structure.

Proposition

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- An affine connection ∇ is called torsionfree if

$$\nabla_X(Y) - \nabla_Y(X) - [X, Y] = 0 \tag{1}$$

for all $X, Y \in \mathfrak{X}$, where $\mathfrak{X}$ denotes the Lie algebra of all differential vector fields on M.

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- An affine connection ∇ is called flat if

$$\nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} = 0 \tag{2}$$

for all $X, Y \in \mathfrak{X}$.

• A torisonfree flat affine connection determines a covariant differentiation $\nabla_X : \mathfrak{X} \to \mathfrak{X}$ via $Y \mapsto \nabla_X(Y)$ for vector fields $X, Y \in \mathfrak{X}$.

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• Setting

$$X \cdot Y := \nabla_X(Y),$$

we obtain an $\mathbb{R}$-bilinear product on $\mathfrak{X}$. Because of (1) and (2) this product turns $\mathfrak{X}$ into a pre-Lie algebra:

$$X \cdot Y - Y \cdot X - [X, Y] = 0,$$
$$X \cdot (Y \cdot Z) - Y \cdot (X \cdot Z) = [X, Y] \cdot Z.$$

Left-invariant affine structures on Lie groups

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• An affine structure on G is called complete, if the universal covering $\widetilde{G}$ is affinely diffeomorphic to $\mathbb{R}^n$.

Theorem

There is a canonical bijection between complete left-invariant affine structures on G and simply transitive actions of G on $\mathbb{R}^n$ by affine transformations.

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- Here a pre-Lie algebra structure on $\mathfrak{g}$ is complete, if all right multiplications $R(x)$ in $\mathrm{End}(\mathfrak{g})$ are nilpotent.

Milnor's question

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Does every solvable n-dimensional Lie group G admit a complete left-invariant affine structure, or equivalently, does the universal covering group $\widetilde{G}$ act simply transitively by affine transformations on $\mathbb{R}^n$?

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Milnor's Question - algebraic version

Does every solvable Lie algebra over a field of characteristic zero admit a (complete) pre-Lie algebra structure?

Positive evidence for Milnor's question

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• Milnor's question has a positive answer for all (connected and simply connected) nilpotent Lie groups of dimension $n \leq 7$.

• Milnor's question has a positive answer for all 2-step solvable Lie groups whose Lie algebra is a semidirect product $\mathfrak{a} \rtimes \mathfrak{b}$ of two abelian Lie algebras.

A negative answer to Milnor's question

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Theorem

Let G be a n-dimensional Lie group with Lie algebra $\mathfrak{g}$. Suppose that G admits a left-invariant affine structure. Then $\mathfrak{g}$ admits a faithful linear Lie algebra representation $\varphi\colon \mathfrak{g} \to \mathfrak{gl}_{n+1}(\mathbb{R})$ of degree $n+1$.

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Proof: The left-invariant affine structure on G induces a pre-Lie algebra structure $x \cdot y = L(x)y$ on $\mathfrak{g}$, so that

$$L\colon \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}),\ x \mapsto L(x)$$

is a linear representation of degree n. The corresponding $\mathfrak{g}$-module $\mathfrak{g}_L$ need not be faithful, but using a nonsingular 1-cocycle we can construct a faithful $\mathfrak{g}$-module of dimension $n+1$ from it.

Because of $[x, y] = x \cdot y - y \cdot x$, the 1-cocycle

$$\omega = \mathrm{id} \in Z^1(\mathfrak{g}, \mathfrak{g}_L)$$

is nonsingular. Hence we have $\ker(\omega) = 0$, and $V_\omega := \mathbb{R} \times \mathfrak{g}_L$ is a faithful $\mathfrak{g}$-module of dimension $n+1$, with action

$$x.(t, v) = (0, x.v + t\omega(x))$$

for $x \in \mathfrak{g}$, $v \in \mathfrak{g}_L$ and $t \in \mathbb{R}$. $\square$

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Definition

Let $\mathfrak{g}$ be a Lie algebra over a field K of dimension n. Denote by $\mu(\mathfrak{g})$ the minimal dimension of a faithful linear representation of $\mathfrak{g}$.

Theorem (B. 1996)

There exists families of 10*-dimensional filiform nilpotent Lie algebras* $\mathfrak{g}$ *such that* $\mu(\mathfrak{g}) \geq 12$*. These algebras give a negative answer to Milnor's question.*

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Theorem (B., Moens 2010)

For every filiform nilpotent Lie algebra of dimension 10 *we have*

$$\boxed{10 \leq \mu(\mathfrak{g}) \leq 18}$$

There is a classification of such algebras satisfying $\mu(\mathfrak{g}) \leq 11$, *respectively* $\mu(\mathfrak{g}) \geq 12$.

Nil-affine transformations

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- $\mathrm{Aff}(N)$ acts on N by $(n, \alpha) \cdot m = n\alpha(m)$.

- For $N = \mathbb{R}^n$ we obtain again $\mathrm{Aff}(\mathbb{R}^n) = \mathbb{R}^n \rtimes Aut(\mathbb{R}^n)$.

• We say that G admits a simply transitively action by nil-affine transformations on N, if there is a homomorphism $\rho\colon G \to \mathrm{Aff}(N)$ letting G act simply transitively on N.

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• In the nil-affine setting, Milnor's question has a positive answer:

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• In the nil-affine setting, Milnor's question has a positive answer:

Proposition (Dekimpe 2003, Baues 2004)

Let G be a solvable Lie group. Then G admits a simply transitive action by nil-affine transformations on some simply connected nilpotent Lie group N. Conversely, assume that G admits such an action. Then G is solvable.

Reduction to the Lie algebra level

Theorem (B-D-V 2012)

Let G and N be nilpotent Lie groups. Then there exists a simply transitive action by nil-affine transformations of G on N if and only if there exists a Lie algebra $\mathfrak{h} \cong \mathfrak{g}$ such that the corresponding pair of Lie algebras $(\mathfrak{h}, \mathfrak{n})$ admits a complete post-Lie algebra structure.

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• In the classical case $N = \mathbb{R}^n$ a complete post-Lie algebra structure on $(\mathfrak{g}, \mathbb{R}^n)$ is just a complete pre-Lie algebra structure on $\mathfrak{g}$; also called an affine structure on $\mathfrak{g}$.

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• In the classical case $N = \mathbb{R}^n$ a complete post-Lie algebra structure on $(\mathfrak{g}, \mathbb{R}^n)$ is just a complete pre-Lie algebra structure on $\mathfrak{g}$; also called an affine structure on $\mathfrak{g}$.

• In the other extreme case $G = \mathbb{R}^n$ a complete post-Lie algebra structure on $(\mathbb{R}^n, \mathfrak{n})$ is a complete LR-structure on $\mathfrak{n}$ [B-D-D 2009].

Post-Lie algebra structures

Definition (B. Vallette 2007)

A post-Lie algebra $(V, \cdot, \{\,,\})$ *is a vectorspace* V *over a field* k *equipped with two* k*-bilinear operations* $x \cdot y$ *and* $\{x, y\}$*, such that* $\mathfrak{g} = (V, \{\,,\})$ *is a Lie algebra, and*

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$$\{x, y\} \cdot z = (y \cdot x) \cdot z - y \cdot (x \cdot z) - (x \cdot y) \cdot z + x \cdot (y \cdot z) \tag{3}$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\} \tag{4}$$

for all $x, y, z \in V$.

- If $\mathfrak{g}$ is abelian then $(V, \cdot)$ is a pre-Lie algebra.

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- We can associate to a post-Lie algebra $(V, \cdot, \{,\})$ a second Lie algebra $\mathfrak{n} = (V, [\,,])$ via

$$[x, y] := x \cdot y - y \cdot x + \{x, y\}. \tag{5}$$

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- This Lie bracket satisfies the following identity

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z), \tag{6}$$

i.e., the post-Lie algebra is a left module over the Lie algebra $\mathfrak{n}$.

Definition (B-D-V 2012)

Let $(\mathfrak{g}, [x, y])$, $(\mathfrak{n}, \{x, y\})$ be two Lie brackets on a vector space V. A post-Lie algebra structure on the pair $(\mathfrak{g}, \mathfrak{n})$ is a k-bilinear product $x \cdot y$ satisfying the identities

$$x \cdot y - y \cdot x = [x, y] - \{x, y\} \tag{7}$$

$$[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z) \tag{8}$$

$$x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\} \tag{9}$$

for all $x, y, z \in V$.

Definition (B-D-V 2012)

Let $(\mathfrak{g}, [x, y])$*,* $(\mathfrak{n}, \{x, y\})$ *be two Lie brackets on a vector space* V*. A post-Lie algebra structure on the pair* $(\mathfrak{g}, \mathfrak{n})$ *is a* k*-bilinear product* $x \cdot y$ *satisfying the identities*

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for all $x, y, z \in V$*.*

• These identities imply (3) – (6), so that $(V, \cdot, [\,,])$ is a post-Lie algebra with associated Lie algebra $\mathfrak{n}$.

- If $\mathfrak{n}$ is abelian then the conditions $(7), (8), (9)$ reduce to

$$\begin{aligned} x \cdot y - y \cdot x &= [x, y], \\ [x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z), \end{aligned}$$

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- If $\mathfrak{g}$ is abelian then the conditions reduce to

$$\begin{aligned} x \cdot y - y \cdot x &= -\{x, y\} \\ x \cdot (y \cdot z) &= y \cdot (x \cdot z), \\ (x \cdot y) \cdot z &= (x \cdot z) \cdot y, \end{aligned}$$

so that $-x \cdot y$ is an LR-structure on $\mathfrak{n}$.

Algebraic structure results

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Let $\mathfrak{n}$ be 2-step nilpotent, or let $\mathfrak{n}$ be 3-step nilpotent with at most 3 generators. Then $\mathfrak{n}$ admits a complete LR-structure.

Theorem (B-D-D 2009)

There are examples of 4-generated 3-step nilpotent Lie algebras $\mathfrak{n}$ of dimension $n \geq 13$ not admitting any LR-structure.

Theorem (B-D-V 2012)

Let $(V, \cdot)$ *be a post-Lie algebra structure on the pair* $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{g}$ *is nilpotent. Then* $\mathfrak{n}$ *is solvable.*

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Let $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras, where* $\mathfrak{g}$ *is semisimple and* $\mathfrak{n}$ *is solvable. Then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

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Theorem (B-D 2013)

Let $\mathfrak{n}$ *be a semisimple Lie algebra and* $\mathfrak{g}$ *be a solvable Lie algebra. Assume that* $\mathfrak{g}$ *is unimodular. Then there is no post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$*.*

Theorem (B-D-V 2012)

Let $(\mathfrak{g}, \mathfrak{n})$ *be a pair of simple Lie algebras. Then there exists a post-Lie algebra structure on* $(\mathfrak{g}, \mathfrak{n})$ *if and only if* $\mathfrak{g} \cong \mathfrak{n}$*, in which case there are only two trivial possibilities:*

$$x \cdot y = 0, \quad [x, y] = \{x, y\},$$
$$x \cdot y = [x, y] = -\{x, y\}.$$

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$$\begin{aligned} x \cdot y &= 0, \quad [x, y] = \{x, y\}, \\ x \cdot y &= [x, y] = -\{x, y\}. \end{aligned}$$

Example

Let $\mathfrak{g} \cong \mathfrak{n} \cong \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Then there exist non-trivial post-Lie algebra structures on $(\mathfrak{g}, \mathfrak{n})$.

• Let $e_1, f_1, h_1, e_2, f_2, h_2$ be a basis of $\mathfrak{n} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ with Lie brackets

$$\begin{array}{ll} \{e_1, f_1\} = h_1, & \{e_2, f_2\} = h_2, \\ \{e_1, h_1\} = -2e_1, & \{e_2, h_2\} = -2e_2, \\ \{f_1, h_1\} = 2f_1, & \{f_2, h_2\} = 2f_2. \end{array}$$

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- The following product defines a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$, with $\mathfrak{g} \cong \mathfrak{n}$:

$$\begin{array}{lll} e_1 \cdot e_2 = -4e_2 + h_2, & f_1 \cdot e_2 = 2e_2 - h_2, & h_1 \cdot e_2 = 6e_2 - 2h_2, \\ e_1 \cdot f_2 = 4f_2 + 4h_2, & f_1 \cdot f_2 = -2f_2 - h_2, & h_1 \cdot f_2 = -6f_2 - 4h_2, \\ e_1 \cdot h_2 = -8e_2 - 2f_2, & f_1 \cdot h_2 = 2e_2 + 2f_2, & h_1 \cdot h_2 = 8e_2 + 4f_2. \end{array}$$

Existence of post-Lie algebra structures - a table

$(\mathfrak{g}, \mathfrak{n})$	$\mathfrak{n}$ abe	$\mathfrak{n}$ nil	$\mathfrak{n}$ sol	$\mathfrak{n}$ sim	$\mathfrak{n}$ sem	$\mathfrak{n}$ red	$\mathfrak{n}$ com
$\mathfrak{g}$ abelian	✓	✓	✓	—	—	—	✓
$\mathfrak{g}$ nilpotent	✓	✓	✓	—	—	—	✓
$\mathfrak{g}$ solvable	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ simple	—	—	—	✓	—	—	—
$\mathfrak{g}$ semisimple	—	—	—	✓	✓	?	—
$\mathfrak{g}$ reductive	✓	?	?	✓	✓	✓	✓
$\mathfrak{g}$ complete	✓	✓	✓	?	?	✓	✓

Commutative post-Lie algebra structures

Definition

A commutative post-Lie algebra structure, or CPA-structure on a Lie algebra $\mathfrak{g}$ is a k-bilinear product $x \cdot y$ satisfying the identities:

$$
\begin{aligned}
x \cdot y &= y \cdot x \\
[x, y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \\
x \cdot [y, z] &= [x \cdot y, z] + [y, x \cdot z]
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for all $x, y, z \in V$.

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for all $x, y, z \in V$.

• A CPA-structure on $\mathfrak{g}$ corresponds to a post-Lie algebra structure on $(\mathfrak{n}, \mathfrak{g})$ with $[x,y] = \{x,y\}$.

Theorem (B-M 2016)

Any CPA-structure on a perfect Lie algebra $\mathfrak{g}$ *is trivial, i.e., satisfies* $\mathfrak{g} \cdot \mathfrak{g} = 0$.

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Definition

A complete Lie algebra $\mathfrak{g}$ is called simply-complete, if no non-trivial ideal in $\mathfrak{g}$ is complete.

Theorem (B-M 2016)

Let $\mathfrak{g}$ be a simply-complete non-metabelian Lie algebra. Suppose that $\mathfrak{g}$ satisfies the condition $\mathrm{nil}(\mathfrak{g}) = [\mathfrak{g}, \mathrm{nil}(\mathfrak{g})]$. Denote by $\mathfrak{z}$ the center of the ideal $I = [\mathfrak{g}, \mathfrak{g}]$. Then there is a bijective correspondence between CPA-structures on $\mathfrak{g}$ and elements $z \in \mathfrak{z}$, given by

$$x \cdot y = [[z, x], y].$$

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Example

The Lie algebra $\mathfrak{aff}(\mathbb{R}^n)$ is simply-complete. All CPA-structures on $\mathfrak{aff}(\mathbb{R}^n)$ are trivial for $n \geq 2$.

Theorem (B-M 2017)

Let $\mathfrak{g}$ be a nilpotent Lie algebra satisfying $Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$. Then any CPA-structure on $\mathfrak{g}$ is complete, i.e., its left multiplication maps $L(x)$ are nilpotent for all $x \in \mathfrak{g}$.

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Conjecture

Every CPA-structure on the free-nilpotent Lie algebra $\mathfrak{g} = F_{g,c}$ with g generators and nilpotency class c, with $c \geq g \geq 3$, satisfies

$$\mathfrak{g} \cdot \mathfrak{g} \subseteq Z(\mathfrak{g}).$$